CRITERION FOR STRONG CONSTRUCTIVIZABILITY OF A HOMOGENEOUS MODEL

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In this paper we give a negative answer to a question posed by Morley in [7, p. 239]: Is it true that for a homogeneous model $\mathfrak{M}$ of a complete decidable theory T to be strongly constructivizable it is sufficient that the family S of all types realizable in $\mathfrak{M}$ be computable?

In Sec. 2 we describe an appropriate counterexample. In Sec. 3 we give a precise criterion for strong constructivizability of $\mathfrak{M}$, which, besides the computability of S, contains a certain effectiveness condition for extensions of types in S.

1. Preliminaries

In this paper we consider only signatures admitting a Gödel enumeration of the set of all formulas of this signature.

A countable or finite model $\mathfrak{M}$ with enumeration $\nu: N \xrightarrow{\text{onto}} |\mathfrak{M}|$ is called strongly constructive if the theory $Th\,\mathfrak{M}_\nu$ is decidable, where $\mathfrak{M}_\nu$ is the enrichment of the signature of $\mathfrak{M}$ by new constant symbols c_i, $i<\omega$, so that c_i can be interpreted by the element $\nu(i) \in |\mathfrak{M}|$. A systematic exposition of the theory of constructive models can be found in Ershov [2].

A diagram is a noncontradictory quantifier-free formula $\mathcal{D}(x_0, x_1, \ldots, x_{n-1})$ of finite predicate signature σ which is a conjunction of atomic formulas or their negations over the variables $x_0, x_1, \ldots, x_{n-1}$, where each such formula (or its negation) occurs in $\mathcal{D}$. If $m<n$, then the diagram $\mathcal{D}_1(x_0, x_1, \ldots, x_{m-1})$ is called a subdiagram of $\mathcal{D}$ if $\mathcal{D}_1$ is obtained from $\mathcal{D}$ by deleting all terms containing the variables $x_m, \ldots, x_{n-1}$.

By an n-type $p(x_0, x_1, \ldots, x_{n-1})$, of a complete theory T we mean a maximal set of formulas consistent with T and containing only formulas whose free variables occur among $x_0, x_1, \ldots, x_{n-1}$. The functions $\langle x, y \rangle$, $\mathcal{J}(n)$, $K(n)$ are standard recursive functions of enumeration of pairs such that $n = \langle \mathcal{J}(n), K(n) \rangle$ for all n (see [4]).

Suppose $\Phi_0, \Phi_1, \ldots, \Phi_s, \ldots$, $s<\omega$, is a Gödel enumeration of all formulas of signature σ. We say that a number s is the recursively enumerable (r.e.) index of the type $p(x_0, x_1, \ldots, x_{n-1})$, if $n = \mathcal{J}(s)$, $p = \{\Phi_i \mid i \in W_{K(s)}\}$, where W_k is the k-th r.e. set in a Post enumeration. A type of index s will be denoted by $P(s)$. A family of types S is called computable if there exists an r.e. set A such that $S = \{P(i) \mid i \in A\}$.

If $p(x_0, x_1, \ldots, x_{n-1})$ is a type, $0 \le i_0 < i_1 < \ldots < i_{s-1} < n$, then we denote by $p \upharpoonright \{x_{i_0}, x_{i_1}, \ldots, x_{i_{s-1}}\}$ the s-type $q(x_0, x_1, \ldots, x_{s-1})$, which is obtained by selecting from p the formulas with the indicated variables and then replacing in the formulas of the obtained set q' the variables $x_{i_0}, x_{i_1}, \ldots, x_{i_{s-1}}$ by the variables $x_0, x_1, \ldots, x_{s-1}$, respectively. We will call q a subtype of p.

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Suppose S is some countable family of types of a complete theory T. There exists a model of T realizing precisely the types in S if and only if S satisfies the following conditions:

1. S is closed under rearrangements of variables in types.
2. S is closed under the taking of a subtype.
3. Any two types $p_1, p_2 \in S$ are subtypes of some type $q \in S$.
4. For any type $p(x_0,\dots,x_{n-1}) \in S$ and any formula $\Phi_0(x_0,\dots,x_{n-1},x_n)$, if $\exists x_n \Phi \in p$, then there exists a type $q(x_0,\dots,x_{n-1},x_n) \in S$, such that $p \cup \{\Phi\} \subseteq q$.

There exists a countable homogeneous model of T realizing precisely the types in S if and only if besides the above conditions we have

5. For any two types $p_1(x_0,\dots,x_{n-1}), p_2(x_0,\dots,x_{m-1}) \in S$ and any $k < \min\{m,n\}$, if $p_1 \upharpoonright \{x_0,\dots,x_{k-1}\} = p_2 \upharpoonright \{x_0,\dots,x_{k-1}\}$, then there exists an $(n+m-k)$-type $q \in S$ such that $p_1 \subseteq q$, $p_2 = q \upharpoonright \{x_0,\dots,x_{k-1}, x_n,\dots,x_{n+m-k}\}$.

It is not difficult to show that it suffices to require the existence of q only in the case $n=m=k+1$. A countable family S having all of the indicated properties is called homogeneous.

The definitions and main results of the theory of homogeneous models can be found in [5].

Suppose A is a nonempty r.e. set and $\varphi : N \xrightarrow{\text{onto}} A$ is a fixed general recursive function (g.r.f.). Let A^s denote the set $\{\varphi(0), \varphi(1), \dots, \varphi(s-1)\}$, and ${}^{\omega}X$ the set of sequences of elements of X.

2. Counterexample to the Question of M. Morley

In this section we describe a complete decidable theory T and a computable homogeneous family of types S such that a homogeneous model $\mathfrak{M}$ of T realizing precisely the types in S is not strongly constructivizable. In fact, $\mathfrak{M}$ is even not constructivizable, since T is a model-complete theory.

The signature of the theory T is

$$\sigma = \{R, P_0, P_1, \dots, P_s, \dots;\ s < \omega\},$$

where R is a binary and the P_i, $i < \omega$, are unary predicate symbols. We will also consider the following finite parts of σ:

$$\sigma_s = \{R, P_0, P_1, \dots, P_s\}, \quad s = 0, 1, 2, \dots .$$

Consider the following propositions of signature σ_0:

$$\text{A1}^{\circ}.\ \forall x \neg R(x,x),$$
$$\text{A2}^{\circ}.\ \forall x \forall y\, [R(x,y) \rightarrow R(y,x)].$$

Suppose $s < \omega$, $\mathcal{D}_0(x_0,\dots,x_{n-1})$, $\mathcal{D}_1(x_0,\dots,x_{n-1},x_n)$ are two diagrams of σ_s-models consistent with A1^0, A2^0, and $\mathcal{D}_0$ is a subdiagram of $\mathcal{D}_1$. The scheme A3$^{\circ}_s$ contains axioms of the following form for all possible such pairs $\mathcal{D}_0$, $\mathcal{D}_1$ with a fixed s:

$$\text{A3}^{\circ}_s.\ \forall x_0 \dots \forall x_{n-1} [\mathcal{D}_0(x_0,\dots,x_{n-1}) \rightarrow \exists x_n \mathcal{D}_1(x_0,\dots,x_{n-1},x_n)].$$

The axioms of T are all of the formulas A1°, A2°, A3$^{\circ}_s$, $s < \omega$.

LEMMA 1. T is a complete decidable theory admitting elimination of quantifiers.

Proof. Let T_s denote the theory of signature σ_s for which the axioms are A1°, A2°, A3$^{\circ}_s$.

We will prove the consistency of T_s by constructing a model. We will call a σ_s -model $\mathfrak{M}$ admissible if $\mathfrak{M}$ satisfies the axioms A1°, A2°.

Suppose $\mathfrak{M}$ is an admissible σ_s -model. Consider one of the axioms of the scheme A3°_s. Suppose $a_0,\ldots,a_{n-1}\in\mathfrak{M}$ are such that $\mathfrak{M}\models\mathcal{D}_0(a_0,\ldots,a_{n-1})$. Then there exists an extension $\mathfrak{M}_1\supseteq\mathfrak{M}$ such that $\mathfrak{M}_1$ is an admissible σ_s -model and $\mathfrak{M}_1\models\exists x_n\mathcal{D}_1(a_0,\ldots,a_{n-1},x_n)$. Indeed, let

$$|\mathfrak{M}_1| = |\mathfrak{M}|\cup\{b\}.$$

On the set $\{a_0,\ldots,a_{n-1},b\}$ we extend the definition of the predicates of signature σ_s so that $\mathcal{D}_1(a_0,\ldots,a_{n-1},b)$. On the pairs (a,b) such that $a\in|\mathfrak{M}_1|\setminus\{a_0,\ldots,a_{n-1},b\}$ we define the predicate R symmetrically in any fashion. The resulting model $\mathfrak{M}_1$ is the desired one.

We can now construct in a standard way a chain

$$\mathfrak{M}_0\subseteq\mathfrak{M}_1\subseteq\ldots\subseteq\mathfrak{M}_t\subseteq\ldots$$

of admissible σ_s -models so that the union $\mathfrak{M}$ of this chain is a model of the theory T_s .

Suppose $\mathfrak{M}$, $\mathfrak{N}$ are two models of T_s . Let $\mu(\mathfrak{M},\mathfrak{N})$ denote the sets of all finite partial isomorphisms of $\mathfrak{M}$ into $\mathfrak{N}$. It follows directly from the axioms A3°_s that $\mu(\mathfrak{M},\mathfrak{N})$ possesses the extension property, i.e., for any $\lambda\in\mu(\mathfrak{M},\mathfrak{N})$ and any $a\in\mathfrak{M}$, $b\in\mathfrak{N}$ there exist $\lambda_1,\lambda_2\in\mu(\mathfrak{M},\mathfrak{N})$ such that $\lambda_1\supseteq\lambda$, $\lambda_2\supseteq\lambda$ and $a\in\mathrm{Dom}\,\lambda_1$, $b\in\mathrm{Rang}\,\lambda_2$. It follows from these same axioms that $\mu(\mathfrak{M},\mathfrak{N})\neq\varnothing$. Consequently, each $\lambda\in\mu(\mathfrak{M},\mathfrak{N})$ is an elementary mapping, hence T_s admits elimination of quantifiers. If $\mathfrak{M}$, $\mathfrak{N}$ are countable, then each $\lambda\in\mu(\mathfrak{M},\mathfrak{N})$ can be extended to an isomorphism, hence T_s is countable-categorical. Obviously, T_s has no finite models. According to Vaught's criterion, T_s is complete.

It remains to observe that $T_s\subseteq T_{s+1}$ for any $s<\omega$.

Thus,

$$T=\bigcup_{s<\omega}T_s,$$

hence T is a complete theory with elimination of quantifiers. By a theorem of Janiczak, T is a decidable theory.

The lemma is proved.

In view of the elimination of quantifiers, any 1-type $q(x)$ of the theory T is uniquely determined by a sequence $\alpha\in{}^{\omega}\{0,1\}$, so that

$$\{P_0^{\alpha_0}(x),\,P_1^{\alpha_1}(x),\ldots,P_s^{\alpha_s}(x),\ldots;\ s<\omega\}\subseteq q. \tag{1}$$

Now suppose $n\geqslant 2$. Any n -type $p(x_0,x_1,\ldots,x_{n-1})$ of the theory T is uniquely determined by the 1-types

$$p_0=p\restriction\{x_0\},\ p_1=p\restriction\{x_1\},\ldots,p_{n-1}=p\restriction\{x_{n-1}\}$$

and some σ_0 -diagram $\mathcal{D}(x_0,x_1,\ldots,x_{n-1})\in p$.

A type $q(x)$ satisfying condition (1) will be called distinguished if

$$\alpha_0=\alpha_1=\ldots=\alpha_s=\ldots=1.$$

A type $q(x)$ of the form (1) will be called admissible if there exists $n<\omega$ such that

$$\alpha_n=\alpha_{n+1}=\ldots=\alpha_{n+i}=\ldots=1.$$

Suppose $A \subseteq \omega$. We will say that a sequence $\alpha \in {}^{\omega}\{0,1\}$ is compatible with A if there exists $s<\omega$ such that

1) $\alpha_{s+i} = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A, \end{cases} \quad i=0,1,\ldots,s-1,$

2) $\alpha_{2s}=0$,

3) $\alpha_k = 1$ for $k>2s$.

A type $q(x)$ is compatible with A if it is determined by a sequence compatible with A.

We now turn to the description of the desired family of types. Suppose $A \subseteq \omega$. We denote by S_A the following set of types of the theory T.

a) A 1-type $q(x) \in S_A \Leftrightarrow q(x)$ is an admissible type.

b) Suppose $p(x_0, x_1)$ is a 2-type consistent with T, $q_0 = p \upharpoonright \{x_0\}, q_1 = p \upharpoonright \{x_1\}$. Then $p \in S_A$ if and only if the following condition is satisfied: if $R(x_0, x_1) \in p$, then

1) q_0 is distinguished $\Longrightarrow$ q_1 is compatible with A,

2) q_1 is distinguished $\Longrightarrow$ q_0 is compatible with A.

c) An n-type $p(x_0,\ldots,x_{n-1}), n \geq 3$ belongs to S_A if and only if each of its 2-subtypes belongs to S_A.

LEMMA 2. For any $A \subseteq \omega$, S_A is a homogeneous family of types of the theory T.

Proof. That S_A is closed under the taking of subtypes and rearrangements of variables in types is obvious.

Let us verify condition 4. Suppose $p(x_0,\ldots,x_{n-1}) \in S_A$, and the σ_S-formula $\Phi(x_0,\ldots,x_{n-1},x_n)$ is such that $\exists x_n \Phi \in p$. There exists a σ_S-diagram $\mathcal{D}(x_0,\ldots,x_{n-1},x_n)$ such that the set of formulas

$$p(x_0,\ldots,x_{n-1}) \cup \{\mathcal{D}(x_0,\ldots,x_{n-1},x_n), \Phi(x_0,\ldots,x_{n-1},x_n)\}$$

is consistent with T. In view of elimination of quantifiers, it follows that

$$\vdash_T \mathcal{D}(x_0,\ldots,x_{n-1},x_n) \longrightarrow \Phi(x_0,\ldots,x_{n-1},x_n).$$

Thus, it suffices to enlarge the set

$$p(x_0,\ldots,x_{n-1}) \cup \{\mathcal{D}(x_0,\ldots,x_{n-1},x_n)\} \tag{2}$$

to a type which belongs to S_A. If $\mathcal{D}$ contains a conjunctive term of the form $x_i = x_n$, $i<n$, then (2) can be uniquely extended to a type, which necessarily belongs to S_A. Otherwise, it suffices to take an enlargement $q(x_0,\ldots,x_{n-1},x_n)$ such that the 1-type $q \upharpoonright \{x_n\}$ is compatible with A.

Finally, let us verify condition 5. Suppose $p_1(x_0,\ldots,x_{n-1},x_n)$, $p_2(x_0,\ldots,x_{n-1},x_n) \in S_A$ are such that $p_1 \upharpoonright \{x_0,\ldots,x_{n-1}\} = p_2 \upharpoonright \{x_0,\ldots,x_{n-1}\}$. Let

$$p_2' = Sb_{x_n}^{x_{n+1}} p_2(x_0,\ldots,x_{n-1},x_n).$$

Consider the set of formulas

$$p^*(x_0,\ldots,x_{n-1},x_n,x_{n+1}) = p_1(x_0,\ldots,x_{n-1},x_n) \cup p_2'(x_0,\ldots,x_{n-1},x_{n+1}).$$

If for some $i<n$ we have $(x_i = x_n) \in p_1$ or $(x_i = x_{n+1}) \in p_2'$, then p^* determines a unique type $q(x_0,\ldots,x_{n-1},x_n,x_{n+1})$ which belongs to S_A and is such that

$$q\upharpoonright\{x_0,\dots,x_{n-1},x_n\}=p_1\,,\ q\upharpoonright\{x_0,\dots,x_{n-1},x_{n+1}\}=p_2\,. \tag{3}$$

In the opposite case we must add to p^* the formula

$$(x_n\neq x_{n+1})\,\&\,\neg R\,(x_n,x_{n+1}),$$

and then the resulting set of formulas determines a unique type q which belongs to S_A and satisfies the relations (3). The condition (3) is the special case of condition (5) with $k=0$.

The lemma is proved.

LEMMA 3. If A is an r.e. set, then S_A is a computable family of types.

Proof. It suffices to prove computability of the set S of 2-types in S_A. For this, obviously, it suffices to find a computable family S' such that $S^*\subset S'\subset S$, where

$$S^*=\left\{p\,(x_0,x_1)\mid R\,(x_0,x_1)\in p\ \&\ p\upharpoonright\{x_0\}\ \text{is distinguished and}\ p\upharpoonright\{x_1\}\ \text{is compatible with}\ A\right\}.$$

Consider an arbitrary finite sequence $\gamma=(\gamma_0,\gamma_1,\dots,\gamma_s)$, $\gamma_i\in\{0,1\}$, and some $n<\omega$. For the pair $\varepsilon=(\gamma,n)$ we will describe the process of enumerating the 2-type $p_\varepsilon\,(x_0,x_1)$. We define two sequences $\alpha,\beta\in{}^{\omega}\{0,1\}$. Put $\alpha_i=\gamma_i$ for $0\le i\le s-1$,

$$\alpha_{s+i}=\begin{cases}1, & \text{if: } i\in A^n,\\ 0, & \text{if } i\notin A^n,\end{cases}\quad 0\le i\le s-1,$$

$\alpha_{2s}=0$, $\alpha_k=1$ for $k>2s$. Clearly, A is compatible with α if for each $k<\omega$ we have

$$X_n^k=(A^k-A^n)\cap\{0,1,\dots,s-1\}=\varnothing.$$

We define the sequence β as follows:

$$\beta_k=\begin{cases}0\ , & \text{if } X_n^{k-1}=\varnothing,\ X_n^k\neq\varnothing,\\ 1 & \text{otherwise.}\end{cases}$$

We construct the type p_ε so that

$$\{R\,(x_0,x_1),P_k^{\beta_k}(x_0),\ P_k^{\alpha_k}(x_1)\ ;\ k<\omega\}\subseteq p_\varepsilon\,(x_0,x_1).$$

Then the set S' of all such types p_ε is the desired one.

The lemma is proved.

For an arbitrary $A\subseteq\omega$ we denote by θ_A the function

$$\theta_A(n)\ -\ \text{the number of elements in}\ \{0,1,\dots,n-1\}\cap A.$$

An r.e. set A will be called approximable if there exists a g.r.f. $f(x)$ such that

$$\left.\begin{aligned}&1)\ \forall x\,f(x)\le\theta_A(x),\\ &2)\ \exists^{\omega}x\,f(x)=\theta_A(x).\end{aligned}\right\} \tag{4}$$

LEMMA 4. If A is an r.e. set and a homogeneous model $\mathfrak{M}$ realizing precisely the types in S_A is constructivizable, then A is approximable.

Proof. We will first show that the family of 1-types

$$Q_A = \{ p \mid p \text{ is compatible with } A \}$$

is computable.

Let p_a denote the type of an element $a \in \mathfrak{M}$. Suppose $a' \in \mathfrak{M}$ realizes a distinguished type, which we denote by p'. We will prove that

$$Q_A = \{ p_a \mid \mathfrak{M} \models R(a', a) \}. \tag{5}$$

If $\mathfrak{M} \models R(a', a)$, then p_a must be in Q_A by definition of S_A. Conversely, suppose $p \in Q_A$. Consider a 2-type $q(x_0, x_1)$ such that

$$q \restriction \{x_0\} = p', \quad q \restriction \{x_1\} = p, \quad R(x, y) \in q.$$

Then $q \in S_A$ and is therefore realized in $\mathfrak{M}$ on some pair (b', b). Since a' and b' realize the type p' and the model $\mathfrak{M}$ is homogeneous, there exists an automorphism $\lambda : \mathfrak{M} \to \mathfrak{M}$ such that $\lambda(b') = a'$. Let $a = \lambda(b)$. Then $\mathfrak{M} \models R(a', a)$, and p_a coincides with p.

The computability of Q_A follows immediately from (5) and the constructivizability of $\mathfrak{M}$. From Q_A we select a computable subfamily α^n, $n < \omega$, such that

the length of the "field of compatibility" of α^n with A is at least n (6)

(which is equivalent to the condition $((\exists i \geq 2n)\ \alpha^n_i = 0)$).

We define g.r.f. $g(t)$, $f(t)$ as follows:

$$\begin{cases} g(0) = 0, \\ g(t) \text{ for } t > 0 \text{ is the smallest number such that } g(t) > \max\{g(t-1), 2t+2\} \text{ and such that for each } n < t \end{cases}$$

there exists a number $s = s(n, t)$ such that $2s < g(t)$ and

$$1)\ \alpha^n_{2s} = 0,\ \alpha^n_{2s+1} = \dots = \alpha^n_{g(t)} = 1,$$
$$2)\ \{ i \mid 0 \leq i \leq s-1 \ \&\ \alpha^n_{s+i} = 1 \} = \{0, 1, \dots, s-1\} \cap A^{g(t)},$$

and $f(t) =$ the number of elements in the set $\{0, 1, \dots, t-1\} \cap A^{g(t)}$.

We can say that at the moment t we survey the initial segments of the sequences $\alpha^0, \alpha^1, \dots, \alpha^{t-1}$ of length $g(t)$ and within the zone of the survey we obtain compatibility of each of these sequences with the set $A^{g(t)} \subseteq A$. However, in certain $\alpha^n, n < t$, there can be zeros outside the zone of the survey. Let t_n denote the smallest number $t > n$ such that there are no zeros outside the segment of length $g(t)$ in α^n. Since the sequence α_n is compatible with A, we have at this moment the exact equality

$$\{ i \mid 0 \leq i \leq s-1 \ \&\ \alpha^n_{s+i} = 1 \} = \{0, 1, \dots, s-1\} \cap A$$

for $s = s(n, t)$. On the other hand, in view of condition 2) in the definition of $g(t)$, at this same moment we have

$$\{ i \mid 0 \leq i \leq s-1 \ \&\ \alpha^n_{s+i} = 1 \} = \{0, 1, \dots, s-1\} \cap A^{g(t)}.$$

Therefore, in order to prove that at this moment we have $f(t) = \theta_A(t)$, it suffices to show that $t_n \leq s(n, t_n)$. If $n = t_n - 1$, this follows from (6). In the case $n < t_n - 1$, note that $g(t_n - 1) > 2t_n$ by definition of g, and by definition of t_n there must be zeros in α^n to the right of $g(t_n - 1)$, from which the desired inequality follows.

Thus, $f(x)$ has the properties (4).

The lemma is proved.

LEMMA 5. There exists a nonapproximate r.e. set.

Proof. We will construct the desired set (which we denote by A) in steps. Suppose A^t is the finite part of A constructed after step t.

Suppose $f_n(x)$, $n<\omega$, is a Kleene enumeration of all partial recursive functions of one variable, f_n^t is the finite part of f_n computed to the moment t, and $f_n^t = \varnothing$ for $n>t$. Assume that the function $<x,y>$ of enumeration of pairs possesses the following property: for any $x<\omega$ the function $\lambda y <x,y>$ is monotone increasing.

We will now describe the construction.

Step $t=0$. Put $A^0 = \varnothing$.

Step $t>0$. Put $A^t = A^{t-1} \cup M^t$, where M^t is the set of all $a<\omega$, for which the following two conditions are satisfied. We first calculate $a = <n,k>$, $b = <n, k+1>$.

1) $\mathcal{D}om\, f_n^t \supseteq [a+1, b]$,

2) $\forall x \in [a+1, b]\; f_n^t(x) \leqslant \theta_{A^{t-1}}(x)$.

We will show that the set A is the desired one. Suppose $f_n(x)$ is a g.r.f. and $\forall x f_n(x) \leqslant \theta_A(x)$. We will prove that

$$\forall x\, [x > <n,0> \Longrightarrow f_n(x) < \theta_A(x)].$$

Suppose $x_0 > <n,0>$. Choose k so that $<n,k> < x_0 \leqslant <n, k+1>$. Let $a = <n,k>$, $b = <n, k+1>$. Consider the first step t where both of the above conditions are satisfied for a. Then $a \notin A^{t-1}$, $a \in A^t$, hence

$$f_n^t(x_0) \leqslant \theta_{A^{t-1}}(x_0) < \theta_{A^t}(x_0) \leqslant \theta_A(x_0).$$

The lemma is proved.

A proof of Lemma 5 was found independently of the author by K. A. Meirembekov.

Remark. It is not difficult to show that if B is approximable and $A \leqslant_m B$ (even if A is r.e. and $A \leqslant_{tt} B$), then A is approximable. It follows from this and Lemma 5 that a creative set K is not approximable.

Now consider $S = S_A$, where A is an r.e. nonapproximable set. By Lemma 2, there exists a homogeneous model $\mathfrak{M}$ of the theory T realizing precisely the types in S. The family S is computable by Lemma 3, but the model $\mathfrak{M}$ cannot be constructivizable in view of Lemma 4.

Also, we can reduce the signature of the theory T to a finite one, using, for example, a method of the author (see [3, Sec. 5, Lemma 4.4]). In this way we can reduce the signature to two binary predicates and then, by some well-known method, to a single binary predicate.

3. Criterion for Strong Constructivizability

Throughout this section we assume given a complete decidable theory T of signature σ, a computable homogeneous family of types S, a homogeneous model $\mathfrak{M}$ of T determined by S, and a Gödel enumeration $\Phi_0, \Phi_1, \ldots, \Phi_s, \ldots;\ s<\omega$, of all formulas of signature σ.

THEOREM 1. The following conditions are equivalent:

1. There exists a g.r.f. $E(\tau)$ such that if $P(\tau)$ is an n -type in S, then $\{P(i) \mid i \in W_{E(\tau)}\}$ is precisely the family of $(n+1)$ -types in S extending $P(\tau)$.

2. There exists a g.r.f. $F(\tau, s)$ such that $\forall \tau, t\, \{P(i) \mid i \in W_{F(\tau,t)}\} \subseteq S$, and if $P(\tau)$ is an n -type in S, then there exists $t_0 = t_0(\tau)$ such that for $t \geq t_0$ the set $\{P(i) \mid i \in W_{F(\tau,t)}\}$ is precisely the family of $(n+1)$ -types in S extending $P(\tau)$.

3. There exists a g.r.f. $G(\tau, s)$ such that if $P(\tau)$ is an n -type in S and $\Phi_s(x_0, \ldots, x_{n-1}, x_n)$ is such that $P(\tau) \cup \{\Phi_s\}$ is consistent with T, then $P(G(\tau, s))$ is an $(n+1)$ -type in S extending $P(\tau)$ and containing Φ_s.

4. There exists a g.r.f. $H(\tau, s, t)$ such that $\forall \tau, s, t\, P(H(\tau, s, t)) \in S$ and for each $\tau < \omega$ there exists $t_0 = t_0(\tau)$ such that if $P(\tau)$ is an n -type in S and $\Phi_s(x_0, \ldots, x_{n-1}, x_n)$ is such that $P(\tau) \cup \{\Phi_s\}$ is consistent with T, then for $t \geq t_0$ we have that $P(H(\tau, s, t))$ is an $(n+1)$ -type in S extending $P(\tau) \cup \{\Phi_s\}$.

5. $\mathfrak{M}$ is strongly constructivizable.

The proof follows the scheme

$$
\begin{tikzcd}
 & (1) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
(2) \arrow[dr, Rightarrow] & (5) \arrow[u, Rightarrow] & (3) \arrow[dl, Rightarrow] \\
 & (4) \arrow[u, Rightarrow] &
\end{tikzcd}
$$

We will prove only the two implications $(4) \Longrightarrow (5) \Longrightarrow (1)$, since the others are obvious.

$\underline{(5) \Longrightarrow (1).}$ Suppose ν is a strong constructivization of $\mathfrak{M}$. Consider any $\tau < \omega$ such that $P(\tau)$ is an n -type in S. We will construct in steps the family

$$S^* = \{q_0, q_1, \ldots, q_m, \ldots;\ m < \omega\}$$

of all $(n+1)$ -types in S extending $P(\tau)$. Let us describe the concepts needed for the construction process.

1. $\Psi_0, \Psi_1, \ldots, \Psi_s, \ldots$ is a Gödel enumeration of all formulas of signature σ whose free variables are included among $x_0, x_1, \ldots, x_n$.

2. $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_s, \ldots, s < \omega$ is an effective enumeration with respect to the ν -numbers of all processions of length n of elements of the model $\mathfrak{M}$.

3. If $\bar{c}$ is a procession of length $(n+1)$ in $\mathfrak{M}$, we put

$$Tp^t(\bar{c}) =_{df} \{\Psi_i \mid 0 \leq i \leq t \ \&\ \mathfrak{M} \models \Psi_i(\bar{c})\}.$$

4. q_i^t is the finite part of the type q_i constructed after step t. We assume that $q_i^t = \varnothing$ for $i \geq t$.

5. Q_i^t is the conjunction of the set q_i^t.

6. $b_0^t, b_1^t, \ldots, b_{t-1}^t$ are specific elements of $\mathfrak{M}$.

7. $f(t)$ is an auxiliary function, whose values are defined in steps.

After step t the following conditions will be satisfied:

1) $q_i^t = T_P^t(\bar{a}_{f(t)}, b_i^t),\ 0 \leq i \leq t-1,$

2) $\exists x_n Q_i^t(x_0, \ldots, x_{n-1}, x_n) \in P(\tau),\ 0 \leq i \leq t-1.$

The construction:

Step $t=0$. Put $q_i^t=\varnothing$ for all $i<\omega$, $f(0)=0$.

Step $t+1$. There are two cases.

Case 1. If $b_i^{t+1}=b_i^t$, $0\leq i\leq t-1$, b_t^{t+1} is the element with smallest number different from all b_i^{t+1}, $i<t$, $f(t+1)=f(t)$, q_i^{t+1} is in accordance with 1), then condition 2) holds (if we replace t by $t+1$). In this case we consider the attempt successful and step $t+1$ has been carried out.

Case 2. The definitions given above lead to a violation of 2). In this case we find the smallest possible value $f(t+1)>f(t)$, for which we can choose a system of elements $b_0^{t+1},\ldots,b_t^{t+1}$ so that conditions 1) and 2) are satisfied and also the following condition: 3) $q_i^t\subseteq q_i^{t+1}$, $0\leq i\leq t-1$. This completes step $t+1$.

Suppose k is the smallest number such that $\bar{a}_k$ realizes the type $P(\tau)$. The idea of the construction is that there occurs a moment t_0 such that $f(t)=k$ for all $t\geq t_0$. Then for all $t\geq t_0$ we are in Case 1, hence S^* will be precisely the set $\{Tp^{\omega}(\bar{a}_k,b)\mid b\in\mathfrak{M}\}$. Since $\mathfrak{M}$ is homogeneous, S^* is as desired.

The process described above depends uniformly on τ, hence there exists a g.r.f. $E(\tau)$, such that $S^*=\{P(i)\mid i\in W_{E(\tau)}\}$.

(4) $\Longrightarrow$ (5). The proof given here is, in essence, only an "elaboration" of M. Morley's proof of the main theorem of [7, Theorem 6.1], which says that if the family of all types consistent with a complete decidable theory T is computable, then a countable saturated model of T is strongly constructivizable.

Suppose A is an r.e. set, $S=\{P(i)\mid i\in A\}$ satisfies the conditions of homogeneity, $H(\tau,s,t)$ is a function satisfying condition 4 of the theorem, and $\langle J(n),K(n)\rangle=n$ is a standard enumeration of the pairs of natural numbers. Fix a g.r.f. $\varphi: N\xrightarrow{\text{onto}}A$. Put $L(s)=P(\varphi(J(s)))$. Then $S=\{L(i)\mid i<\omega\}$ and in this list each type $p\in S$ has an infinite set of numbers.

We will assume without loss of generality that the signature σ of the theory T is purely predicate. Consider

$$\sigma^*=\sigma\cup\{c_0,c_1,\ldots,c_s,\ldots;\ s<\omega\}.$$

Suppose $\theta_0,\theta_1,\ldots,\theta_s,\ldots;\ s<\omega$ is a Gödel enumeration of all propositions of signature σ^*. We will construct by steps with respect to t certain propositions $\theta(t)$ such that $\theta(0)$ is an identically true formula and $\theta(t+1)$ is either $\theta(t)\&\theta_t$ or $\theta(t)\&\neg\theta_t$. The resulting sequence $\theta(t)$ is a set of axioms of a complete decidable Henkin theory, and a canonical model of this theory is the desired one.

At the same time, we will construct auxiliary functions $f(n,t)$, $g(n,t)$, $h(n,t)$, $X(n,t)$, $Y(n,t)$, $R(n,t)$. For brevity, we will sometimes omit the second argument, replacing it if necessary by the phrase "at the moment t."

At the moment $t=0$ the functions $f(n)$, $g(n)$, $h(n)$ will be general recursive. At most one value of each of these functions can vary in a step.

The objects X,Y at the moment t depend on f as follows:

$$X(n)=\langle c_0,\ldots,c_{m-1},c_{f(0)},\ldots,c_{f(n)}\rangle,$$
$$Y(n)=\langle c_0,\ldots,c_{m-1},c_{f(0)},\ldots,c_{f(n)},c_m\rangle,$$

where $m = max\{\mathcal{J}(n') \mid n' \leq n\}$ and $Y(n)$ is considered only for those n such that $\mathcal{J}(n+1) = m+1$, i.e., for $n = 1, 4, 8, 13, 19, \ldots, 1 + \frac{1}{2}s(s+5)$.

The objects R are the following assertions concerning $\theta = \theta(t)$:

$$R(2n) = R'(2n) \& R''(2n),$$

where $R'(2n) =$ "either it follows from θ that $< c_0, \ldots, c_{\mathcal{J}(n)-1} >$ has no type $L(K(n)) \upharpoonright \{x_0, \ldots, x_{\mathcal{J}(n)-1}\}$, or θ is consistent with the fact that $< c_0, \ldots, c_{\mathcal{J}(n)-1}, c_{f(n)} >$ has type $L(K(n))$,"

$R''(2n) =$ " θ is consistent with the fact that $X(n)$ has type $L(g(n))$,"

$R(2n+1) =$ " θ is consistent with the fact that $Y(n)$ has type $P(h(n))$."

In the conditions $R''(2n), R(2n+1)$, the number of variables of a type is always equal to the length of the corresponding procession. The conditions $R'(2n)$ for which this is violated and also the conditions $R(2n+1)$ for which $Y(n)$ is not considered are taken to be true by definition.

The inductive construction is as follows.

Step $t=0$. $\theta(0)$ is a tautology, $f(n,0) = 0$ for all n, $g(n,0), h(n,0)$ are defined to be the recursive functions assuming the smallest possible values such that the number of variables of the types they determine correspond to the lengths of the processions.

Step $t>0$. Put $\theta(t) = \theta(t-1) \& \theta^{\tau}_{t-1}$, where τ is chosen so that the largest possible initial segment of conditions is satisfied.

Find the smallest $s < t$ such that $R(s)$ is false. If there is no such s, then step t is complete (the values of all functions at the moment t are the same as at the moment $t-1$). If there is such an s, then we consider two cases.

Case 1. The first violated condition is $R(2n)$. Then we put $f(n,t)$ equal to the smallest τ such that c_τ does not occur in $\theta(t)$ (thus condition $R'(2n)$ will be satisfied). We next put $g(n,t)$ equal to the smallest number greater than $g(n,t-1)$ for which $R''(2n)$ is satisfied.

Case 2. The first violated condition is $R(2n+1)$. Suppose α is a function such that

$$Y(n) = < c_{\alpha(0)}, \ldots, c_{\alpha(m+n+1)} >.$$

We construct a formula θ' by replacing in $\theta(t)$ each constant c_s occurring in $Y(n)$ by the variable x_k where $k = \mu\tau\,(\alpha(\tau) = s)$. Suppose θ'' is obtained from θ' by replacing all remaining constants by different variables and quantifying them by means of $\exists$. Finally, let

$$\psi(x_0, \ldots, x_{m+n+1}) = \theta'' \& \wedge \{(x_i = x_j) \mid \alpha(i) = \alpha(j)\}.$$

Suppose τ is the r.e. index of the type $L(g(n))$, and s is the Gödel number of the formula ψ. Put $h(n) = H(\tau, s, t')$, where t' is the smallest number such that $t' \geq t$ and the number of variables of the type $P(H(\tau, s, t'))$ is equal to the length of the procession $Y(n)$. Step t is complete.

We will show that for any fixed $n < \omega$ the value of each of the functions f, g, h stabilizes as t increases.

At the moment $t=0$ we have $f(0) = 0$ and $g(0) =$ is equal to the smallest s such that $L(s)$ is a 1-type. These values cannot change, since in the construction of $\theta(t)$ there is always the possibility of choosing τ so that the condition $R(0)$ is not violated.

Now assume that that $f(k), g(k), h(k)$ stabilize for all $k < n$, beginning with the moment t'. We will show that $f(n)$ and $g(n)$ stabilize. If $Y(n-1)$ is not defined, put $Z(n) = X(n-1), p = L(g(n-1))$; otherwise, suppose $Z(n) = \langle c_0,\dots,c_n, c_{f(0)},\dots,c_{f(n-1)}\rangle$ and p is the type of $Z(n)$ under the condition that $P(h(n-1))$ is the type of $Y(n-1)$ (p is obtained from the type $P(h(n-1))$ by a rearrangement of variables). Let s denote the number of variables of the type $L(K(n))$. We consider several cases.

Case 1. Suppose $s = \mathcal{I}(n)+1$ and

$$L(K(n)) \restriction \{x_0,\dots,x_{\mathcal{I}(n)-1}\} = p \restriction \{x_0,\dots,x_{\mathcal{I}(n)-1}\}. \tag{7}$$

Stabilization is achieved at the moment when the following conditions are satisfied (here k is the length of the procession $Z(n)$):

$$\left.\begin{array}{l} 1.\ L(g(n)) \restriction \{x_0,\dots,x_{k-1}\} = p, \\ 2.\ L(g(n)) \restriction \{x_0,\dots,x_{s-1}\} = L(K(n)). \end{array}\right\} \tag{8}$$

If $g(n)$ at some moment $t > t'$ does not satisfy these conditions, then the condition $R(2n)$ is violated. But then we must change the value $f(n)$, and thus we remove from $X(n)$ any additional conditions imposed by the increasing proposition $\theta(t)$, except for the conditions (8). Therefore, a value $g(n)$ satisfying the conditions (8) will ultimately be found.

Case 2. $s = \mathcal{I}(n)+1$. In this case the condition $R'(2n)$ is true by definition. Therefore, only the condition (8.1) is imposed on $g(n)$. As in the previous case, after a series of violations of the condition $R''(2n)$ we will find a value $g(n)$ satisfying this condition, and from this moment $g(n)$ and $f(n)$ no longer vary.

Case 3. $s = \mathcal{I}(n)+1$, but condition (7) is violated. In this case, the increase of $\theta(t)$ implies that the condition $R'(2n)$ will be satisfied constantly from some moment on. Then the stabilization of $f(n)$ and $g(n)$ occurs as in Case 2.

If $Y(n)$ is defined, then the stabilization of $h(n)$ occurs at the moment $t = 1 + max\{t_0(\tau), t'\}$, where τ is the index of the type $L(g(n))$, $t_0(\tau)$ is the number in condition 4 of the theorem, and t' is the moment of stabilization.

The desired model $\mathfrak{M} = \langle M; \sigma\rangle$ is constructed as follows. Suppose $M = \{c_0, c_1,\dots,c_s,\dots;\ s<\omega\}$. The predicates of σ are defined on M in correspondence with the sequence $\theta(t)$. If we consider the mapping $\nu : N \xrightarrow{\text{onto}} M$ defined by the rule $\nu(i) = c_i$, then we can show by a standard argument that $(\mathfrak{M}, \nu)$ is a strongly constructive model of T. The conditions $R'(2n)$ guarantee the homogeneity of $\mathfrak{M}$, the realization in $\mathfrak{M}$ of all types in S, and also the Henkin condition, from which it follows that $\mathfrak{M}$ is a model of T. The conditions $R(2n+1)$ do not allow the realization in $\mathfrak{M}$ of extraneous types. The conditions $R''(2n)$ play a determining role in the stabilization of the functions.

It is easy to see that in the example of Sec. 2 there is a violation of condition 1, namely, if τ is the index of a distinguished type, then any value for $E(\tau)$ is unsuitable, since the family of 2-types in S extending a distinguished type is not computable.

We now give two corollaries of our theorem. We assume that T is a complete decidable theory.

COROLLARY 1.1. (Goncharov and Nurtazin [1], Harrington [6]). If T has a prime model $\mathfrak{M}$ and the family S of principal types is computable, then $\mathfrak{M}$ is strongly constructivizable.

Proof. Condition 4 of Theorem 1 is satisfied for S. Indeed, in the role of $H(\tau, s, t)$ we take the index of the first type found in S containing $P(\tau)^t$ and Φ_s, where $P(\tau)^t$ is the finite part of the type $P(\tau)$

computed after t steps. If $P(\tau)^t$ with Φ_s is inconsistent or if Φ_s contains superfluous variables, we define $H(\tau, s, t)$ trivially in any fashion. The role of $t_0(\tau)$ is played by the moment when the generating type $P(\tau)^t$ occurs in $P(\tau)$.

COROLLARY 1.2 (Morley [7]). If T has a saturated model $\mathfrak{M}$ and the family S of all types is computable, then $\mathfrak{M}$ is strongly constructivizable.

Proof. Condition 3 of Theorem 1 is obviously satisfied for S.

There naturally arises the following question. Suppose T is a complete decidable theory and S is a computable family of types possessing the properties 1–4 given in Sec. 1. Such families can be called admissible. Does there always exist at least one strongly constructive model of T realizing precisely the types in S ? The answer turns out to be negative, since we have the following

THEOREM 2. Suppose S is a computable admissible family of types. Then conditions 1, 2, 3, 4 of Theorem 1 are equivalent to each other and to

5'. There exists a strongly constructive model of T realizing precisely all types in S.

The proof of this theorem follows the same scheme as that of Theorem 1.

The proof of the implication $(5') \Longrightarrow (1)$ is somewhat more complicated. This is related to the fact that the family of all $(n+1)$ -types in S extending a given n -type $P(\tau)$ may not be realized over some one realization of the type $P(\tau)$. Therefore, it is necessary to organize a search of the points of realization of the type $P(\tau)$.

The proof of the implication $(4) \Longrightarrow (5)$ is an almost verbatim repetition of the proof in Theorem 1. The only thing requiring any essential alteration is the condition $R'(2n)$ and everything connected with it. It must be adjusted so that we obtain a Henkin theory and can realize all types in S.

COROLLARY 2.1. Suppose $(\mathfrak{M}, \nu)$ is a strongly constructive model of a complete theory T and the family S' of types realizable in $\mathfrak{M}$ is homogeneous. Then the homogeneous model $\mathfrak{M}'$, defined by the family S is strongly constructivizable.

It follows that in the example of Sec. 2 there exists no strongly constructive model realizing precisely the types in S.

In conclusion, the author would like to mention that the main results of this paper were obtained independently by Goncharov [8]. Conditions 1 and 3 of Theorem 1 were found by the author later, as a result of a joint discussion.

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