STRUCTURE OF THE UPPER SEMILATTICE OF RECURSIVELY ENUMERABLE m-DEGREES AND RELATED QUESTIONS. I

S. D. Denisov

UDC 517:11,518:5

In the first part of this paper we consider the following upper semilattices; the semilattice $\mathcal{L}^e$ of recursively enumerable m-degrees, the semilattice ${}_a\mathcal{L}=\{b\in\mathcal{L}^e \mid a\leq b\}$, where $a\in\mathcal{L}^e$ and a is not equal to the largest element of $\mathcal{L}^e$, and the semilattices $\mathcal{L}(S_n)$ of computable enumerations of the classes $S_n=\{\emptyset,\{1\},\ldots,\{n\}\}$, where $n=1,2,\ldots$. We prove (Theorem 1) that it is possible to provide the semilattice $\mathcal{L}^e$ (${}_a\mathcal{L}$, $\mathcal{L}(S_n)$) with an enumeration π (ζ, ξ respectively) such that in a suitable category of enumerated semilattices $\mathcal{L}^e_\pi$ (${}_a\mathcal{L}_\zeta$, $\mathcal{L}(S_n)_\xi$) possesses the "morphism extension property." Theorem 1 and Theorem 2, which asserts, roughly speaking, the isolation of the largest element of $\mathcal{L}^e_\pi$ (${}_a\mathcal{L}_\zeta$, $\mathcal{L}(S_n)_\xi$), characterize the semilattice $\mathcal{L}^e$ (${}_a\mathcal{L}$, $\mathcal{L}(S_n)$) uniquely to within isomorphism. It follows, in particular, that the above-mentioned semilattices are isomorphic: $\mathcal{L}^e\cong{}_a\mathcal{L}\cong\mathcal{L}(S_n)$. It had been conjectured that these semilattices are isomorphic.

In the second part of this paper ("Structure ... II") we investigate by the methods of this first part the semilattice $\mathcal{L}^d=\{d_m(A)\mid A\in\Delta^0_2\}$ and the semilattices of computable enumerations $\mathcal{L}(S)$, where S is a computable family of general recursive functions containing exactly one limit point and is such that the semilattice $\mathcal{L}(\tilde{S})$, where $\tilde{S}$ is the set of isolated points of S, is a one-element set. We will prove that $\overline{\mathcal{L}^d}\cong\overline{\mathcal{L}(S)}\cong\mathcal{L}^e$, where $\overline{\mathcal{L}^d}$ (respectively $\overline{\mathcal{L}(S)}$) is obtained from the semilattice $\mathcal{L}^d$ (respectively $\mathcal{L}(S)$) by externally adjoining a largest element. We begin a more detailed exposition.

1. Preliminary Facts

As a working definition we adopt the following definition of m-reducibility. Suppose $A,B\subset N$; we say that the set A is m-reducible to the set B, $A\leq_m B$, if either A is recursive or there exists a general recursive function f such that $\forall x\in N\,(x\in A\leftrightarrow f(x)\in B)$. The relation $\leq_m$ is obviously a preorder on the set of all subsets of N; we denote by $\sim_m$ the corresponding equivalence relation: $A\sim_m B \leftrightharpoons A\leq_m B\ \&\ B\leq_m A$. The equivalence class of the set A relative to $\sim_m$ is denoted by $d_m(A)$ and is called the m-degree of A; an m-degree containing a recursively enumerable set is called recursively enumerable. The relation $\leq_m$ induces an order on the set of m-degrees, and this ordered set is an upper semilattice, i.e., any two elements have a least upper bound. In the sequel, instead of "upper semilattice" we will simply write "semilattice." We denote the semilattice of m-degree by $\mathcal{L}^m$, and the set of recursively enumerable m-degrees by $\mathcal{L}^e$.

Translated from Algebra i Logika, Vol. 17, No. 6, pp. 643-683, November-December, 1978. Original article submitted August 30, 1978.

 0002-5232/78/1706-0418$07.50

Let us establish some conventions. We will denote a semilattice and its underlying set by the same letter, and the operation of taking the least upper bound by $\cup$; thus, $a \leq b \leftrightarrow a \cup b = b$. Suppose $\mathcal{L} = \langle \mathcal{L}, \cup \rangle$ is a semilattice. The smallest element of $\mathcal{L}$ (if it exists) will be denoted by $\mathbf{O}$, and the largest (if it exists) by $\mathbf{I}$; sometimes these elements will be denoted more explicitly: $\mathbf{O}_{\mathcal{L}}, \mathbf{I}_{\mathcal{L}}$. A subset $A \subset \mathcal{L}$ is called an ideal of the semilattice $\mathcal{L}$ if for all $a, b \in \mathcal{L}$ we have the relations $a, b \in A \rightarrow a \cup b \in A$, $a \in A \,\&\, b \leq a \rightarrow b \in A$. For recursively enumerable m-degrees we will use the following abbrevations. If $a, b \in \mathcal{L}^e$, then

$$_a\mathcal{L}_b = \{c \in \mathcal{L}^e \mid a \leq c \leq b\}, \quad \mathcal{L}_b = \{c \in \mathcal{L}^e \mid c \leq b\},$$
$$_a\mathcal{L} = \{c \in \mathcal{L}^e \mid a \leq c\}.$$

It is easy to see that $\mathcal{L}^e$ is an ideal of the semilattice $\mathcal{L}^m$ and that $d_m(\phi)$ is the smallest element of $\mathcal{L}^m$ and $\mathcal{L}^e$. It follows from the computability of the family of all recursively enumerable subsets of N that the semilattice $\mathcal{L}^e$ possesses a largest element. We will also consider the semilattices $_a\mathcal{L} = \{b \in \mathcal{L}^e \mid a \leq b\}$, where $a \in \mathcal{L}^e$ and a is not equal to the largest element of $\mathcal{L}^e$, and the semilattices of computable enumerations $\mathcal{L}(S_n)$, where $S_n = \{\phi \; , \{1\}, \ldots, \{n\}\}$ and $n = 1, 2, \ldots$. Suppose S is a computable family of recursively enumerable sets and $\mathcal{L}(S)$ is the semilattice of computable enumerations of S (see [1]); by analogy with m-degrees, the element of $\mathcal{L}(S)$ defined by a computable enumeration $f: N \xrightarrow{\text{onto}} S$ will be denoted by $d_m(f)$. It can be shown that the semilattice $\mathcal{L}(S_n)$ possesses largest and smallest elements (see [1]) and that the semilattice $\mathcal{L}^e$ is (naturally) isomorphic to the semilattice $\mathcal{L}(S_1)$.

The concept of m-reducibility was introduced by Post [4]. In that same paper he introduced the concept of a creative set; it turns out (Myhill [5]) that $d_m(A) = I_{\mathcal{L}^e}$ if and only if A is a creative set. Yany (see [3]) observed that the m-degree of a so-called maximal set $\mathcal{M}$ is minimal, i.e., satisfies the condition $d_m(\mathcal{M}) \neq 0 \;\&\; \forall b \in \mathcal{L}^e (\mathbf{O} \leq b \leq d_m(\mathcal{M}) \rightarrow b = \mathbf{O} \vee b = d_m(\mathcal{M}))$. Lachlan [6] proved that the largest element of $\mathcal{L}^e$ is indecomposable, i.e., $a \cup b = I \rightarrow a = I \cup b = I$. Ershov [7] showed that

1) $\mathcal{L}^e$ contains infinitely many minimal elements;

2) there exist elements $(\neq \mathbf{O})$ under which there are no minimal ones;

3) $\mathcal{L}^e$ is not a lattice;

4) the elementary theory of the semilattice $\mathcal{L}^e$ is undecidable.

It is proved in [8] that for any $a \in \mathcal{L}^e \setminus \{\mathbf{O}, I\}$ there exists $b \in \mathcal{L}^e$ such that $a \nleq b \;\&\; b \nleq a$, and that for any $a \in \mathcal{L}^e$ we have $a < I \rightarrow \exists b \in \mathcal{L}^e (a < b < I)$. It is proved in [11] that for any $a \in \mathcal{L}^e$ we have

$$a < I \rightarrow \exists b \in \mathcal{L}^e (a < b \;\&\; \forall c \in \mathcal{L}^e (c \leq b \rightarrow c \leq a \vee c = b)).$$

Lachlan's paper [12] was a significant advance in the study of $\mathcal{L}^e$, namely Lachlan proved that if $\mathcal{L}_\theta$ is an L-semilattice (denoted by $\mathcal{L}_\theta : \mathcal{L} = \langle \mathcal{L}, \cup \rangle$, where L is a semilattice and θ is an enumeration of $\mathcal{L}$; the definition of a Lachlan semilattice (L-semilattice) is given below), then there exists $a \in \mathcal{L}^e$ such that the semilattice $\mathcal{L}_a = \{b \in \mathcal{L}^e \mid b \leq a\}$ is isomorphic to $\mathcal{L}$; conversely, for each $a \in \mathcal{L}^e$ there exists an enumeration $\theta: N \xrightarrow{\text{onto}} \mathcal{L}_a$ such that $(\mathcal{L}_a)_\theta$ is an L-semilattice. The last results on the semilattice $\mathcal{L}^e$ (and also $\mathcal{L}(S_n)$) are the theorems of Ershov–Lavrov [13] and V'yugin [14]. Let us recall what they are.

THEOREM (Ershov–Lavrov [13]). If $A \subset \mathcal{L}^e$, $A \neq \emptyset$ is a computable ideal, $B \subset \mathcal{L}^e$ is a computable family of m-degrees such that $A \cap B = \emptyset$ and $I \notin A \cup B$, then there exists $a \in \mathcal{L}^e$ such that $\forall b \in \mathcal{L}^e (b < a \leftrightarrow b \in A)$ and $\forall b \in B$ (a is comparable with b).

THEOREM (V'yugin [14]). For any $a \in \mathcal{L}^e$ different from I and for an arbitrary L-semilattice $\mathcal{L}_\theta$, there exist $b \in \mathcal{L}^e$ such that $a \leq b$, the semilattice ${}_a\mathcal{L}_b = \{c \in \mathcal{L}^e \mid a \leq c \leq b\}$ is isomorphic $\mathcal{L}$ and $\forall c \in \mathcal{L}^e (c \leq b \rightarrow c \leq a \vee a \leq c)$.

A complete description of the semilattice $\mathcal{L}^m$ is contained in Ershov [15] with the addendum of Palyutin [16].

2. Definitions and Statements of Theorems

A pair consisting of a (no more than countable) semilattice $\mathcal{L} = \langle \mathcal{L}, \cup \rangle$ and an enumeration $\theta: N \xrightarrow{\text{onto}} \mathcal{L}$ of the underlying set $\mathcal{L}$ will be denoted by $\mathcal{L}_\theta$ and called an enumerated semilattice. We introduce the following category K : the object of K are the enumerated semilattices, and a morphism $a: \mathcal{L}^1_\theta \longrightarrow \mathcal{L}^2_\nu$ of an enumerated semilattice $\mathcal{L}^1_\theta = (\langle \mathcal{L}^1, \cup \rangle, \theta)$ into an enumerated semilattice $\mathcal{L}^2_\nu = (\langle \mathcal{L}^2, \cup \rangle, \nu)$ is a mapping $a: \mathcal{L}^1 \longrightarrow \mathcal{L}^2$ of the underlying set $\mathcal{L}^1$ into the underlying set $\mathcal{L}^2$ such that

1) a is a multivalent;

2) a is a semilattice homomorphism;

3) $a(\mathcal{L}^1)$ is an ideal of $\mathcal{L}^2$;

4) there exists a general recursive function f such that $\forall x \in N \, (a\theta(x) = \nu f(x))$ (i.e., a is a morphism of the corresponding enumerated sets (see [1])).

Suppose $\mathcal{L}_\theta$ is an enumerated semilattice. We will say that $\mathcal{L}_\theta$ is a Lachlan semilattice (L-semilattice) if there exists a sequence of finite preordered sets $\langle D_0, \leq_0 \rangle \subset \langle D_1, \leq_1 \rangle \subset \ldots$, where $D_i \subset N$, such that

L0) $\theta(x) \leq \theta(y) \longleftrightarrow \exists i \in N \; (x \leq_i y)$;

L1) $\{D_i\}_{i \geq 0}$ is a strongly computable sequence of finite sets (we will use the following abbrevations:

$$x \sim_i y \rightleftharpoons x \leq_i y \,\&\, y \leq_i x, \; [x]_i = \{y \in N \mid x \sim_i y\}, \; \tilde{D}_i = \{[x]_i \mid x \in D_i\}),$$

L2) the ordered set $\tilde{D}_i$ is a distributive lattice;

L3) the mapping $\tilde{D}_i \longrightarrow \tilde{D}_{i+1}$ induced by the embedding $\langle D_i, \leq_i \rangle \subset \langle D_{i+1}, \leq_{i+1} \rangle$ preserves the least upper bound and the largest and smallest elements;

L4) there exist general recursive functions $u(x, y, i)$, $v(x, y, i)$ such that

$$x,y \in D_i \longrightarrow u(x,y,i),\ v(x,y,i) \in D_i,$$
$$[x]_i \cup [y]_i = [u(x,y,i)]_i,$$
$$[x]_i \cap [y]_i = [v(x,y,i)]_i, \text{ where } x,y \in D_i;$$

L5) there exists a recursive predicate $P(x,y,i,a,b)$ such that for all x,y,i we have $x \leq_i y \longleftrightarrow \forall a \exists b P(x,y,i,a,b)$.

We mention one property of Lachlan semilattices that will be needed to prove Theorem 1. Suppose $\mathcal{L}_\theta$ is an L-semilattice and f is a general recursive function such that $f(N)=N$. Then $\mathcal{L}_{\theta \circ f}$ is an L-semilattice. Indeed, suppose $\langle D_0, \leq_0 \rangle \subset \langle D_1, \leq_1 \rangle \subset \ldots$ is a sequence of preordered sets satisfying conditions L1)-L5) and such that $\theta(x) \leq \theta(y) \longleftrightarrow \exists i \in N\ (x \leq_i y)$. Suppose $g(x) = \mu y (f(y)=x)$ (here μ is the minimization operator). Since $f(N) = N$, the function g is general recursive and $fg(x) = x$. Put $D_i' = \{x \in N \mid f(x) \in D_i \ \&\ x \leq \sup(g(D_i))\}$, $x \leq_i' y \rightleftarrows x,y \in D_i' \ \&\ f(x) \leq_i f(y)$. It is easy to see that the sequence of preordered sets $\langle D_0', \leq_0' \rangle \subset \langle D_1', \leq_1' \rangle$ ... satisfies conditions L1)-L5) and that $\theta f(x) \leq \theta f(y) \longleftrightarrow \exists i \in N\ (x \leq_i' y)$. Thus, we have proved that $\mathcal{L}_{\theta \circ f}$ is an L-semilattice.

Suppose $\{f_i\}_{i \geq 0}$ is a principal enumeration of the set of all one-place partial recursive functions. If we let Π_i be the domain of f_i, it is clear that $\{\Pi_i\}_{i \geq 0}$ is a principal enumeration of the class of all recursively enumerable subsets of N. We introduce an enumeration of the semilattice $\mathcal{L}^e$: $\pi(i) = d_m(\Pi_i)$ and an enumeration of the semilattice ${}_a\mathcal{L}$: $\zeta(i) = a \cup \pi(i)$ (the dependence of ζ on a is not indicated, but this will not lead to complications). We also introduce an enumeration of the semilattice $\mathcal{L}(S_n)$ as follows. Let $\bar{f}_i(0) = \emptyset$, $\bar{f}_i(x) = \{x\}$ for $1 \leq x \leq n$, $\bar{f}_i(n+x+1) = \{f_i(x)\}$ if $f_i(x)$ is defined and $f_i(x) \in \{1,\ldots,n\}$, $\bar{f}_i(n+x+1) = \emptyset$; otherwise, put $\xi(i) = d_m(\bar{f}_i)$ (the dependence of the enumeration ξ on n is not indicated).

We are now in a position to state Theorems 1, 1', and 2. We fix an enumerated semilattice $\mathcal{L}_\varphi \in \{\mathcal{L}^e_\pi, {}_a\mathcal{L}_\zeta, \mathcal{L}(S_n)_\xi\}$.

<u>THEOREM 1.</u> Suppose in the diagram

$$\begin{tikzcd}
\mathcal{L}_\theta^1 \arrow[r, dashed, "c"] & \mathcal{L}_\varphi \\
\mathcal{L}_\nu^0 \arrow[u, "a"] \arrow[ur, "b"'] &
\end{tikzcd}$$

that $a, b \in K$, $I \notin b(\mathcal{L}^0)$, and $\mathcal{L}_\theta^1$ is an L-semilattice. Then there exists $c \in K$ making the diagram commutative.

<u>THEOREM 1'.</u> Suppose the diagram

$$\begin{tikzcd}
\mathcal{L}_\theta^1 \arrow[r, dashed, "e"] & \mathcal{L}_\varphi \\
\mathcal{L}_\nu^0 \arrow[u, "a"] \arrow[ur, "b"] \arrow[r, "d"'] & \gamma \arrow[u, "c"']
\end{tikzcd}$$

that $a, b \in K$, c, d are morphisms of enumerated sets, $b = c \circ d$, $I \notin c(y)$ and $\mathcal{L}'_\theta$ is an L-semilattice. Then there exists $e \in K$ making the diagram commutative and such that $e(\mathcal{L}') \cap c(y) = b(\mathcal{L}^\circ)$.

THEOREM 2. Suppose $a : y \to \mathcal{L}_\varphi$ is a morphism of enumerated sets such that $I \notin a(y)$. Then there exist an L-semilattice $\mathcal{L}'_\theta$, a morphism of enumerated sets $b : y \to \mathcal{L}'_\theta$, and a K-morphism $c : \mathcal{L}'_\theta \to \mathcal{L}_\varphi$ such that $a = c \circ b$ and $I \notin c(\mathcal{L}')$.

3. Proof of Theorems 1 and 2

Recall that $\{f_i\}_{i \geq 0}$ is a principal enumeration of the set of all one-place partial recursive functions (p.r.f.), Π_i is the domain of f_i, and, therefore, $\{\Pi_i\}_{i \geq 0}$ is a principal enumeration of the set of all recursively enumerable subsets of N. Fix a general recursive function (g.r.f.) $c(x, y)$ effecting a one-to-one correspondence $N \leftrightarrow N^2$ and such that $c(x, y)$ is nondecreasing in x and y, in particular, $\sup(x, y) \leq c(x, y)$. Let $c(x, y, z) = c(x, c(y, z))$. We give the definition of the Lachlan ψ-operator (see [10]). Suppose $U \subset N$ is a set and $A \subset N$ is a recursively enumerable (r.e.) set. Then we denote by $\psi(U, A)$ the following m-degree: if $A = \phi$, then $\psi(U, A) = d_m(\phi)$; if $A \neq \phi$ and f is a g.r.f. such that $f(N) = A$, then $\psi(U, A) = d_m(f^{-1}(U))$. This definition is obviously correct, i.e., does not depend on the choice of f. The following are the main properties of the Lachlan ψ-operator.

01) The ψ-operator $A \mapsto \psi(U, A)$ maps the set of r.e. subsets of N onto the set of m-degrees $\leq d_m(U)$; $\psi(U, N) = d_m(U)$;

02) $\psi(U, A \cup B) = \psi(U, A) \cup \psi(U, B)$;

03) If $\psi(U, A) \leq \psi(V, B)$ and $B \cap V \neq \phi$, $B \cap (N \setminus V) \neq \phi$, then there exists a p.r.f. f with domain A such that $f(A) \subset B$ and $x \in A \to (x \in U \leftrightarrow f(x) \in V)$; conversely, the existence of a p.r.f. f with these properties implies that $\psi(U, A) \leq \psi(V, B)$; in particular, if $A \cap U$, $A \cap (N \setminus U)$ are recursively enumerable, then $\psi(U, A) = d_m(\phi)$;

04) If A, B are r.e. sets, $\sim$ is a r.e. equivalence relation on A such that for any $x \in A$ there exists y: $y \in A \cap B$ & $x \sim y$, and for any $x, y \in A$ we have $x \sim y \to (x \in U \leftrightarrow y \in U)$, then $\psi(U, A) \leq \psi(U, B)$.

For example, let us prove 04). Suppose $C \rightleftharpoons \{(x, y) | x \sim y \,\&\, y \in B\}$. The set C is recursively enumerable, $(x, y) \in C \to (x \in U \leftrightarrow y \in U)$, $x \in A \leftrightarrow \exists y ((x, y) \in C)$, and $(x, y) \in C \to y \in B$. In view of the first and third properties of C, there exists a p.r.f. f with domain A such that $x \in A \to (x, f(x)) \in C$, and it follows from the second and fourth properties that the p.r.f. f also satisfies the relations $x \in A \to (x \in U \leftrightarrow f(x) \in U)$ and $f(A) \subset B$. In view of 03), $\psi(U, A) \leq \psi(U, B)$.

Let us recall some facts about finite distributive lattices (see [12]). Suppose D is a finite distributive lattice. An element $a \in D$ is called an atom if $a \leq b \cup c \to a \leq b \vee a \leq c$. Suppose D_1, D_2 are finite distributive lattices and $\varphi : D_1 \to D_2$ is a mapping preserving the least upper bound and the largest and smallest elements. If $a \in D_2$ is an atom, we denote by

$C(a)$ the set of minimal elements of the set $B(a) \rightleftharpoons \{b \in D_1 \mid a \leq \varphi(b)\}$. We claim the following.

D1) The set $C(a)$ is nonempty and each element of $C(a)$ is an atom.

D2) If $a, b \in D_2$ are atoms and $a \leq b$, then there exists a mapping $\psi: C(b) \to C(a)$ such that $\psi(d) \leq d$.

That $C(a)$ is nonempty follows from the fact that φ preserves the largest element. We will show that each element of $C(a)$ is an atom. If $b \in C(a)$ and $b \leq c \cup d$, then $b = (b \cap c) \cup (b \cap d)$, $a \leq \varphi(b) = \varphi((b \cap c) \cup (b \cap d)) = \varphi(b \cap c) \cup \varphi(b \cap d)$; but a is an atom, hence $a \leq \varphi(b \cap c)$ or $a \leq \varphi(b \cap d)$. If, for definiteness, $a \leq \varphi(b \cap c)$, then $b \cap c \in B(a)$ and $b \cap c \leq b$; but b is a minimal element of $B(a)$, hence $b \cap c = b$ and $b \leq c$. Let us prove D2). Since $a \leq b$, it follows that $C(b) \subset B(a)$ and, since $B(a)$ is finite, under each element there is a minimal one, i.e., the desired $\psi: C(b) \subset C(a)$ exists.

We also introduce the concept of convergence. Suppose A, B are sets and $\text{Ч}(A,B)$ is the set of all partial mappings from A into B. If $f \in \text{Ч}(A,B)$, $a \in A$, then $f(a)!$ is an abbreviation for "f is defined at the point a." Suppose $\{g_s\}_{s \geq 0}$ is a sequence of elements of $\text{Ч}(A,B)$. We will say that the sequence $\{g_s\}_{s \geq 0}$ converges if

1) $\exists s\, \forall u, v\, \forall a \in A\, (s \leq u \leq v \;\&\; g_u(a)! \to g_v(a)!)$,

2) $\forall a \in A\, \exists s\, [\forall t \geq s\, (\neg g_t(a)!)$ or $\forall t \geq s\, (g_t(a)! \;\&\; g_t(a) = g_s(a))]$.

If the sequence $\{g_s\}_{s \geq 0}$ converges and $g \in \text{Ч}(A,B)$, we will say that g is the limit of $\{g_s\}_{s \geq 0}$, $g = \lim\limits_{s \to \infty} g_s$, if for any a:

1) $\neg g(a)! \to \exists s\, \forall t \geq s\, (\neg g_t(a)!)$,

2) $g(a)! \to \exists s\, \forall t \geq s\, (g_t(a)! \;\&\; g_t(a) = g(a))$.

Obviously, for a convergent sequence the limit exists and is uniquely defined. Note that if a sequence $\{g_s\}_{s \geq 0}$ converges and its limit $g = \lim\limits_{s \to \infty} g_s$ is a function with finite domain, then there exists s such that $g_t = g$ for all $t \geq s$. Indeed, suppose s_0 is such that $\forall u, v\, \forall a \in A\, (s_0 \leq u \leq v \;\&\; g_u(a)! \to g_v(a)!)$, and suppose $A_0 \subset A$ is the domain of g. Since A_0 is finite, there exists $s_1 \geq s_0$ such that for $t \geq s_1$ and $a \in A_0$: $g_t(a)$ is defined and $g_t(a) = g(a)$. Obviously, $g_t = g$ for $t \geq s_1$. For functions $f: N \to C$ (C an arbitrary set) and $f: N \to N$ the equalities $\lim\limits_{s \to \infty} f(s) = c$ (where $c \in C$) and $\lim\limits_{s \to \infty} f(s) = \infty$ have the usual meaning, namely $\lim\limits_{s \to \infty} f(s) = c \leftrightarrow$ (there exists $n \in N$ such that $s \geq n \to f(s) = c$), $\lim\limits_{s \to \infty} f(s) = \infty \leftrightarrow$ (for each $n \in N$ there exists $m \in N$ such that $s \geq m \to f(s) \geq n$). Note that if C is $\text{Ч}(A,B)$, then an equality $\lim\limits_{s \to \infty} f(s) = c$ in the sense of the second definition implies the equality $\lim\limits_{s \to \infty} f(s) = c$ in the sense of the first, but not conversely.

<u>Other Conventions.</u> The totality of subsets of a given set A will be denoted by $S(A)$. As usual, a partition $\mathbb{P}$ of a set A is a subset of $S(A)$, $\mathbb{P} \subset S(A)$, such that each element $\mathbb{P}$ is nonempty, the elements of $\mathbb{P}$ are pairwise disjoint, and the union of the elements of $\mathbb{P}$ is A. If $\mathbb{P}, \mathbb{Q} \subset S(A)$ are two partitions of A, then $\mathbb{P}$ is called a refinement of $\mathbb{Q}$ if each element of $\mathbb{P}$ is a subset of a suitable element of $\mathbb{Q}$. If $B \subset A$ and $\mathbb{P} \subset S(A)$

is a partition of A, then we will denote by $\mathcal{P}\mid B$ the following partition of B: $\mathcal{P}\mid B = \{C\cap B \mid C\in\mathcal{P}\ \&\ C\cap B\neq\emptyset\}$.

To avoid obscuring the main ideas with complex notation we analyze only the case $\mathcal{L}_\varphi = \mathcal{L}^e_\pi$ in Theorem 1. The changes required for ${}_a\mathcal{L}_\xi$, $\mathcal{L}(S_n)_\xi$ will be indicated later.

<u>THEOREM 1.</u> Suppose in the diagram

$$\begin{tikzcd} \mathcal{L}^1_\theta \arrow[r, dashed, "c"] & \mathcal{L}^e_\pi \\ \mathcal{L}^0_\nu \arrow[u, "a"] \arrow[ur, "b"'] & \end{tikzcd}$$

that $a, b\in K$, $I\notin b(\mathcal{L}^0)$ and $\mathcal{L}^1_\theta$ is an L-semilattice. Then there exists $c\in K$ making the diagram commutative.

Suppose $\bar\theta$ is the cylindrification of the enumeration θ; by definition, there exists a g.r.f. g such that $\bar\theta = \theta\circ g$, $g(N)=N$, and g assumes each of it values infinitely often (g is a function of large amplitude). In view of the remark immediately following the definition of L-semilattice, $\mathcal{L}^1_{\bar\theta}$ is an L-semilattice. Obviously, the identity mapping $\mathcal{L}^1_\theta \longleftrightarrow \mathcal{L}^1_{\bar\theta}$ is a K-isomorphism. Therefore, we may assume without loss of generality that the enumeration θ is itself a cylinder, i.e., $\theta=\theta\cdot g$ for some function g of large amplitude. Suppose $\langle D^1_0, \leqslant^1_0\rangle \subset \langle D^1_1, \leqslant^1_1\rangle \subset \ldots$ is a sequence of finite preordered sets satisfying conditions L1)-L5) and such that $\theta(x)\leqslant\theta(y) \longleftrightarrow \exists i\in N\ (x\leqslant^1_i y)$ and suppose $u^1(x,y,i)$, $v^1(x,y,i)$ are g.r.f. satisfying L4) (in connection with our sequence). Let $\mathcal{L}$ be a semilattice obtained from $\mathcal{L}^1$ by externally adjoining a largest element. We define an enumeration of $\mathcal{L}$, $\mu: N \xrightarrow{\text{onto}} \mathcal{L}$, as follows: $\mu(0)=I_{\mathcal{L}}$, $\mu(x+1) = \theta(x)$. We also define a sequence of preordered sets $\langle D_0,\leqslant_0\rangle\subset\langle D_1,\leqslant_1\rangle\subset\ldots$ and g.r.f. $u(x,y,i)$, $v(x,y,i)$;

$$(*)\quad D_i = \{x\in N \mid x=0 \vee x\geqslant 1\ \&\ x-1\in D^1_i\},$$

$$x\leqslant_i y \rightleftarrows x,y\in D_i\ \&\ [y=0 \vee x,y\geqslant 1\ \&\ (x-1)\leqslant^1_i (y-1)],$$

$$u(x,y,i) = \begin{cases} u^1(x-1,y-1,i)+1, & \text{if } x,y\geqslant 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$v(x,y,i) = \begin{cases} x, & \text{if } y=0, \\ y, & \text{if } x=0, \\ v^1(x-1,y-1,i)+1, & \text{otherwise.} \end{cases}$$

It is easy to see that the sequence $\langle D_0,\leqslant_0\rangle \subset \langle D_1,\leqslant_1\rangle\subset\ldots$ and the function u, v satisfy conditions L1)-L5) and that $\mu(x)\leqslant\mu(y) \longleftrightarrow \exists i\in N\ (x\leqslant_i y)$, in particular, $\mathcal{L}_\mu$ is an L-semilattice. We emphasize that throughout the proof of Theorem 1, $\langle D_i, \leqslant_i\rangle, u, v$ are the objects introduced in (*). We will also assume that $x\leqslant i \longrightarrow x\in D_i$.

It is clear that the natural embedding $\mathcal{L}^1_\theta\subset\mathcal{L}_\mu$ is a K-morphism. Suppose $\bar a$ is the composite mapping $\mathcal{L}^0_\nu \xrightarrow{a} \mathcal{L}^1_\theta \subset \mathcal{L}_\mu$. It suffices to prove that there exists $c\in K$, making the diagram

$$\begin{tikzcd} \mathcal{L}_\mu \arrow[r, dashed, "c"] & \mathcal{L}^e_\pi \\ \mathcal{L}^0_\nu \arrow[u, "\bar a"] \arrow[ur, "b"'] & \end{tikzcd}$$

commutative. Since the enumeration θ is a cylinder, there exists a g.r.f. f such that $a\nu(x)=\theta f(x)$ and $f(x)\geq x$. Let $\bar f(x)\rightleftharpoons f(x)+1$. Obviously, $\bar a\nu(x)=\mu\bar f(x)$ and $\bar f(x)\geq x$. The latter relation implies that the set $\bar f(N)$ is recursive (we denote it by H), and the first relation implies the equality $\mu(H)=\bar a(\mathcal{L}^0)$. This set H will be needed later.

We will use (until the end of the proof of Theorem 1) the following abbreviations: $x\sim_i y \rightleftharpoons x\leq_i y \,\&\, y\leq_i x$, $[x]_i=\{y\in N \mid x\sim_i y\}$, $\tilde D_i=\{[x]_i \mid x\in D_i\}$. Suppose A is a subset of D_i. We will say that A is an atom of D_i if the distributive lattice $\tilde D_i$ contains an atom a such that $A=\{x\in D_i \mid a\leq [x]_i\}$. We introduce, following Lachlan (see [12]), frames and towers. By a frame of length i we mean a sequence $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_i)$, where $\mathfrak{A}_j\subset S(D_j)$ ($S(D_j)$ is the totality of subsets of D_j), such that

K1) $\mathfrak{A}_i$ is a singleton;

K2) $\mathfrak{A}_j=\cup\{C(B)\mid B\in\mathfrak{A}_{j+1}\}$, $j<i$;

K3) for $B\in\mathfrak{A}_{j+1}$, $D_j\cap B=\cap\{U\mid U\in C(B)\}$, $j<i$;

here $C(B)$ is the totality of maximal (with respect to inclusion) elements of the set $\{U\in\mathfrak{A}_j \mid U\supset B\cap D_j\}$.

We will denote the length of a frame $\mathfrak{A}$ by $\ln(\mathfrak{A})$. A frame $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_i)$ will be called good if, for each $j\leq i$, each element of $\mathfrak{A}_j$ is an atom D_j. It follows from conditions D1) and D2) that if $A\subset D_i$ is an atom of D_i, then there exists a unique good frame $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_i)$ such that $\mathfrak{A}_i=\{A\}$; it is also easy to see that $\{0\}$ is an atom of D_i for all $i\geq 0$, hence the sequence $(\{\{0\}\},\dots,\{\{0\}\})$ is a good frame. If $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_i)$, $\mathfrak{L}=(\mathfrak{L}_0,\dots,\mathfrak{L}_j)$ are two frames, we will say that $\mathfrak{A}$ is a subframe of $\mathfrak{L}$ if $i\leq j$, $\mathfrak{A}_e\subset\mathfrak{L}_e$ when $e\leq i$, and for $B\in\mathfrak{A}_{e+1}$ the set $C(B)$ computed in $\mathfrak{A}$ is equal to $C(B)$ computed in $\mathfrak{L}$. We now define a tower. Suppose $\mathcal{F}\subset N$ is a finite set. A tower with base $\mathcal{F}$ and length i is a sequence $\mathbb{A}=(\mathbb{A}_0,\dots,\mathbb{A}_i,\varphi_0,\dots,\varphi_i)$ of partitions of $\mathcal{F}$ and mappings $\varphi_j:\mathbb{A}_j\to S(D_j)$ such that

B1) the partition $\mathbb{A}_i$ is a singleton: $\mathbb{A}_i=\{\mathcal{F}\}$;

B2) the partition $\mathbb{A}_j$ is a refinement of the partition $\mathbb{A}_{j+1}$, $j<i$;

B3) for $Q\in\mathbb{A}_{j+1}$ the restriction of φ_j to $\{P\in\mathbb{A}_j \mid P\subset Q\}=\mathbb{A}_j|Q$ is a bijection of this set onto $C(\varphi_{j+1}(Q))$, where $C(\varphi_{j+1}(Q))$ is the totality of maximal (with respect to inclusion) elements of the set $\{B\in\varphi_j(\mathbb{A}_j)\mid B\supset\varphi_{j+1}(Q)\cap D_j\}$, $j<i$;

B4) the sequence $(\varphi_0(\mathbb{A}_0),\dots,\varphi_i(\mathbb{A}_i))$ is a frame.

The frame in B4) will be called the frame of the tower $\mathbf{A}$. The length of the tower $\mathbf{A}$ will be denoted by ln $(\mathbf{A})$, the frame by fr $(\mathbf{A})$, and the base by bs $(\mathbf{A})$. It is not difficult to show (see [12]) that for any frame $\mathfrak{A}$ and any finite set $\mathcal{F} \subset N$ containing sufficiently many elements there exists a tower $\mathbf{A}$ with base $\mathcal{F}$ and frame $\mathfrak{A}$. Suppose $\mathbf{A} = (\mathbf{A}_0, \ldots, \mathbf{A}_i, \varphi_0, \ldots, \varphi_i)$ is a tower, $j \leq i$ and $P \in \mathbf{A}_j$. We denote by tw $(\mathbf{A}, j, P)$ the tower $(\mathbf{A}_0 | P, \ldots, \mathbf{A}_j | P, \ \bar{\varphi}_0, \ldots, \bar{\varphi}_j)$, where $\bar{\varphi}_\kappa$, $\kappa < j$, is the restriction of φ_κ to $\mathbf{A}_\kappa | P$ (in view of condition B2), $\mathbf{A}_\kappa | P$ is a subset of $\mathbf{A}_\kappa$); we denote the frame of tw $(\mathbf{A}, j, P)$ by fr $(\mathbf{A}, j, P)$. We introduce a partial order on the frames. Suppose $\mathfrak{A} = (\mathfrak{A}_0, \ldots, \mathfrak{A}_i)$, $\mathcal{L} = (\mathcal{L}_0, \ldots ,\mathcal{L}_i)$ are two frames of the same length, $\mathfrak{A}_i = \{A\}$, $\mathcal{L}_i = \{B\}$. We will say that $\mathfrak{A}$ is less than $\mathcal{L}$, $\mathfrak{A} \preccurlyeq \mathcal{L}$, if 1) $A \supset B$ and 2) for any $j < i$, $\mathcal{D} \in \mathfrak{A}_{j+1}$, $\mathcal{E} \in \mathcal{L}_{j+1}$, if $\mathcal{D} \supset \mathcal{E}$, then there exists a mapping $\psi : C(\mathcal{D}) \to C(\mathcal{E})$ such that $\psi(U) \supset U$, $U \in C(\mathcal{D})$, where $C(\mathcal{D})$ is totality of maximal (with respect to inclusion) elements of the set $\{U \in \mathfrak{A}_j \mid U \supset \mathcal{D}_j \cap \mathcal{D}\}$; $C(\mathcal{E})$ is defined analogously. It is easy to see that if $\mathfrak{A} = (\mathfrak{A}_0, \ldots, \mathfrak{A}_i)$, $\mathcal{L} = (\mathcal{L}_0, \ldots, \mathcal{L}_i)$ are good frames and $\mathfrak{A}_i = \{A\}$, $\mathcal{L}_i = \{B\}$, then, in view of D2), $\mathfrak{A} \preccurlyeq \mathcal{L}$, if and only if $A \supset B$. Suppose $\mathbf{A} = (\mathbf{A}_0, \ldots, \mathbf{A}_i, \varphi_0, \ldots, \varphi_i)$, $\mathcal{B} = (\mathcal{B}_0, \ldots, \mathcal{B}_j, \ \psi_0, \ldots, \psi_j)$ are towers with bases $\mathcal{F}, G$ respectively, where $\mathcal{F} \cap G = \emptyset$, and suppose $\kappa \leq \inf(i,j)$, $P \in \mathbf{A}_\kappa$, $Q \in \mathcal{B}_\kappa$ and fr $(\mathbf{A}, \kappa, P) \preccurlyeq$ fr $(\mathcal{B}, \kappa, Q)$. Then there exist mappings $\theta_0 : \mathcal{B}_0 | Q \to \mathbf{A}_0 | P, \ldots, \theta_\kappa : \mathcal{B}_\kappa | Q \to \mathbf{A}_\kappa | P$ such that $\varphi_e \theta_e(R) \supset \psi_e(R)$ for $e \leq \kappa$, $R \in \mathcal{B}_e | Q$. Indeed, since the sets $\mathcal{B}_\kappa | Q$, $\mathbf{A}_\kappa | P$ are singletons, there exists a unique mapping θ_κ : $\mathcal{B}_\kappa | Q \to \mathbf{A}_\kappa | P$ and this mapping satisfies our condition by virtue of the relation fr $(\mathbf{A}, \kappa, P) \preccurlyeq$ fr $(\mathcal{B}, \kappa, Q)$. Assume that we have constructed a mapping θ_{e+1} satisfying our condition. Using condition 2) in the definition of $\preccurlyeq$ and condition B3), we can easily define the desired θ_e (not necessarily uniquely, of course). We will now construct a tower $\mathcal{C} = (\mathcal{C}_0, \ldots, \mathcal{C}_i, \bar{\varphi}_0, \ldots, \bar{\varphi}_i)$ with base $\mathcal{F} \cup Q$. For $e \leq \kappa$ the partition $\mathcal{C}_e$ is obtained from the partition $\mathbf{A}_e$ by replacing each element $R \in \mathbf{A}_e$ by $R^* \rightleftharpoons R \cup (\cup\{T \in \mathcal{B}_e | Q \mid \theta_e(T) = R\})$, and for $e > \kappa$ by replacing each element $R \in \mathbf{A}_e$ by

$$R^* \rightleftharpoons \begin{cases} R \cup Q \ , \text{ if } & R \cap P \neq \emptyset, \\ R \ , \text{ if } & R \cap P = \emptyset; \end{cases}$$

$\bar{\varphi}_e(R^*) \rightleftharpoons \varphi_e(R)$. We denote this tower $\mathcal{C}$ by tw $(\mathbf{A}, \mathcal{B}, \kappa, P, Q)$.

Let H be the recursive set introduced earlier with the property that $\mu(H) = \bar{a}(\mathcal{L}^\circ)$. In the sequel we will consider only those frames $\mathfrak{A} = (\mathfrak{A}_0, \ldots, \mathfrak{A}_i)$, $\mathfrak{A}_i = \{A\}$, that satisfy the condition

(**) $A \cap H = \emptyset$.

We now introduce a set of pairs $\mathcal{Q}$. A pair $\alpha = (\mathfrak{A}, V)$ is an element of $\mathcal{Q}$ if and only if the first component of α is a frame $\mathfrak{A} = (\mathfrak{A}_0, \ldots ,\mathfrak{A}_i)$, $\mathfrak{A}_i = \{A\}$, $i = c(m,n,e)$, and the second component is either 1) the symbol I and then $n \in A$ (a pair of the first kind), or 2) V is the symbol II (a pair of the second kind), or 3) V is a set B such that $A \subset B \subset \mathcal{D}_{i+1}$ and $B \cap H \neq \emptyset$ (a pair of the third kind). The length of a pair α (ln (α)) is the length of the first component of α . We define the norm of the pair α at step s (nr (α, s)). Suppose $\mathcal{U}, \mathcal{V}$ are the g.r.f. introduced in (*), $a \in D_0$ is the smallest element of $\langle D_0, \le_0 \rangle$ (it is the smallest in all $\langle D_i, \le_i \rangle$), and $A \subset N$ is a finite set. We define $u(A,i)$, $v(A,i)$ by induction on the number of elements in A: $u(\emptyset,i)=a$, $v(\emptyset,i)=0$, $u(A\cup\{x\},i)=u(x,u(A,i),i)$, $v(A\cup\{x\},i)=v(x,v(A,i),i)$, where x is greater than all elements of A. It is easy to see that we have an equivalence

$$[A\subset D_i \ \&\ A \text{ is an atom of } D_i] \leftrightarrow [A\subset D_i \ \&\ \forall x,y\in D_i\,(u(x,y,i)\in A \to$$
$$\to x\in A \vee y\in A) \ \&\ \forall x\in D_i\,(v(A,i)\le_i x \to x\in A)].$$

It follows from L5) that the second member of the equivalence is a $\exists\forall$-predicate, hence there exists a g.r.f. $p((A,i),s)$ that is nondecreasing in s and such that $\lim_{s\to\infty} p((A,i),s)\neq\infty$ if and only if $A\subset D_i$ and A is an atom of D_i. Suppose $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_i)$ is a frame. Put $p(\mathfrak{A},s)=\sup\{p((A,j),s) \mid j\le i \ \&\ A\in\mathfrak{A}_j\}$. Fix an effective one-to-one correspondence $\omega: \Omega\leftrightarrow N$ such that if $\omega(\mathfrak{A},V_1)=i$, $\omega(\mathfrak{L},V_2)=j$, and ln $(\mathfrak{A})\le$ ln $(\mathfrak{L})$, then $i\le j$, and for $\alpha=(\mathfrak{A},V)$ put nr $(\alpha,s)=c(p(\mathfrak{A},s),\omega(\alpha))$. We emphasize that if $\mathfrak{A}$ is a subframe of $\mathfrak{L}$, $\alpha=(\mathfrak{A},V_1)$, and $\beta=(\mathfrak{L},V_2)$, then nr $(\alpha,s)\le$ nr (β,s) (and nr $(\alpha,s)=(\beta,s)\leftrightarrow\alpha=\beta$). Put nr$(\alpha)=\lim_{s\to\infty}$ nr (α,s); nr(α) can assume the value ∞, and nr$(\alpha)\in N$ is equivalent to saying that the first component of α is a good frame. We also introduce a computable sequence of r.e. sets $\{B_i\}_{i\ge 0}$ with the following property: if $\mu(i)=\mu\bar f(j)$, then $d_m(B_i)=\delta\nu(j)$, where $\bar f$ is the previously fixed g.r.f. representing the morphism $\bar a: \mathcal{L}^0_\nu \to \mathcal{L}_{\mu_e}$. Suppose g is a g.r.f. representing the morphism $b: \mathcal{L}^0_\nu \to \mathcal{L}_\pi$, and suppose $h(x)=\mu y(\bar f(y)=x)$ (here μ is the minimization operator). Put $A_i=\emptyset$, if $i\notin\bar f(N)$, and $A_i=\Pi_{gh(i)}$, if $i\in\bar f(N)$. Obviously, the sequence $\{A_i\}_{i\ge 0}$ is computable. In view of L5), there exists a g.r.f. $p(x,y,i,s)$, that is nondecreasing in s and such that $x\sim_i y\leftrightarrow\lim_{s\to\infty}p(x,y,i,s)=\infty$. Suppose $p(x,y,i)=\lim_{s\to\infty}p(x,y,i,s)$ ($p(x,y,i)$ can assume the value ∞), and suppose $p(x,0)=0$, $p(x,i+1)=\sup\{p(x,y,i)\mid y\in D_i\cap\bar f(N)\}$ ($\sup(\emptyset)=0$). Put $B_x=\{c(i,y,j)\mid y\in D_i \ \&\ [j\in A_y \ \&\ j<p(x,y,i)\vee j<p(x,i)]\}$. The computability of the sequence $\{B_x\}_{x\ge 0}$ and the fact that it satisfies our condition can be verified directly.

We fix an effective procedure which at the even steps 0, 2, 4,... yields:

either 1) a triple $(\mathbf{A},i,P)$, where $\mathbf{A}=(\mathbf{A}_0,\dots,\mathbf{A}_j,\varphi_0,\dots,\varphi_j)$ is a tower, $i\le j$, and $P\in\mathbf{A}_i$,

or 2) a pair $\alpha\in\Omega$ of the first kind,

or 3) a natural number $i\in N$,

or 4) a pair $\alpha\in\Omega$ of the third kind, each object occurring infinitely often,

at the odd steps 1, 3, 5,... yields elements of Ω, each $\alpha\in\Omega$ occurring nr (α) times.

We will describe, in general terms, a construction which leads to a proof of the existence of the desired morphism $c:\mathcal{L}_\mu\to\mathcal{L}^e_\pi$. At step s we will define for each $\alpha\in\Omega$ a partial mapping G^α_{s+1} from N into the set of all towers and transfer certain elements into a set U; that which we include in U up to step s will be denoted by U_s. The following relations will be satisfied:

1) $G_s^\alpha(x)! \;\&\; G_s^\beta(y)! \;\&\;$ $\mathrm{bs}(G_s^\alpha(x)) \cap \mathrm{bs}(G_s^\beta(y)) \neq \emptyset \to \alpha = \beta \;\&\; x = y$;

2) $G_s^\alpha(x)! \to \mathrm{bs}(G_s^{(\alpha)}(x)) \subset U_s \vee \mathrm{bs}(G_s^\alpha(x)) \subset N \setminus U_s$;

3) $G_s^\alpha(x)! \to$ (the frame of the tower $G_s^\alpha(x)$ is equal to the first component of α).

We will say that the tower A exists to step s if there exist α, x (uniquely determined by virtue of 1)) such that $G_s^\alpha(x)! \;\&\; G_s^\alpha(x) = A$. The number x is said to be used to step s if either $x \in \{0,1\}$ or x has been used to step s in the base of the tower, i.e., $\exists t \leq s$, $\alpha \in \Omega, y \in N(G_t^\alpha(y)! \;\&\; x \in \mathrm{bs}(G_t^\alpha(y)))$. Before turning to a detailed description of the construction we define several auxiliary functions.

Suppose $\mathcal{M}$ is a creative set. Suppose $\{f_{i,s}\}, \{\Pi_{i,s}\}, \{B_{i,s}\}, \{M_s\}$ are strongly computable sequences of finite functions and sets that are nondecreasing in s and such that $f_i = \cup \{f_{i,s} \mid s \geq 0\}$, and so on.

We define the so-called indicators and heights. The indicator for pairs of the first kind. Suppose $\alpha = (\mathcal{Ol}, I)$ is a pair of the first kind, $\ln(\mathcal{Ol}) = c(m,n,e) = j$, $K = \sup(m,n)$. We define a function in (α, s). Let $s_0 < s_1 < s_2 < \ldots$ be those even steps at which our procedure yields α. If $s \leq s_0$, then in $(\alpha, s) = 0$. If $s_i < s \leq s_{i+1}$, then in $(\alpha, s) =$ in $(\alpha, s_i + 1)$. Suppose $s = s_i + 1$. If

1) the function $f_{e,s}$ is defined on the set

$$B = \cup \{P \mid \exists y \; (G_{s-1}^\alpha(y)! \;\&\; G_{s-1}^\alpha(y) = (A_0, \ldots, A_j, \varphi_0, \ldots, \varphi_j) \;\&\; P \in A_K \;\&\; n \in \varphi_K(P)\},$$

2) for each $y \in B$ the number $f_e(y)$ is used to step $s-1$,

3) for each $y \in B$ we have $y \in U_{s-1} \leftrightarrow f_e(y) \in U_{s-1}$,

4) $f_e(B) \cap \widetilde{B} = \emptyset$, where $\hat{B} = \cup \{\mathrm{bs}(G_{s-1}^\alpha(y)) \mid G_{s-1}^\alpha(y)!\}$,

then we put in $(\alpha, s) =$ in $(\alpha, s-1) + 1$. Otherwise, in $(\alpha, s) =$ in $(\alpha, s-1)$.

The indicator for natural numbers. Suppose $i \in N$, $i = c(n,e)$. Let $s_0 < s_1 < s_2 < \ldots$ be those steps at which our procedure yields i. We define a function in (i, s). If $s \leq s_0$, then in $(i,s) = 0$. If $s_j < s \leq s_{j+1}$, then in $(i,s) =$ in $(i, s_j + 1)$. Suppose $s = s_j + 1$, $a =$ in $(i, s-1)$. If

1) the function $f_{e,s}$ is defined on the set $\{0, 1, \ldots, a\}$,

2) for each $x \leq a$ the number $f_e(x)$ is used to step $s-1$,

3) for each $x \leq a$ we have $x \in \Pi_{g(n),s} \leftrightarrow f_e(x) \in U_{s-1}$ (recall that g is a g.r.f. representing the morphism $b: \mathcal{L}_\nu^0 \to \mathcal{L}_\pi^e$),

then we put in $(i,s) =$ in $(i, s-1) + 1$. Otherwise in $(i,s) =$ in $(i, s-1)$.

Suppose $\mathcal{Ol}$ is a frame. We define a function ht $(\mathcal{Ol}, s)$. Let $s_0 < s_1 < s_2 < \ldots$ be those steps at which our procedure yields triples (A, i, P). If $s \leq s_0$, put ht $(\mathcal{Ol}, s) = 0$. If $s_j < s \leq s_{j+1}$ then ht $(\mathcal{Ol}, s) =$ ht $(\mathcal{Ol}, s_j + 1)$. Suppose $s = s_j + 1$ and at step s_j the procedure yields (A, i, P).

If

1) $\mathcal{O}\!\mathcal{l} = \mathrm{fr}\,(A, i, P)$,

2) the tower A exists to step $s-1$ and $P \cap \Pi_{i,s} \neq \emptyset$ & $\mathrm{bs}(A) \cap U_{s-1} = \emptyset$,

3) $\ln(A) \geq \mathrm{ht}(\mathcal{O}\!\mathcal{l}, s-1)$,

then we put $\mathrm{ht}\,(\mathcal{O}\!\mathcal{l}, s) = \mathrm{ht}\,(\mathcal{O}\!\mathcal{l}, s-1) + 1$. Otherwise, $\mathrm{ht}\,(\mathcal{O}\!\mathcal{l}, s) = \mathrm{ht}\,(\mathcal{O}\!\mathcal{l}, s-1)$.

We can now describe the construction. Before step 0 we assume the numbers 0, 1 to be used and transfer 1 into U , and for each $\alpha \in \Omega$ we put $G_0^{\alpha} = \emptyset$.

Step s . a) s is even.

1) Our procedure at step s yields a triple (A, i, P). Suppose $\mathcal{L} = \mathrm{fr}(A, i, P)$, $\mathrm{ht}\,(\mathcal{L}, s) = a$. If $\mathrm{ht}\,(\mathcal{L}, s+1) = a$ or at step a our procedure yields an element of $\Omega \cup N$, then we change nothing: $G_{s+1}^{\alpha} = G_s^{\alpha}$ for all $\alpha \in \Omega$. Suppose $\mathrm{ht}\,(\mathcal{L}, s+1) = a+1$ and therefore the tower A exists to step s ; $G_s^{\alpha}(x) = A$ and suppose at step a our procedure yields (B, j, Q) and $\mathcal{C} = (B, j, Q)$. If $i = j$, $Q \cap \Pi_{i,s} = \emptyset$, $\mathcal{C} \preccurlyeq \mathcal{L}$, $\ln(B) < \ln(A)$, the tower B exists to step s : $G_s^{\beta}(y) = B$, $\mathrm{bs}\,(B) \cap U_s = \emptyset$, then we put $G_{s+1}^{\beta}(y)$ equal to $\mathrm{tw}\,(B, A, i, Q, P)$, $G_{s+1}^{\alpha}(x)$ is not defined, and there are no changes at the other points.

2) Our procedure at step s yields a pair $\alpha = (\mathcal{O}\!\mathcal{l}, I)$ of the first kind. Suppose $\mathrm{in}\,(\alpha, s) = a$. If $\mathrm{in}\,(\alpha, s+1) = a$, then we change nothing. Assume that $\mathrm{in}\,(\alpha, s+1) = a+1$. Suppose x is the first point at which the function G_s^{α} is undefined. We take a sufficiently large initial segment of unused numbers $\mathcal{F}$, construct a tower A with base $\mathcal{F}$ and frame $\mathcal{O}\!\mathcal{l}$, and put $G_{s+1}^{\alpha}(x) = A$, and for $y \neq x$ we put $G_{s+1}^{\alpha}(y) = G_s^{\alpha}(y)$. For the $\beta \in \Omega$ such that $\mathrm{nr}\,(\alpha, s) < \mathrm{nr}\,(\beta, s)$, we put $G_{s+1}^{\beta} = \emptyset$, and for the remaining $\beta\,(\neq \alpha)$ there are no changes: $G_{s+1}^{\beta} = G_s^{\beta}$.

3) Our procedure at step s yields a natural number $i = c(n, e)$. Suppose $\mathrm{in}\,(i, s) = a$. If $\mathrm{in}\,(i, s+1) = a$, then we change nothing. If $\mathrm{in}\,(i, s+1) = a+1$ in particular, $f_{e,s}(a)!$, suppose $f_e(a) = b$. If to step s there exists no tower B such that $b \in \mathrm{bs}\,(B)$, $i < \ln(B)$, $\mathrm{bs}\,(B) \cap U_s = \emptyset$, then we change nothing. Suppose such a tower B exists: $B = G_s^{\beta}(x) = (B_0, \ldots, B_j, \varphi_0, \ldots, \varphi_j)$, and suppose $b \in P$, $P \in B_i$, $\mathcal{O}\!\mathcal{l} = \mathrm{fr}\,(B, i, P)$. We form a pair $\alpha = (\mathcal{O}\!\mathcal{l}, I)$ of the second kind and let y be the first point at which the function G_s^{α} is undefined. We put $G_{s+1}^{\alpha}(y) = \mathrm{tw}\,(B, i, P)$, $G_{s+1}^{\beta}(x)$ is undefined, and there are no changes at the other points.

4) Our procedure at step s yields a pair $\alpha = (\mathcal{O}\!\mathcal{l}, B)$ of the third kind. Let x be the first point at which the function G_s^{α} is undefined. We take a sufficiently large initial segment of unused numbers $\mathcal{F}$, construct a tower A with base $\mathcal{F}$ and frame $\mathcal{O}\!\mathcal{l}$, and put $G_{s+1}^{\alpha}(x) = A$. There are no changes at the other points.

b) s is odd and at step s our procedure yields a pair α . Put $G_{s+1}^{\alpha} = \emptyset$, $G_{s+1}^{\beta} = G_s^{\beta}$ for $\beta \neq \alpha$. Consider the elements of Ω . If γ is a pair of the first or second kind, $G_{s+1}^{\gamma}(x)!$ and $x \in \mathcal{M}_s$, then we transfer the base of the tower $G_{s+1}^{\gamma}(x)$ into U . If $\gamma = (\mathcal{O}\!\mathcal{l}, B)$ is a pair of the third kind, $\ln(\mathcal{O}\!\mathcal{l}) = i, j = \sigma(B, i+1)$, then for those x such that $G_{s+1}^{\gamma}(x)!$ & $x \in B_{j,s}$ we transfer the base of the tower $G_{s+1}^{\gamma}(x)$ into U . This completes the description of step s of the construction.

Let $U = \cup\{U_s \mid s \geq 0\}$. Obviously, the set U is recursively enumerable. We will prove several lemmas.

LEMMA 1. Suppose that $\mathrm{nr}(\alpha) \neq \infty$ for a pair $\alpha \in \Omega$. Then the sequence $\{G_s^\alpha\}_{s\geq 0}$ converges, and if α is a pair of the first or second kind, then $G^\alpha = \lim_{s\to\infty} G_s^\alpha$ is a function with finite domain.

The proof will be carried out by induction on $\mathrm{nr}(\alpha) \in N$. Suppose the lemma is true for the elements $\Omega_0 = \{\beta \in \Omega \mid \mathrm{nr}(\beta) < \mathrm{nr}(\alpha)\}$ and let $\Omega_1 = \{\beta \in \Omega_0 \mid \beta$ be a pair of the first or second kind $\}$. It is obvious that the set Ω_0 is finite. Suppose s_0 is such that $s \geq s_0 \,\&\, \mathrm{nr}(\beta) \leq \mathrm{nr}(\alpha) \rightarrow \mathrm{nr}(\beta, s) = \mathrm{nr}(\beta)$, $s \geq s_0 \,\&\, \mathrm{nr}(\beta) > \mathrm{nr}(\alpha) \rightarrow \mathrm{nr}(\beta, s) > \mathrm{nr}(\alpha)$. In view of the property of our convergence mentioned directly after the definition, there exists $s_1 \geq s_0$ such that $s \geq s_1 \,\&\, \beta \in \Omega_1 \rightarrow G_s^\beta = G^\beta$. Put $\mathrm{ht}(\mathcal{L}) = \lim_{s\to\infty} \mathrm{ht}(\mathcal{L}, s)$ ($\mathrm{ht}(\mathcal{L})$ can assume the value ∞). Let $K_0 = \{\mathcal{L} \mid \mathcal{L}$ be a subframe of $\mathfrak{A}\}$, where $\mathfrak{A}$ is the first component of the pair α and $K_1 = \{\mathcal{L} \in K_0 \mid \mathrm{ht}(\mathcal{L}) = \infty\}$. Suppose $s_2 \geq s_1$ is such that $\mathcal{L} \in K_0 \setminus K_1 \,\&\, s \geq s_2 \rightarrow$ $\mathrm{ht}(\mathcal{L}) = \mathrm{ht}(\mathcal{L}, s)$, $\mathcal{L} \in K_1 \,\&\, s \geq s_2$ —$\mathrm{ht}(\mathcal{L}, s) > \mathrm{ln}(\mathfrak{A})$. Fix $s_3 \geq s_2$ such that $s \geq s_3$ (our procedure at step s yields the pair α) $\longrightarrow$ (s is even). We claim that if $s_3 \leq s \leq t \,\&\, G_s^\alpha(x)!$, then $G_t^\alpha(x)!$. Obviously, it suffices to consider the case $t = s+1$. Assume the contrary: $s \geq s_3 \,\&\, G_s^\alpha(x)!$, but $G_{s+1}^\alpha(x)$ is undefined. If at step s of the construction we are in case a1), then there exists a frame $\mathcal{L} \in K_0$ such that $\mathrm{ht}(\mathcal{L}, s) \neq \mathrm{ht}(\mathcal{L}, s+1) \,\&\, \mathrm{ln}(\mathfrak{A}) \geq \mathrm{ht}(\mathcal{L}, s)$; but this is impossible in view of the choice of s_0. If we are in case a2) or a3), then, by choice of s_0, for some $\beta \in \Omega_1$ we can extend the definition of G_s^β, but this is impossible in view of the choice of s_1. Case a4) is obviously impossible, and case b) is impossible by the choice of s_3. Contradiction. Now consider $s \geq s_3$ and x such that $G_s^\alpha(x)!$, and suppose $G_s^\alpha(x) = (A_0, \ldots, A_i, \varphi_0, \ldots, \varphi_i)$. Consider $t \geq s$; as shown above, $G_t^\alpha(x)!$, so let $G_t^\alpha(x) = (B_0, \ldots, B_i, \psi_0, \ldots, \psi_i)$. From the description of the construction it is easy to see that for each $e \leq i$ there exists a bijection $\theta_{e,t}: A_e \rightarrow B_e$ such that $\theta_{e,t}(P) \supset P$ (obviously, $\theta_{e,t}$ is uniquely determined). For $e \leq i$ we put $A_e^\circ = \{P \in A_e \mid \theta_{e,t}(P) \cap \Pi_{e,t} \neq \emptyset$ for some $t \geq s\}$. Suppose $t_0 \geq s$ is such that

$$\{P \in A_e \mid \theta_{e,t_0}(P) \cap \Pi_{e,t_0} \neq \emptyset\} = A_e^\circ$$

for all $e \leq i$. From the description of a1) it now follows immediately that $t_0 \leq t \rightarrow G_{t_0}^\alpha(x) = G_t^\alpha(x)$. The convergence of the sequence $\{G_s^\alpha\}_{s\geq 0}$ is proved.

Before proving the second half of the lemma for α we make several remarks. Suppose $\beta \in \Omega_0 \cup \{\alpha\}$, $G^\beta = \lim_{s\to\infty} G_s^\beta$. We define a partial function g^β as follows: $g^\beta(x) = y \leftrightarrow$ $\leftrightarrow G^\beta(y)! \,\&\, x \in \mathrm{bs}(G^\beta(y))$ The sequence of finite functions $\{G_s^\beta\}_{s\geq 0}$ has the following properties: a) it is strongly computable, b) it converges to G^β, and c) $G_s^\beta(x)! \,\&\, G_{s+1}^\beta(x)! \rightarrow$ $\mathrm{bs}(G_s^\beta(x)) \subset \mathrm{bs}(G_{s+1}^\beta(x))$; therefore, the function g^β is partial recursive and the domain of g^β, which we denote by H^β, is a recursively enumerable set. If $\beta \in \Omega_0$ is a pair of the first or second kind, then H^β is finite, hence $\psi(U, H^\beta) = 0$. Suppose $\beta \in \Omega_0$ is a pair of the third kind, $\beta=(\mathcal{L},B)$. We calculate $\psi(U,H^{\beta})$. Suppose $\kappa=\ln(\mathcal{L})$, $j=\sigma(B,\kappa+1)$. We claim that $\psi(U,H^{\beta})=d_m(B_j)$ (where $\{B_e\}_{e\geq 0}$ is the computable sequence introduced earlier). Indeed, it is obvious, in the first place, that $g^{\beta}(H^{\beta})=N$ (see case a4) of the construction), and, secondly, it follows from the description of the second part of case b) that for $x\in H^{\beta}$ we have $x\in U\leftrightarrow g^{\beta}(x)\in B_j$, which, in conjunction with property 03) of the ψ-operator, yields the equality $\psi(U,H^{\beta})=d_m(B_j)$.

We will now prove the second half of the lemma for α. We first analyze the case where α is a pair of the first kind, $\alpha=(\mathfrak{A},I)$, $\ln(\mathfrak{A})=c(m,n,e)=i$, $\kappa=\sup(m,n)$. Assume that the function $G^{\alpha}=\lim_{s\to\infty} G^{\alpha}_s$ has an infinite domain. Then it follows from the description of case a2) of the construction that the domain of G^{α} is N, hence $g^{\alpha}(H^{\alpha})=N$. Let $\widetilde{H}=\cup\{P\mid\exists x\,(G^{\alpha}(x)=(A_0,\ldots,A_i,\varphi_0,\ldots,\varphi_i)\,\&\,P\in A_\kappa\,\&\,n\in\varphi_\kappa(P)\}$; it is clear that set $\widetilde{H}$ is recursively enumerable and $\widetilde{H}\subset H^{\alpha}$. It follows from the definition of pairs of the first kind that for each x we have $\widetilde{H}\cap\mathrm{bs}(G^{\alpha}(x))\neq\emptyset$, hence $g^{\alpha}(\widetilde{H})=N$, and it follows from the description of the second part of case b) of the construction that for $x\in H^{\alpha}$ we have $x\in U\leftrightarrow g^{\alpha}(x)\in\mathcal{M}$; this, in conjunction with property 03) of the ψ-operator, yields the equality $\psi(U,\widetilde{H})=d_m(\mathcal{M})=I$. We claim that the function f_e is defined on the set $\widetilde{H}$, $f_e(\hat{H})\cap H^{\alpha}=\emptyset$ and for each $x\in\widetilde{H}$ we have $x\in U\leftrightarrow f_e(x)\in U$. Indeed, we would otherwise have $\lim_{s\to\infty}\mathrm{in}(\alpha,s)\in N$, while our assumption "the function G^{α} has an infinite domain" implies, as is easily seen, the equality $\lim_{s\to\infty}\mathrm{in}(\alpha,s)=\infty$. We call a tower $\mathbb{A}$ final if there exists s_0 such that $s\geq s_0\longrightarrow$ (the tower $\mathbb{A}$ exists to step s). Put $V=N\setminus(\cup\{\mathrm{bs}(\mathbb{A})\mid\mathbb{A}$ is a final tower$\})$ The set V is recursively enumerable, as is the set $V\cap(N\setminus U)$. Therefore, by 03), $\psi(U,V)=0$. It follows from the description of case a2) of the construction that $N=V\cup(\cup\{H^{\beta}\mid\beta\in\Omega_0\cup\{\alpha\}\}$, and it follows from the properties of f_e and 02) and 03) that $I=\psi(U,\widetilde{H})\leq\psi(U,V\cup(\cup\{H^{\beta}\mid\beta\in\Omega_0\}))=\cup\{\psi(U,H^{\beta})\mid\beta\in\Omega_0\}\in\mathcal{B}(\mathcal{L}^{o})$, which contradicts the assumptions of Theorem 1.

We now analyze the case where α is a pair of the second kind, $\alpha=(\mathfrak{A},\mathbb{I})$, $\ln(\mathfrak{A})=i=c(n,e)$. Assume that the function $G^{\alpha}=\lim_{s\to\infty}G^{\alpha}_s$ has an infinite domain. Then it follows from the description of case a3) of the construction that the domain of G^{α} is N, hence $g^{\alpha}(H^{\alpha})=N$. It is also easy to see that for $x\in H^{\alpha}$ we have $x\in U\leftrightarrow g^{\alpha}(x)\in\mathcal{M}$. Now consider the function f_e. We claim that f_e is a g.r.f. and that for each x we have $x\in\Pi_{g(n)}\leftrightarrow f_e(x)\in U$ (g is the previosuly fixed g.r.f. representing the morphism $\mathcal{b}:\mathcal{L}^{o}_{v}\longrightarrow\mathcal{L}^{e}_{\pi}$). Indeed, in the contrary case we have $\lim_{s\to\infty}\mathrm{in}(i,s)\in N$, and our assumption "the function G^{α} has infinite domain" implies that $\lim_{s\to\infty}\mathrm{in}(i,s)=\infty$. It follows from consideration of case a3) of the construction that for each x we have $\mathrm{bs}(G^{\alpha}(x))\cap f_e(N)\neq\emptyset$, hence the image of the p.r.f. $g^{\alpha}\cdot f_e$ is N. This last fact, in conjunction with the relations $x\in H^{\alpha}\rightarrow(x\in U\leftrightarrow g^{\alpha}(x)\in\mathcal{M})$, $x\in\Pi_{g(n)}\longleftrightarrow f_e(x)\in U$ and 03), yields the inequality $d_m(\mathcal{M})\leq d_m(\Pi_{g(n)})\in\mathcal{B}(\mathcal{L}^{o})$, which contradicts the assumptions of Theorem 1.

Lemma 1 is proved.

Suppose $\Omega_0 = \{\alpha \in \Omega \mid \mathrm{nr}(\alpha) \in N\}$ and $G^\alpha = \lim_{s\to\infty} G_s^\alpha$ for $\alpha \in \Omega_0$ (the sequence $\{G_s^\alpha\}_{s\geq 0}$ converges by Lemma 1). Obviously, the tower A is final if and only if there exist $\alpha \in \Omega_0$ and $x \in N$ such that $G^\alpha(x)! \,\&\, A = G^\alpha(x)$. Recall that $V = N \setminus (\cup\{\mathrm{bs}(A) \mid A$ is a final tower$\})$ and $x \leq l \longrightarrow x \in D_i$ (where $\{D_i\}_{i\geq 0}$ is our sequence from (*). For each triple (α, x, i) such that $\alpha \in \Omega_0 \,\&\, x \in D_i \,\&\, i \leq \mathrm{ln}(\alpha)$ we introduce the set $R^\alpha_{xi} = \cup\{P \mid$ there exists $y \in N$ such that $G^\alpha(y)! \,\&\, G^\alpha(y) = (A_0, \ldots, A_j, \varphi_0, \ldots, \varphi_j) \,\&\, P \in A_i \,\&\, x \in \varphi_i(P)\}$; for each triple (α, x, i) such that $i = \mathrm{ln}(\alpha) \,\&\, \alpha \in \Omega_0 \,\&\, x \in D_i$ we introduce the set $\tilde{R}^\alpha_{xi}$: $\tilde{R}^\alpha_{xi} = \emptyset$ if α is a pair of the first or second kind, while if $\alpha = (\mathcal{O}, B)$ is a pair of the third kind, then $\tilde{R}^\alpha_{xi} = \emptyset$, if $x \notin B$, and $\tilde{R}^\alpha_{xi} = \cup\{\mathrm{bs}(G^\alpha(y)) \mid y \geq 0\}$, if $x \in B$. We also put

$$R_{xi} = V \cup (\cup\{R^\alpha_{xi} \mid \alpha \in \Omega_0 \,\&\, i \leq \mathrm{ln}(\alpha)\}) \cup (\cup\{\tilde{R}^\alpha_{xi} \mid \alpha \in \Omega_0 \,\&\, i = \mathrm{ln}(\alpha)\}).$$

LEMMA 2. The set R_{xi} is recursively enumerable and $\psi(U, R_{xi}) = \psi(U, R_{xi+1})$.

Let $K_0 = \{\mathcal{L} \mid \mathcal{L}$ is a frame $\&\ \mathrm{ln}(\mathcal{L}) \leq i\}$, $K_1 = \{\mathcal{L} \in K_0 \mid \mathrm{ht}(\mathcal{L}) \neq \infty\}$. Suppose s_0 is such that $s \geq s_0 \longrightarrow [\mathrm{ht}(\mathcal{L}) = \mathrm{ht}(\mathcal{L}, s)$ for $\mathcal{L} \in K_1] \,\&\, [G^\alpha_s = G^\alpha$ for pairs $\alpha \in \Omega_0$ of the second kind and of length $\leq i]$. Put $Q'_{xi} = \cup\{P \mid$ there exist $s \geq s_0$, $\alpha \in \Omega$, $y \in N$ such that

1) $G^\alpha_s(y)! \,\&\, G^\alpha_s(y) = (A_0, \ldots, A_j, \varphi_0, \ldots, \varphi_j) \,\&\, i \leq j \,\&\, P \in A_i \,\&\, x \in \varphi_i(P)$;

2) $\mathrm{ht}(\mathcal{L}, s) > \mathrm{ln}(G^\alpha_s(y))$ for all $\mathcal{L} \in K_0 \setminus K_1\}$.

We also put $R'_{xi} = \cup\{R^\alpha_{xi} \mid \alpha \in \Omega_0 \,\&\, i \leq \mathrm{ln}(\alpha)\}$. We claim that $V \cup R'_{xi} = V \cup Q'_{xi}$. The first set is obviously contained in the second. Let us prove the reverse inclusion. Suppose $a \in Q'_{xi} \setminus V$; and suppose $s \geq s_0$, $\alpha_0 \in \Omega$, $y_0 \in N$, P_0 satisfy conditions 1) and 2) in the definition of Q'_{xi} and $a \in P_0$. Since $a \notin V$, it follows that for uniquely determined $\alpha_1 \in \Omega$, $y_1 \in N$ we have $G^{\alpha_1}_{s+1}(y_1)! \,\&\, a \in \mathrm{bs}(G^{\alpha_1}_{s+1}(y_1))$. Let $G^{\alpha_1}_{s+1}(y_1) = (B_0, \ldots, B_k, \psi_0, \ldots, \psi_k)$. Looking at the description of the construction, it is easy to see that by virtue of the choice of s_0 and condition 2) we have $\mathrm{ln}(\alpha_0) \geq \mathrm{ln}(\alpha_1) \geq i$, and if $\alpha_1 \neq \alpha_0$, then $\mathrm{ln}(\alpha_0) > \mathrm{ln}(\alpha_1)$, but if $\alpha_0 = \alpha_1$, then $y_0 = y_1$. Let P_1 be the element of B_i containing a. Again by the choice of s_0 and condition 2), $P_1 \supset P_0 \,\&\, \psi_i(P_1) \supset \varphi_i(P_0)$, hence $x \in \psi_i(P_1)$. Thus, $s+1, \alpha_1, y_1, P_1$ satisfy conditions 1) and 2) and $P_0 \subset P_1$. Continuing this argument, we obtain in t steps a sequence $(\alpha_0, y_0, P_0), \ldots, (\alpha_t, y_t, P_t)$ such that $\mathrm{ln}(\alpha_0) \geq \ldots \geq \mathrm{ln}(\alpha_t) \geq i$ and if $\alpha_{j+1} \neq \alpha_j$, then $\mathrm{ln}(\alpha_{j+1}) < \mathrm{ln}(\alpha_j)$, but if $\alpha_{j+1} = \alpha_j$, then $y_{j+1} = y_j$, the set $s+t, \alpha_t, y_t, P_t$ satisfies conditions 1) and 2) and $P_0 \subset P_1 \subset \ldots \subset P_t$, and so on. The sequence $\{(\alpha_t, y_t, P_t)\}_{t\geq 0}$ obviously converges; let (γ, z, S) be its limit. Clearly, $\gamma \in \Omega_0 \,\&\, \mathrm{ln}(\gamma) \geq i$ and $a \in R^\gamma_{xi}$. Therefore, the equality $V \cup R'_{xi} = V \cup Q'_{xi}$ is proved and with it the recursive enumerability of the set $V \cup R'_{xi}$, since the set $V \cup Q'_{xi}$ is obviously recursively enumerable. Suppose $\alpha \in \Omega_0 \,\&\, i = \mathrm{ln}(\alpha)$. It is easy to see that the set $\tilde{R}^\alpha_{xi}$ is either empty or equal to H^α (H^α is the set introduced in the proof of Lemma 1, where we proved that it is recursively enumerable). But $R_{xi} = V \cup R'_{xi} \cup (\cup\{\tilde{R}^\alpha_{xi} \mid \alpha \in \Omega_0 \,\&\, i = \mathrm{ln}(\alpha)\}$, hence the set R_{xi} is recursively enumerable.

We will now prove the equality $\psi(U, R_{xi}) = \psi(U, R_{xi+1})$.

$$R^1_{xi} = \cup\{R^\alpha_{xi} \mid \alpha\in\Omega_0 \ \&\ i+1 \leqslant \mathrm{ln}\ (\alpha)\};$$
$$R^2_{xi} = \cup\{\tilde{R}^\alpha_{xi} \mid \alpha\in\Omega_0 \ \&\ i = \mathrm{ln}\ (\alpha)\};$$
$$R^3_{xi} = \cup\{R^\alpha_{xi} \mid \alpha\in\Omega_0 \ \&\ i = \mathrm{ln}\ (\alpha)\};$$
$$R^1_{xi+1} = \cup\{P \mid \text{there exist } \alpha\in\Omega_0 \text{ and } y\in N \text{ such that } G^\alpha(y)! \ \&\ G^\alpha(y) = (A_0,\dots,A_j,\varphi_0,\dots,\varphi_j)\ \&$$
$$\&\ i+1\leqslant j \ \&\ P\in A_{i+1} \ \&\ x\in\varphi_{i+1}(P)\} \left(=\cup\{R^\alpha_{xi+1} \mid \alpha\in\Omega_0 \ \&\ i+1\leqslant \mathrm{ln}(\alpha)\}\right);$$
$$R^2_{xi+1} = \cup\{\tilde{R}^\alpha_{xi+1} \mid \alpha\in\Omega_0 \ \&\ i+1 = \mathrm{ln}\ (\alpha)\}.$$

It is obvious that $R_{xi} = V\cup R^1_{xi}\cup R^2_{xi}\cup R^3_{xi}$, $R_{xi+1} = V\cup R^1_{xi}\cup R^2_{xi}$. The recursive enumerability of the sets $V\cup R^1_{xi}$, $V\cup R^1_{xi+1}$ is proved in the same way as the recursive enumerability of $V\cup R^1_{xi}$ was proved in the first part. Obviously, $V\cup R^1_{xi}\subset V\cup R^1_{xi+1}$, hence, in view of 02), $\psi(U, V\cup R^1_{xi}) \leqslant \psi(U, V\cup R^{(1)}_{xi+1})$. Consider the partition $\mathbb{P}$ of the set $V\cup R^1_{xi+1}$: $\mathbb{P} = \{V\cap U\}\cup\{V\cap(N\setminus U)\}\cup$ (the set appearing after the symbol $\cup$ in the definition of R^1_{xi+1}), and the equivalence relation connected with $\mathbb{P}$ on $V\cup R^1_{xi+1}$: $a\sim b \longleftrightarrow (a,b)\in\cup\{P\times P \mid P\in\mathbb{P}\}$. It is obvious that for each $a\in V\cup R^1_{xi+1}$ there exists $b\in V\cup R^1_{xi}$ such that $a\sim b$ and for $a,b\in V\cup R^1_{xi+1}$: $a\sim b \rightarrow (a\in U \longleftrightarrow b\in U)$. Therefore, if we can prove the recursive enumerability of the equivalence $\sim$, then, in view of 04), we would have $\psi(U, V\cup R^1_{xi+1}) \leqslant \psi(U, V\cup R^1_{xi})$. Let $K_0 = \{\mathcal{L} \mid \mathcal{L}$ is a frame & $\mathrm{ln}\ (\mathcal{L})\leqslant i+1\}$, $K_1 = \{\mathcal{L}\in K_0 \mid \mathrm{ht}\ (\mathcal{L})\neq\infty\}$. Suppose s_0 is such that $s\geqslant s_0 \rightarrow [\mathrm{ht}(\mathcal{L},s) = \mathrm{ht}(\mathcal{L})$ for $\mathcal{L}\in K_1]$ & $[G^\alpha_s = G^\alpha$ for pairs α of the second kind and of length $\leqslant i+1]$. Consider the family of sets $\mathbb{Q}$: $\mathbb{Q} = \{V\cap U\}\cup\{V\cap(N\setminus U)\}\cup\{Q \mid$ there exist $s\geqslant s_0$, $\alpha\in\Omega$, $y\in N$ such that

1) $G^\alpha_s(y)!\ \&\ G^\alpha_s(y) = (A_0,\dots,A_j,\varphi_0,\dots,\varphi_j)\ \&\ j\geqslant i+1\ \&\ Q\in A_{i+1}\ \&\ x\in\varphi_{i+1}(Q)$,

2) $\mathrm{ht}(\mathcal{L},s) > \mathrm{ln}(G^\alpha_s(y))$ for all $\mathcal{L}\in K_0\setminus K_1\}$.

Obviously, the family $\mathbb{Q}$ is computable and $\mathbb{P}\subset\mathbb{Q}$. We will prove that for each $Q\in\mathbb{Q}$ there exists $P\in\mathbb{P}$ such that $Q\subset P$. If $Q = V\cap U$ or $Q = V\cap(N\setminus U)$, then this is so. Suppose that for certain $s\geqslant s_0$, $\alpha_0\in\Omega$, $y_0\in N$ the set s,α_0,y_0,Q satisfies the above conditions 1) and 2). Fix $a\in Q$ and denote Q by Q_0. If $a\notin V$, then arguing as in the first part, we obtain a sequence $\{(\alpha_t,y_t,Q_t)\}_{t\geqslant 0}$ converging to some triple (y,z,P), where $Q_0\subset Q_1\subset\dots$ hence $Q\subset P$, and for some $t\geqslant 0$ the set $s+t,\alpha_t,y_t,Q_t$ satisfies conditions 1) and 2), hence $P\in\mathbb{P}$. It remains to analyze the case $a\in V$.

Suppose $t\geqslant 0$ is such that a lies in the base of some tower to step $s+t$ but not to step $s+t+1$. Arguing as in the first part, we obtain a sequence (α_0,y_0,Q_0), (α_1,y_1,Q_1), $\dots,(\alpha_t,y_t,Q_t)$ such that $Q_0\subset Q_1\subset\dots\subset Q_t$ and the set $s+t$, α_t,y_t,Q_t satisfies conditions 1) and 2). If we now look at the description of the construction and take into account the choice of s_0 and condition 2), we see easily that either $Q\subset Q_t\subset V\cap U$ or $Q\subset Q_t\subset V\cap(N\setminus U)$. It follows from what has been proved that $\cup\{P\times P \mid P\in\mathbb{P}\} = \cup\{Q\times Q \mid Q\in\mathbb{Q}\}$; but the second set is recursively enumerable in view of the computability of the family $\mathbb{Q}$, hence the equivalence $\sim$ is recursively enumerable. Thus, the inequality $\psi(U, V\cup R^1_{xi+1})\leqslant$

$\psi(U, V\cup R^{1}_{xi})$, hence also the equality $\psi(U, V\cup R^{1}_{xi}) = \psi(U, V\cup R^{1}_{xi+1})$ is proved. In a completely analogous way we can prove that $\psi(U, R^{\alpha}_{xi+1}) = \psi(U, R^{\alpha}_{xi})$ for pairs $\alpha : \alpha\in\Omega_0$ & $i+1 \leq \ln(\alpha)$.

It is easy to see that the set $R^{3}_{xi} \setminus R^{2}_{xi}$ is finite, hence, the view of 02) and 03), $\psi(U, R^{3}_{xi}) \leq \psi(U, R^{2}_{xi})$. Thus, it remains to prove the inequalities

$$\psi(U, R^{2}_{xi}) \leq \psi(U, R_{xi+1}),\ \psi(U, R^{2}_{xi+1}) \leq \psi(U, R_{xi}).$$

Let $\Omega_1 = \{\alpha\in\Omega_0 \mid \alpha = (\mathfrak{A}, B)$ be a pair of the third kind & $\ln(\alpha) = i$ & $x\in B\}$, $\Omega_2 = \{\alpha\in\Omega_0 \mid \alpha = (\mathfrak{A}, B)$ be a pair of the third kind & $\ln(\alpha) = i+1$ & $x\in B\}$. It is obvious that $R^{2}_{xi} = \cup\{\widetilde{R}^{\alpha}_{xi} \mid \alpha\in\Omega_1\}$, $R^{2}_{xi+1} = \cup\{\widetilde{R}^{\alpha}_{xi+1} \mid \alpha\in\Omega_1\}$. Therefore, it suffices to prove that $\psi(U, \widetilde{R}^{\alpha}_{xi}) \leq \psi(U, R_{xi+1})$, $\alpha\in\Omega_1$ and $\psi(U, \widetilde{R}^{\alpha}_{xi+1}) \leq \psi(U, R_{xi})$, $\alpha\in\Omega_2$.

Suppose $\alpha\in\Omega_1$; $\alpha = (\mathfrak{A}, B)$, $\ln(\mathfrak{A}) = i$, $B\subset D_{i+1}$, $x\in B$, $B\cap H \neq \emptyset$; let $j = \sigma(B, i+1)$. It follows at once from the definitions that $\widetilde{R}^{\alpha}_{xi} = H^{\alpha}$ (the set H^{α} was introduced in the proof of Lemma 1), hence $\psi(U, \widetilde{R}^{\alpha}_{xi}) = \psi(U, H^{\alpha}) = d_m(B_j)$ ($\{B_e\}_{e\geq 0}$ is the sequence introduced earlier, and the computation of $\psi(U, H^{\alpha})$ is given in the proof of Lemma 1). Let $\widetilde{B} = \{y\in D_{i+2} \mid j \leq_{i+2} y\}$, $q = \sigma(\widetilde{B}, i+2)$. It is obvious that a) $j \sim_{i+2} q$, hence $d_m(B_j) = d_m(B_q)$; b) $x\in\widetilde{B}$; c) $\widetilde{B}\cap H \neq \emptyset$. Consider a pair of the third kind, $\beta = (\mathcal{L}, \widetilde{B})$, where $\mathcal{L}$ is the sequence $(\{\{0\}\}, \ldots, \{\{0\}\})$ of length $i+1$. As we have already noted, $\lim\limits_{s\to\infty} \rho(\mathcal{L}, s) \in N$, hence nr $(\beta)\in N$. Therefore, $\beta\in\Omega_2$, and in view of a),

$$\psi(U, \widetilde{R}^{\alpha}_{xi}) = d_m(B_j) = d_m(B_q) = \psi(U, \widetilde{R}^{\beta}_{x\,i+1}) \leq \psi(U, R_{xi+1}).$$

Suppose $\alpha\in\Omega_2$; $\alpha = (\mathfrak{A}, B)$, $B\cap H \neq \emptyset$, $\ln(\mathfrak{A}) = i+1$, $B\subset D_{i+2}$, $x\in B$; let $j = \sigma(B, i+2)$. We decompose the element $[j]_{i+2}$ of the distributive lattice $\widetilde{D}_{i+2}$ into atoms: $[j]_{i+2} = [j_1]_{i+2}\cup\ldots\cup[j_n]_{i+2}$. Obviously, $\psi(U, \widetilde{R}^{\alpha}_{xi+1}) = d_m(B_j) = \cup\{d_m(B_{j_e}) \mid 1\leq e\leq n\}$. Therefore, it suffices to prove that $d_m(B_{j_e}) \leq \psi(U, R_{xi})$. Denote j_e by q. Let $\{[K_1]_{i+1}, \ldots, [K_m]_{i+1}\}$ be the totality of minimal elements of the set $\{[y]_{i+1} \mid y\in\widetilde{D}_{i+1}$ & $q \leq_{i+2} y\}$. In view of D2), each $[K_s]_{i+1}$ is an atom of the distributive lattice $\widetilde{D}_{i+1}$, and since $q \leq_{i+2} x$, it follows that for some s_0 we have $K_{s_0} \leq_{i+1} x$. Denote K_{s_0} by w. Suppose $A = \{y\in D_{i+1} \mid w \leq_{i+1} y\}$. If $A\cap H \neq \emptyset$, consider the pair $\beta = (\mathcal{L}, A)$ of the third kind, where $\mathcal{L}$ is the sequence $(\{\{0\}\}, \ldots, \{\{0\}\})$ of length i. In view of our assumptions, $\beta\in\Omega_1$ and $\psi(U, \widetilde{R}^{\beta}_{xi}) = d_m(B_w) \geq d_m(B_q)$. It remains to analyze the case $A\cap H = \emptyset$. Suppose $\mathcal{L} = (\mathcal{L}_0, \ldots, \mathcal{L}_{i+1})$ is the uniquely determined good frame such that $\mathcal{L}_{i+1} = \{A\}$ and let $\widetilde{B} = \{y\in D_{i+2} \mid q \leq_{i+2} y\}$. Obviously, $x\in A\subset\widetilde{B}$. Consider the pair $\beta = (\mathcal{L}, \widetilde{B})$ of the third kind. The following chain of equalities is a consequence of the definitions and the first part of the proof of Lemma 2: $d_m(B_q) = \psi(U, \widetilde{R}^{\beta}_{xi+1}) = \psi(U, R^{\beta}_{xi+1}) = \psi(U, R^{\beta}_{xi}) \leq \psi(U, R_{xi})$. Thus, the proof of Lemma 2 is complete.

We will use the following notation up to the end of the proof of Theorem 1: if $x\in\widetilde{D}_i$, then

$$R^1_{xi} \rightleftharpoons (\cup\{R^{\alpha}_{xi} \mid \alpha\in\Omega_0 \,\&\, i \leq \ln(\alpha)\})\cup V,$$
$$R^2_{xi} \rightleftharpoons \cup\{\tilde{R}^{\alpha}_{xi} \mid \alpha\in\Omega_0 \,\&\, i = \ln(\alpha)\};$$

clearly, $R_{xi} = R^1_{xi} \cup R^2_{xi}$. We define a mapping $c:\mathcal{L}\to\mathcal{L}^e$ as follows: $c\mu(x) = \psi(U, R_{xx})$. Let us verify the correctness of the definition. Suppose $\mu(x)=\mu(y)$. Then, in view of L0), for some i we have $x,y\in D_i \,\&\, x\sim_i y$. By Lemma 2, $\psi(U,R_{xi}) = \psi(U,R_{xx})$, $\psi(U,R_{yi}) = \psi(U,R_{yy})$. Therefore, it suffices to prove that $\psi(U,R_{xi}) = \psi(U,R_{yi})$. Since $x\sim_i y$, it follows that $R^1_{xi} = R^1_{yi}$. Now suppose $\alpha = (\mathfrak{A},B)$ is a pair or third kind, $\ln(\mathfrak{A}) = i$, $\alpha\in\Omega_0$, $x\in B$; let $j=\sigma(B,i+1)$. Also, let $\tilde{B} = \{z\in D_{i+1} \mid j \leq_{i+1} z\}$, $q=\sigma(\tilde{B},i+1)$. Obviously, $j\sim_{i+1} q$, $y\in\tilde{B}$, $\tilde{B}\cap H\neq\emptyset$. Consider the pair $\beta = (\mathcal{C},\tilde{B})$ of the third kind, where $\mathcal{C}$ is the sequence $(\{\{0\}\},\ldots,\{\{0\}\})$ of length i; it is clear that β lies in Ω_0. It follows from all of the above that $\psi(U,\tilde{R}^{\alpha}_{xi}) = d_m(B_j) = d_m(B_q) = \psi(U,\tilde{R}^{\beta}_{yi})$. In view of the symmetry of the situation, the equality $\psi(U,R_{xi}) = \psi(U,R_{yi})$ is proved, hence also the correctness of the definition of the mapping c.

LEMMA 3. The mapping $c:\mathcal{L}\to\mathcal{L}^e$ is an upper semilattice homomorphism, and the diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, "c"] & \mathcal{L}^e \\ & \mathcal{L}^0 \arrow[ul, "a"] \arrow[u, "b"'] \end{tikzcd}$$

is commutative.

We must prove that for all x,y we have $c(\mu(x)\cup\mu(y)) = c\mu(x)\cup c\mu(y)$. Fix x,y; in view of L0) and L3), there exists i such that $x,y\in D_i$ and $\mu u(x,y,i) = \mu(x)\cup\mu(y)$; let $x\cup y \rightleftharpoons u(x,y,i)$. It follows immediately from the definition of an atom of a finite distributive lattice that $R^1_{x\cup y,i} = R^1_{xi}\cup R^1_{yi}$, so it suffices to prove that $\psi(U,R^2_{x\cup y,i}) = \psi(U,R^2_{xi})\cup\psi(U,R^2_{yi})$. We have $c(\mu(x)\cup\mu(y)) = \psi(U,R_{x\cup y,i}) = \psi(U,R^1_{x\cup y,i})\cup\psi(U,R^2_{x\cup y,i}) = \psi(U,R^1_{xi})\cup\psi(U,R^1_{yi})\cup\psi(U,R^2_{xi})\cup\psi(U,R^2_{yi}) = \psi(U,R_{xi})\cup\psi(U,R_{yi}) = c\mu(x)\cup c\mu(y)$.

Suppose $\alpha=(\mathfrak{A},B)$ is a pair of the third kind such that $\ln(\mathfrak{A}) = i$, $\alpha\in\Omega_0$, $x\cup y\in B$; let $j=\sigma(B,i+1)$. We decompose the element $[j]_{i+1}$ of the distributive lattice $\tilde{D}_{i+1}$ into atoms: $[j]_{i+1} = [j_1]_{i+1}\cup\ldots\cup[j_n]_{i+1}$. Obviously, $\psi(U,\tilde{R}^{\alpha}_{x\cup y,i}) = d_m(B_{j_1})\cup\ldots\cup d_m(B_{j_n})$. Suppose $1\leq \ell\leq n$. Since $j_\ell \leq_{i+1} x\cup y$ and $[j_\ell]_{i+1}$ is an atom of $\tilde{D}_{i+1}$, it follows that either $j_\ell\leq_{i+1} x$ or $j_\ell\leq_{i+1} y$. Suppose $\tilde{B} = \{z\in D_{i+1} \mid j_\ell\leq_{i+1} z\}$, and $\mathcal{C}$ is the sequence $(\{\{0\}\},\ldots,\{\{0\}\})$ of length i; if $\beta = (\mathcal{C},B)$ it is obvious that $\beta\in\Omega_0$. If $j_\ell\leq_{i+1} x$, then $\psi(U,\tilde{R}^{\beta}_{xi}) = d_m(B_{j_\ell})$, and if $j_\ell\leq_{i+1} y$, then $\psi(U,\tilde{R}^{\beta}_{yi}) = d_m(B_{j_\ell})$. Consequently, $\psi(U,\tilde{R}^{\alpha}_{x\cup y,i}) \leq \psi(U,R^2_{xi})\cup\psi(U,R^2_{yi})$, hence $\psi(U,R^2_{x\cup y,i}) \leq \psi(U,R^2_{xi})\cup\psi(U,R^2_{yi})$. The inequalities.

$$\psi(U,R^2_{xi}) \leq \psi(U,R^2_{x\cup y,i}),\ \psi(U,R^2_{yi}) \leq \psi(U,R^2_{x\cup y,i})$$

can be proved in a completely analogous fashion. Thus, the first part of the lemma is proved. We will now prove that $c\circ\bar{a} = b$.

Let $\bar{f}$ be the g.r.f. fixed earlier such that $\mu\bar{f}(x)=\bar{a}\nu(x)$ (recall that the H in condition (**) is $\bar{f}(N)$). It suffices to show that $c\bar{a}\nu(x)=b\nu(x)$ or, taking into account the equality $\bar{a}\nu(x)=\mu\bar{f}(x)$, that $c\mu\bar{f}(x)=b\nu(x)$. Fix x and denote $\bar{f}(x)$ by y ; suppose $y\in D_i$. Then $c\mu(y)=\psi(U,R_{yi})$. It is easy to see that $R^1_{yi}=\emptyset$ (the notation was introduced before the statement of Lemma 3), since our frames satisfy condition (**). We will now prove that $\psi(U,R^2_{yi})=b\nu(x)$. Suppose $\tilde{B}=\{z\in D_{i+1} \mid y\leqslant_{i+1} z\}$, $j=\sigma(\tilde{B},i+1)$, and $\mathcal{L}$ is the sequence $(\{\{0\}\},\dots,\{\{0\}\})$ of length i ; let $\beta=(\mathcal{L},\tilde{B})$. Obviously, $\beta\in\Omega_0$, $j\sim_{i+1} y$ and $\psi(U,R^{\beta}_{yi})=d_m(B_j)=b\nu(x)$ (see the definition of $\{B_e\}_{e\geqslant 0}$), hence $b\nu(x)\leqslant\psi(U,R^2_{yi})$. Suppose $\alpha=(\mathfrak{A},B)$ is a pair of the third kind such that ln $(\mathfrak{A})=i$, $y\in B$, $\alpha\in\Omega_0$, and let $j=\sigma(B,i+1)$. Then $\psi(U,\tilde{R}^{\alpha}_{yi})=d_m(B_j)\leqslant d_m(B_y)=b\nu(x)$. Therefore, $\psi(U,R^2_{yi})=b\nu(x)$ and the equality $c\circ\bar{a}=b$ is proved.

LEMMA 4. The mapping $c:\mathcal{L}\to\mathcal{L}^e$ is one-to-one.

We will first prove that $b\nu(x)\leqslant c\mu(y)\longleftrightarrow\bar{a}\nu(x)\leqslant\mu(y)$. The right to left implication holds by virtue of Lemma 3. Let us verify the left to right implication. We have $d_m(\Pi_{g(x)})=b\nu(x)\leqslant c\mu(y)=\psi(U,R_{yy})$. Therefore, by 03), there exists a g.r.f. f_e such that $f_e(N)\subset R_{yy}$ and $a\in\Pi_{g(x)}\leftrightarrow f_e(a)\in U$. Let $i=c(n,e)$. It follows from the definition of the indicator for natural numbers and our assumptions that $\lim_{s\to\infty}$ in $(i,s)=\infty$. Let $R=(\cup\{R^{\alpha}_{yy}\mid\alpha\in\Omega_0\,\&\, y\leqslant \ln(\alpha)\leqslant i\})\cup(\cup\{\tilde{R}^{\alpha}_{yy}\mid\alpha\in\Omega_0\,\&\,\ln(\alpha)=y\})$. We claim that $f_e(N\setminus\Pi_{g(x)})\subset R\cup V$. Assume the contrary and let a be the first element of the set $N\setminus\Pi_{g(x)}$ for which $b=f_e(a)$ does not lie in $R\cup V$. Since $b\notin V$, there exists a final tower $\mathbb{A}$ such that $b\in$ bs$(\mathbb{A})$ (bs$(\mathbb{A}\cap U=\emptyset)$; since $b\in R_{yy}\setminus R$, we have ln $(\mathbb{A})>i$. The following property of the construction is immediate; if a tower $\mathbb{B}$ exists to step t , a tower $\mathbb{C}$ exists to step $t+1$, and bs $(\mathbb{B})\cap$ bs$(\mathbb{C})\neq\emptyset$, then ln $(\mathbb{B})\geqslant$ ln$(\mathbb{C})$. Now suppose s is such that in$(i,s)=a$, in$(i,s+1)=a+1$. Let us see what must be done as step s of the construction. First of all, it is obvious that s is even, and at step s our procedure yields the number i and we have satisfied part a3) of the construction. Secondly (since in in$(i,s)\neq$ in$(i,s+1)$), $f_{es}(a)!$, $b=f_e(a)\notin U_s$ and there exists to step s a tower $\mathbb{B}$ such that $b\in$ bs$(\mathbb{B})$. This tower $\mathbb{B}$ must also possess the following properties ln$(\mathbb{B})>$ln$(\mathbb{A})>i$ and bs$(\mathbb{B})\cap U_s=\emptyset$. Consequently, at step s we must satisfy the second part of a3), from which follows the inequality ln $(\mathbb{A})\leqslant i$; but this contradicts our assumptions. Thus, the inclusion $f_e(N\setminus\Pi_{g(x)})\subset R\cup V$ is proved. This inclusion easily implies the inequality $d_m(\Pi_{g(x)})\leqslant\psi(U,R\cup V)=\psi(U,R)$. We will now compute $\psi(U,R)$. Suppose $\alpha\in\Omega_0$, $y\leqslant$ ln$(\alpha)=j\leqslant i$. If α is a pair of the first or second kind, then the set R^{α}_{yy} is finite and $\psi(U,R^{\alpha}_{yy})=0$, so suppose $\alpha=(\mathfrak{A},B)$ is a pair of the third kind, $\mathfrak{A}=(\mathfrak{A}_0,\dots,\mathfrak{A}_j)$, $\mathfrak{A}_j=\{A\}$. If $y\notin A$, then $R^{\alpha}_{yy}=\emptyset$. Suppose $y\in A\subset B$, $q\rightleftharpoons\sigma(B,j+1)$. We have $q\leqslant_{j+1} y$ (hence, $q\leqslant_{i+1} y$), $\mu(q)\in\bar{a}(\mathcal{L}^o)$ (since $B\cap H\neq\emptyset$), $\psi(U,R^{\alpha}_{yy})=d_m(B_q)$, $c\mu(q)=d_m(B_q)$ (the latter equality is proved by means of computations analogous to those of Lemma 3), and therefore $\psi(U,R^{\alpha}_{yy})=c\mu(q)$. In a similar way we can compute $\psi(U,\tilde{R}^{\alpha}_{yy})$ for $\alpha\in\Omega_0$ and ln $(\alpha)=y$. Finally, there exists $q\in D_{i+1}$ such that $q\leqslant_{i+1} y$, $\psi(U,R)=c\mu(q)$, and $\mu(q)\in\bar{a}(\mathcal{L}^o)$. We have $c\bar{a}\nu(x)=b\nu(x)\leqslant\psi(U,R)=c\mu(q)$ and $\mu(q)\leqslant\mu(y)$; but the restriction of c to $\bar{a}(\mathcal{L}^o)$ is an isomorphic embedding, hence $\bar{a}\nu(x)\leqslant\mu(q)$ and $\bar{a}\nu(x)\leqslant\mu(y)$, as required.

We will also need a property of distributive semilattices. The concept of distributive semilattice and the following lemma are due to Ershov [15] (in that paper he proved the equivalence of the concept of a distributive lattice and the concept of a semilattice satisfying the "closure condition," which had been introduced earlier by Lachlan [10]). A semilattice $\mathcal{L}=\langle\mathcal{L},\cup\rangle$ is called distributive if for $x,y,z\in\mathcal{L}$ it follows from $x\leqslant y\cup z$ that there exist $y_1\leqslant y, z_1\leqslant z$ such that $x=y_1\cup z_1$.

LEMMA (Ershov [15]). Suppose $\mathcal{L}=\langle\mathcal{L},\cup\rangle$ is a distributive semilattice and $A\subset\mathcal{L}$ is a (nonempty) ideal. Suppose $x\sim y\,(\mathrm{mod}\,A)\rightleftarrows$ (there exists $z\in A$ such that $x\cup z=y\cup z$), x/A is the class of the element x relative to the equivalence relation $x\sim y\,(\mathrm{mod}\,A)$, $\mathcal{L}/A=\{x/A\mid x\in\mathcal{L}\}$, $\hat{x}=\{y\in\mathcal{L}\mid y\leqslant x\}$, and $I(\mathcal{L})$ is the totality of ideals of $\mathcal{L}$. Then the mapping of $\mathcal{L}$ into $(\mathcal{L}/A)\times I(\mathcal{L})$ that sends x into $(x/A,\hat{x}\cap A)$ is multivalent.

We have the following easily verifiable implication: ($\mathcal{M}_1$ is a Lachlan semilattice) $\longrightarrow$ ($\mathcal{M}$ is a distributive semilattice). Therefore, our semilattice $\mathcal{L}$ is distributive. We now turn to the proof of Lemma 4.

Assume that $\mu(\bar{x})\neq\mu(\bar{y})$, but $c\mu(\bar{x})=c\mu(\bar{y})$. We will show that there exist $x,y\in N$ such that $\mu(x)\neq\mu(y)$, $c\mu(x)=c\mu(y)$ and $x\leqslant_\kappa y$, where $\kappa=\sup(x,y)$. Suppose $m=\sup(\bar{x},\bar{y})$, and denote $u(\bar{x},\bar{y},m)$ by y. Then $c\mu(\bar{x})=c\mu(\bar{y})=c\mu(y)$, and either $\mu(\bar{x})\neq\mu(y)$ or $\mu(\bar{y})\neq\mu(y)$. Suppose, for definiteness, that $\mu(\bar{x})\neq\mu(y)$; obviously, we then have $\bar{x}\neq 0$. Since the enumeration $\theta: N\xrightarrow{\text{onto}}\mathcal{L}'$ is a cylinder, we may assume that for all j,z such that $z\in\mathcal{D}_j$ & $z\neq 0$ the set $\{\tau\in\mathcal{D}_j\mid \tau\sim_j z\}$ contains at least $j+1$ elements. Suppose $j=\sup(\bar{x},y,m)$, and $x\in\mathcal{D}_j$ is such that $x\sim_j\bar{x}$ & $j\leqslant x$. It is clear that x,y satisfy our conditions. Fix a triple $x,y,\kappa\in N$ such that $\mu(x)\neq\mu(y)$, $c\mu(x)=c\mu(y)$, $\kappa=\sup(x,y)$, $x\leqslant_\kappa y$ We have $c\mu(x)=\psi(U,R_{x\kappa})$, $c\mu(y)=\psi(U,R_{y\kappa})$, hence, in view of 03), there exists a p.r.f. f_e such that the domain of f_e is equal to $R_{y\kappa}$, $f_e(R_{y\kappa})\subset R_{x\kappa}$, $z\in R_{y\kappa}\longrightarrow(z\in U\longleftrightarrow f_e(z)\in U)$. If $i=c(x,y,e)$, then $\kappa\leqslant i$ and $x\leqslant_i y$ (here c is the previously fixed g.r.f. effecting a one-to-one correspondence $N^3\longleftrightarrow N$). Let $[\kappa_1]_i,\ldots,[\kappa_f]_i$ be all atoms of the finite distributive lattice $\tilde{\mathcal{D}}_i$ lying under $[x]_i$ and let $[\kappa_1]_i,\ldots,[\kappa_w]_i$ $(f<w)$ be all atoms of $\tilde{\mathcal{D}}_i$ lying under $[y]_i$. We claim that there exists p, $f<p\leqslant w$ such that $\{z\in\mathcal{D}_i\mid\kappa_p\leqslant_i z\}\cap H=\emptyset$. Indeed, otherwise we would have $\mu(\kappa_{f+1}),\ldots,\mu(\kappa_w)\in\bar{a}(\mathcal{L}^0)$; if $z=\mu(\kappa_{f+1})\cup\ldots\cup\mu(\kappa_w)\in\bar{a}(\mathcal{L}^0)$, then $\mu(x)\cup z=\mu(y)\cup z$, i.e., $\mu(x)/\bar{a}(\mathcal{L}^0)=\mu(y)/\bar{a}(\mathcal{L}^0)$; on the other hand, from the first part of the proof of Lemma 4 we obtain the chain of equalities

$$\{\bar{a}(z)\mid z\in\mathcal{L}^0\ \&\ \bar{a}(z)\leqslant\mu(x)\}=\{\bar{a}(z)\mid z\in\mathcal{L}^0\ \&$$

$$\&\ b(z)\leqslant c\mu(x)\}=\{\bar{a}(z)\mid z\in\mathcal{L}^0\ \&\ b(z)\leqslant c\mu(y)\}=\{\bar{a}(z)\mid z\in\mathcal{L}^0\ \&\ \bar{a}(z)\leqslant\mu(y)\},$$

i.e., $\hat{\mu}(x)\cap\bar{a}(\mathcal{L}^0)=\hat{\mu}(y)\cap\bar{a}(\mathcal{L}^0)$; by Ershov's lemma, $\mu(x)=\mu(y)$, which contradicts our assumptions. Consequently, the desired p exists. If $A=\{z\in\mathcal{D}_i\mid\kappa_p\leqslant_i z\}$, then $y\in A$, $x\notin A$, $A\cap H=\emptyset$. Suppose $\mathcal{O}\!\mathcal{L}=(\mathcal{O}\!\mathcal{L}_0,\ldots,\mathcal{O}\!\mathcal{L}_i)(\mathcal{O}\!\mathcal{L}_i=\{A\})$ is the good frame determined by the atom A, and $\alpha=(\mathcal{O}\!\mathcal{L},I)$ is a pair of the first kind with first component equal to $\mathcal{O}\!\mathcal{L}$. It follows from our assumptions concerning x,y,e that $\lim_{s\to\infty}$ in $(\alpha,s)=\infty$, and from part a2) of the construction that $G^\alpha=\lim_{s\to\infty}G^\alpha_s$ has infinite domain, which contradicts Lemma 1.

LEMMA 5. The image $c(\mathcal{L})$ of the mapping $c: \mathcal{L} \to \mathcal{L}^e$ is an ideal of the semilattice $\mathcal{L}^e$.

We begin with two preliminary remarks. First, suppose $\mathfrak{A}$ is a frame, $i = \ln(\mathfrak{A})$, and for each $j \geq i$ there exist a final tower $\mathbb{A} = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K)$ and a subset $P \subset N$ such that $K \geq j$, $\operatorname{bs}(\mathbb{A}) \cap U = \emptyset$, $P \in \mathbb{A}_i$, $\mathfrak{A} = \operatorname{fr}(\mathbb{A}, i, P)$, and $P \cap \Pi_i \neq \emptyset$. Then $\operatorname{bs}(\mathfrak{A}) = \infty$. Secondly, suppose $\mathfrak{A}$ is a frame, $i = \ln(\mathfrak{A})$, $\operatorname{bs}(\mathfrak{A}) = \infty$, $\mathbb{A} = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K)$ is a final tower of length $\geq i$, $\operatorname{bs}(\mathbb{A}) \cap U = \emptyset$, and suppose $P \in \mathbb{A}_i$ and $\operatorname{fr}(\mathbb{A}, i, P) \preccurlyeq \mathfrak{A}$. Then $P \cap \Pi_i \neq \emptyset$. The proof of these two assertions is easy and is omitted.

In view of property 01) of the ψ-operator, it suffices to prove that for each i there exists x such that $c\mu(x) = \psi(U, \Pi_i)$. Fix i. Suppose $A_1, \ldots, A_e$ are all atoms of $\mathcal{D}_i$ that do not meet H $(A_1 \cap H = \ldots = A_e \cap H = \emptyset)$, $\mathfrak{A}^1, \ldots, \mathfrak{A}^e$ are the good frames determined by these atoms. Consider those atoms A_p, such that for each $j \geq i$ there exist a final tower $\mathbb{A} = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K)$ and a subset $P \subset N$ such that $K \geq j$, $P \in \mathbb{A}_i$, $\mathfrak{A}^p = \operatorname{fr}(\mathbb{A}, i, P)$, and $P \cap \Pi_i \neq \emptyset$. We may assume without loss of generality that $A_1, \ldots, A_w$ $(w \leq e)$ are precisely those atoms satisfying this condition. Suppose $K_1 = v(A_1, i), \ldots, K_w = v(A_w, i)$, and $x \in \mathcal{D}_i$ is such that $[x]_i = [K_1]_i \cup \ldots \cup [K_w]_i$. We have $\operatorname{ht}(\mathfrak{A}^1) = \ldots = \operatorname{ht}(\mathfrak{A}^w) = \infty$ (the "first remark"), and if $x \in A_q$, then there exists p, $1 \leq p \leq w$, such that $A_q \supset A_p$, hence $\mathfrak{A}^q \preccurlyeq \mathfrak{A}^p$; therefore,

(***) if the final tower $\mathbb{A} = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K)$ and subset $P \subset N$ satisfy the conditions $K \geq i$, $P \in \mathbb{A}_i$, $x \in \varphi_i(P)$, and $\operatorname{bs}(\mathbb{A}) \cap U = \emptyset$, then $P \cap \Pi_i \neq \emptyset$ (the "second remark").

On the other hand, there exists $j_0 \geq i$ such that if the final tower $\mathbb{A} = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K)$ and subset $P \subset N$ satisfy the conditions $K \geq j_0$, $P \in \mathbb{A}_i$, $\varphi_i(P) = A_q$, where $w < q \leq e$, $\operatorname{bs}(\mathbb{A}) \cap U = \emptyset$, then $P \cap \Pi_i = \emptyset$.

Suppose $R = \cup\{\operatorname{bs}(G^\alpha(y)) \mid \alpha \in \Omega_0 \ \&\ \ln(\alpha) \leq j_0 \ \&\ G^\alpha(y)!\}$. Then $\Pi_i \subset R^1_{xi} \cup R \cup U$ and $\psi(U, R) \in \mathcal{b}(\mathcal{L}^0)$ (the notation R^1_{xi}, R^2_{xi} was introduced before the statement of Lemma 3, and $R_{xi} = R^1_{xi} \cup R^2_{xi}$). We will prove that $\psi(U, R^2_{xi}) \leq \psi(U, R^1_{xi})$, hence $\psi(U, R_{xi}) = \psi(U, R^1_{xi})$, and also that $\psi(U, \Pi_i \cap R^1_{xi}) = \psi(U, R^1_{xi})$. Suppose $\alpha = (\mathfrak{A}, \mathcal{B})$ is a pair of the third kind, $i = \ln(\alpha)$, $x \in \mathcal{B}$, and suppose $j = v(\mathcal{B}, i+1)$, $a[j]_{i+1} = [j_1]_{i+1} \cup \ldots \cup [j_d]_{i+1}$ is a decomposition of the element $[j]_{i+1}$ of the finite distributive lattice $\widetilde{\mathcal{D}}_{i+1}$ into the atoms. We have

$$\psi(U, \widetilde{R}^\alpha_{xi}) = d_m(B_j) = d_m(B_{j_1}) \cup \ldots \cup d_m(B_{j_d}),$$

$$[K_1]_{i+1} \cup \ldots \cup [K_w]_{i+1} = [x]_{i+1} \geq [j_1]_{i+1} \cup \ldots \cup [j_d]_{i+1}.$$

Fix p, $1 \leq p \leq d$; since $[j_p]_{i+1}$ is an atom of $\widetilde{\mathcal{D}}_{i+1}$, it follows that for some q, $1 \leq q \leq w$, we have $j_p \leq_{i+1} K_q$. Let $\widetilde{\mathcal{B}} = \{z \in \mathcal{D}_{i+1} \mid j_p \leq_{i+1} z\}$, $\beta = (\mathfrak{A}^q, \widetilde{\mathcal{B}})$. Then $\beta \in \Omega_0$ and $\psi(U, R^\beta_{xi}) = d_m(B_{j_p})$, hence $\psi(U, \widetilde{R}^\alpha_{xi}) \leq \psi(U, R^1_{xi})$ and $\psi(U, R^2_{xi}) \leq \psi(U, R^1_{xi})$. Now consider the partition of the set R^1_{xi}: $\mathcal{P} = \{R^1_{xi} \cap U\} \cup \{V \cap (N \setminus U)\} \cup (\{P \mid$ there exist $\alpha \in \Omega_0$ and $y \in N$ such that $G^\alpha(y)! \ \&\ G^\alpha(y) = (\mathbb{A}_0, \ldots, \mathbb{A}_K, \varphi_0, \ldots, \varphi_K) \ \&\ K \geq i \ \&\ P \in \mathbb{A}_{i_1} \ \&\ x \in \varphi_i(P) \ \&$ $\operatorname{bs}(\mathbb{A}) \cap U = \emptyset\})$ and the equivalence relation connected with $\mathcal{P}$ on R^1_{xi}: $a \sim b \rightleftarrows (a, b) \in \cup\{P$

$\times P \mid P \in \mathcal{P}\}$. The recursive enumerability of the equivalence $\sim$ can be proved by the methods of Lemma 2; we also have $a \sim b \to (a \in U \leftrightarrow b \in U)$. In condition (***) it is actually asserted that for each $a \in R'_{xi}$ there exists $b \in (\Pi_i \cap R'_{xi}) \cup V$ such that $a \sim b$, hence, according to 04),

$$\psi(U, \Pi_i \cap R'_{xi}) = \psi(U, (\Pi_i \cap R'_{xi}) \cup V) = \psi(U, R'_{xi}).$$

since $\psi(U,R) \in \mathcal{b}(\mathcal{L}^{\circ})$, for some y we have $c\mu(y) = \psi(U, \Pi_i \cap R)$. We now have the chain of equalities

$$\psi(U, \Pi_i) = \psi(U, \Pi_i \cap R'_{xi}) \cup \psi(U, \Pi_i \cap R) =$$
$$= \psi(U, R'_{xi}) \cup c\mu(y) = \psi(U, R_{xi}) \cup c\mu(y) = c\mu(x) \cup c\mu(y) = c(\mu(x) \cup \mu(y)),$$

which proves Lemma 5.

<u>LEMMA 6.</u> There exists a general recursive function h such that $c\mu(x) = \pi h(x)$ for each $x \in N$.

Fix $x \in N$. Suppose $K_0 = \{\mathcal{L} \mid \mathcal{L}$ is a frame & $\ln(\mathcal{L}) \leqslant x\}$, $K_1 = \{\mathcal{L} \in K_0 \mid \mathrm{ht}(\mathcal{L}) \neq \infty\}$, $S(K_0)$ is the totality of subsets of K_0. For $A \in S(K_0)$ we introduce the notation $\widehat{\mathrm{ht}}(A, s) = \inf\{\mathrm{ht}(\mathcal{L}, s) \mid \mathcal{L} \in A\}$, $\check{\mathrm{ht}}(A, s) = \sup\{\mathrm{ht}(\mathcal{L}, s) \mid \mathcal{L} \in A\}$; and $\widehat{\mathrm{ht}}(A) = \lim\limits_{s \to \infty} \widehat{\mathrm{ht}}(A, s)$, $\check{\mathrm{ht}}(A) = \lim\limits_{s \to \infty} \check{\mathrm{ht}}(A, s)$. Let $A_1, \ldots, A_K$ be some linear ordering of $S(K_0)$. For i, $1 \leqslant i \leqslant K$, put $R^i_x = V \cup (\cup\{P \mid$ there exist $s < \widehat{\mathrm{ht}}(K_0 \setminus A_i)$, $\alpha \in \Omega$, $y \in N$ such that $G^{\alpha}_s(y)!\ \&\ G^{\alpha}_s(y) = (A_0, \ldots, A_j, \varphi_0, \ldots, \varphi_j)\ \&\ [x \leqslant j < \widehat{\mathrm{ht}}(K_0 \setminus A_i, s)\ \&$ $P \in A_x\ \&\ x \in \varphi_x(P) \vee \alpha = (\mathcal{A}, \mathcal{B})$ is a pair of the third kind $\&\ j = x\ \&\ P = \mathrm{bs}(G^{\alpha}_s(y))\ \&$ $x \in \mathcal{B}]\})$, $U^i_x = \{y \in R^i_x \mid y < \check{\mathrm{ht}}(A_i)\} \cup U$; also, put

$$R_x = (\ldots((R^1_x \oplus R^2_x) \oplus R^3_x) \oplus \ldots) \oplus R^K_x,$$
$$U_x = (\ldots((U^1_x \oplus U^2_x) \oplus U^3_x) \oplus \ldots) \oplus U^K_x,$$

where, as usual, $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$; obviously, $\psi(U_x, R_x) = \cup\{\psi(U^i_x, R^i_x) \mid 1 \leqslant i \leqslant K\}$. We will prove that $\psi(U, R_{xx}) = \psi(U_x, R_x)$. Suppose $A_i \neq K_1$. Then either $\widehat{\mathrm{ht}}(K_0 \setminus A_i) \neq \infty$ or $\check{\mathrm{ht}}(A_i) = \infty$. If $\check{\mathrm{ht}}(A_i) = \infty$, then obviously $R^i_x \subset U^i_x$, hence $\psi(U^i_x, R^i_x) = 0$; if $\check{\mathrm{ht}}(A_i) \neq \infty$, then $\widehat{\mathrm{ht}}(K_0 \setminus A_i) = \infty$, and the sets $R^i_x \setminus V$, $U^i_x \setminus U$ are finite, which implies that $\psi(U^i_x, R^i_x) = 0$. Now suppose $A_i = K_1$. Then $\widehat{\mathrm{ht}}(K_0 \setminus A_i) = \infty$, $\check{\mathrm{ht}}(A_i) \neq \infty$, hence $R_{xx} \subset R^i_x$ and the set $U^i_x \setminus U$ is finite. It is easy to see that the set $R^i_x \setminus R_{xx}$ is also finite. Consequently, $\psi(U^i_x, R^i_x) = \psi(U, R_{xx})$, hence $\psi(U_x, R_x) = \psi(U, R_{xx}) = c\mu(x)$. In view of the uniform effectiveness of the construction and the fact that the enumeration $\{\Pi_i\}_{i \geqslant 0}$ is principal, there exists a g.r.f. h such that for each $x \in N$ we have $c\mu(x) = \psi(U_x, R_x) = d_m(\Pi_{h(x)})$.

Thus, Theorem 1 is proved for the enumerated semilattice $\mathcal{L}^e_{\pi}$. Note that we have proved more than was required. Indeed, let $\bar{c}$ be the composite mapping $\mathcal{L}'_{\theta} \subset \mathcal{L}_{\mu} \xrightarrow{c} \mathcal{L}^e_{\pi}$. Then $\bar{c} \in K$, $\bar{c} \cdot a = b$, and $I \notin \bar{c}(\mathcal{L}')$. We will use Theorem 1 in this strengthened form. Let us now indicate the changes that must be made in the proof of Theorem 1 for the semilattices

${}_a\mathcal{L}_\xi$, $\mathcal{L}(S_n)_\xi$. The changes for ${}_a\mathcal{L}_\xi$: in the definition of the indicator for natural numbers we must consider $\Pi_{g(n)} \oplus A$, where $a = d_m(A)$, and in the proof of Lemma 5 we must assume that $a \leq \psi(U, \Pi_i)$. The changes for $\mathcal{L}(S_n)_\xi$ are as follows. First note that the set of computable enumerations of S_n is in a natural one-to-one correspondence with the set of sequences $(U_1, \ldots, U_n)$ of pairwise disjoint, recursively enumerable sets such that $U_i \neq \emptyset$ and $N \setminus (U_1 \cup \ldots \cup U_n) \neq \emptyset$, namely,

$$f \longmapsto (f^{-1}(\{1\}), \ldots, f^{-1}(\{n\}));$$

instead of U we must construct the sequence $(U_1, \ldots, U_n)$. Before step 0 we regard the numbers $0, 1, \ldots, n$ as used, and transfer 1 into $U_1, \ldots$, and n into U_n. Instead of the creative set $\mathcal{M}$ we must use a sequence $(\mathcal{M}_1, \ldots, \mathcal{M}_n)$ such that the corresponding computable enumeration $f\colon N \xrightarrow{\text{onto}} S_n$ lies in the largest element of $\mathcal{L}(S_n)$, $d_m(f) = I$; the other changes are obvious. We give only the definition of the ψ-operator for $\mathcal{L}(S_n)$. Suppose $f\colon N \xrightarrow{\text{onto}} S_n$ is a computable enumeration and $A \subset N$ is a recursively enumerable set. If $A = \emptyset$, then $\psi(f, A) = 0$. Suppose $A \neq \emptyset$ and g is a general recursive function such that $g(N) = A$. Put $\bar{g}(0) = \emptyset$, $\bar{g}(i) = \{i\}$, $1 \leq i \leq n$, $\bar{g}(x+n+1) = fg(x)$ and $\psi(f, A) = d_m(\bar{g})$.

The proof of Theorem 1' is analogous to that of Theorem 1, but in the definition of the indicator for natural numbers we must take as g a g.r.f. representing the morphism $c\colon \mathcal{Y} \longrightarrow \mathcal{L}_\varphi$.

We will now prove Theorem 2. Again, in order to avoid cumbersome notation that obscures the essence of the matter we analyze only the case $\mathcal{L}_\varphi = \mathcal{L}_\pi^e$. The changes for ${}_a\mathcal{L}_\xi$, $\mathcal{L}(S_n)_\xi$ will be given later.

<u>THEOREM 2.</u> Suppose $a\colon \mathcal{Y} \longrightarrow \mathcal{L}_\pi^e$ is a morphism of enumerated sets such that $I \notin a(\mathcal{Y})$. Then there exist an L-semilattice $\mathcal{L}_\theta^1$, a morphism of enumerated sets $b\colon \mathcal{Y} \longrightarrow \mathcal{L}_\theta^1$, and a K-morphism $c\colon \mathcal{L}_\theta^1 \longrightarrow \mathcal{L}_\pi^e$ such that $a = c \circ b$ and $I \notin c(\mathcal{L}^1)$.

<u>Proof.</u> Let $4 = \{f \mid f$ is a p.r.f. & $\forall x, y \in N\ (x \leq y \;\&\; f(y)! \longrightarrow f(x)!)\}$ and suppose $\{\tilde{f}_i\}_{i \geq 0}$ is a principal enumeration of 4; let g be a general recursive function representing the morphism a. Put $A_0 = \Pi_{g(0)}$, $A_{i+1} = A_i \oplus \Pi_{g(i+1)}$; $B_n = \tilde{f}_i^{-1}(A_j)$, where $n = c(i,j)$. Clearly, $\{B_n\}_{n \geq 0}$ is a computable sequence of r.e. sets and $A = \{d_m(B_n) \mid n \geq 0\}$ is the smallest ideal of $\mathcal{L}^e$ containing $a(\mathcal{Y})$. Since the largest element of $\mathcal{L}^e$ is indecomposable, $I \notin A$. We equip the semilattice A with the enumeration $\nu\colon \nu(i) = d_m(B_i)$. In view of the computability of $\{B_i\}_{i \geq 0}$, the natural embedding $A_\nu \subset \mathcal{L}_\pi^e$ is a K-morphism and, since $\{\tilde{f}_i\}_{i \geq 0}$ is principal $a\colon \mathcal{Y} \longrightarrow A_\nu$ is a morphism of enumerated sets. By a theorem of Lachlan [12], $\mathcal{L}^e$, equipped with the enumeration μ, $\mu(i) = \psi(\mathcal{M}, \Pi_i)$, where $\mathcal{M}$ is a creative set, is an L-semilattice. But the enumeration μ is equivalent to the enumeration π and, since π is complete, is isomorphic to it, i.e., for some recursive permutation p we have $\pi = \mu \circ p$ (see [2, p. 201]). Thus, $\mathcal{L}_\pi^e$ is an L-semilattice. By Theorem 1, there exists a K-morphism $c\colon \mathcal{L}_\pi^e \longrightarrow \mathcal{L}_\pi^e$ such that 1) $I \notin c(\mathcal{L}^e)$ and 2) the composite mapping $A_\nu \subset \mathcal{L}_\pi^e \xrightarrow{c} \mathcal{L}_\pi^e$ is an embedding $A_\nu \subset \mathcal{L}_\pi^e$. Taking $\mathcal{L}_\pi^e$ in the role of $\mathcal{L}_\theta'$ and the composite mapping $\mathcal{Y} \xrightarrow{a} A_\nu \subset \mathcal{L}_\pi^e$ in the role of b, we obtain everything we need.

Remarks for ${}_a\mathcal{L}_\zeta$, $\mathcal{L}(S_n)_\xi$: the indecomposability of the largest element of $\mathcal{L}(S_n)$ follows from the theorem of Ershov [9] on the indecomposability of precomplete enumerations representing the largest element of $\mathcal{L}(S_n)$ (see [2, p. 210]); that ${}_a\mathcal{L}_\zeta$,$\mathcal{L}(S_n)_\xi$ are L-semilattice was proved in [14].

3. Some Corollaries

We now deduce several corollaries of our theorems.

COROLLARY 1. The Ershov—Lavrov Theorem [13] (see p. 4).

We first prove an auxiliary assertion. Suppose $\mathcal{L}_\nu$ is an enumerated semilattice and the semilattice $\bar{\mathcal{L}}$ is obtained from $\mathcal{L}$ by extremely adjoining a largest element. Assume there exists a K-morphism $\alpha : \mathcal{L}_\nu \longrightarrow \mathcal{L}^o_\mu$ of the enumerated semilattice $\mathcal{L}_\nu$ into the L-semilattice $\mathcal{L}^o_\mu$. We claim that there then exists an enumeration $\theta : N \xrightarrow{\text{onto}} \bar{\mathcal{L}}$ of the semilattice $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}_\theta$ is an L-semilattice and the natural embedding $\mathcal{L}_\nu \subset \bar{\mathcal{L}}_\theta$ is a K-morphism. Suppose f is a general recursive function representing the morphism α , i.e., $\forall x \in N\ (\alpha\nu(x) = \mu f(x))$, and suppose $\langle D_0, \leqslant_0 \rangle \subset \langle D_1, \leqslant_1 \rangle \subset \ldots$ is a sequence of preordered sets satisfying conditions L1)-L5) in the definition of an L-semilattice and such that $\mu(x) \leqslant \mu(y) \longleftrightarrow \exists i \in N\ (x \leqslant_i y)$ and $\{f(0), \ldots, f(i)\} \subset D_i$. Finally, let $A_i = \{f(0), \ldots, f(i)\}$. $g(i) = u(A_i, i)$, $\bar{D}_i = \{v(x, g(i), i) \mid x \in D_i\}$, $\tilde{D}_i = \{x \mid x = 0 \vee 1 \leqslant x \ \&\ (x-1) \in D_j \ \&\ j \leqslant i\}$, where u, v are the general recursive functions in L4). We introduce preorders on $\tilde{D}_i : x \tilde{\leqslant}_i 0, \neg(0 \tilde{\leqslant}_i (x+1))$ and $(x+1) \tilde{\leqslant}_i (y+1) \longleftrightarrow x \leqslant_i y$. We also define general recursive functions $\tilde{u}, \tilde{v} : \tilde{u}(x, 0, i) = \tilde{u}(0, y, i) = 0,\ \tilde{u}(x+1, y+1, i) = v(u(x, y, i), g(i), i) + 1\ \tilde{v}(x, 0, i) = x ,\ \tilde{v}(0, y, i) = y,\ \tilde{v}(x+1, y+1, i) = v(v(x, y, i), g(i), i) + 1$. It is easy to see that the sequence $\langle \tilde{D}_0, \tilde{\leqslant}_0 \rangle \subset \langle \tilde{D}_1, \tilde{\leqslant}_1 \rangle \subset \ldots$ and the g.r.f. $\tilde{u}, \tilde{v}$ satisfy L1)-L5). Let $A = \cup \{\tilde{D}_i \mid i \geqslant 0\}$; we introduce an enumeration $\bar{\theta}$ of the semilattice $\bar{\mathcal{L}}$: the domain of $\bar{\theta}$ is A and $\bar{\theta}(0) = I_{\bar{\mathcal{L}}}$, $\bar{\theta}(x+1) = \mu(x)$. It follows from the above that the enumerated semilattice $\bar{\mathcal{L}}_{\bar{\theta}}$ is an L-semilattice (except that the domain of $\bar{\theta}$ is the recursively enumerable set A , and not all of N) and the g.r.f. $i \longmapsto v(f(i), g(i), i)$ represents the natural embedding $\mathcal{L}_\nu \subset \bar{\mathcal{L}}_{\bar{\theta}}$. Passage from $\bar{\theta}$ to an enumeration θ with domain N is obvious. We now begin the proof proper of the Ershov—Lavrov theorem. Suppose $A \subset \mathcal{L}^e$, $A \neq \emptyset$ is a computable ideal, and $B \subset \mathcal{L}^e$ is a computable family of m-degree such that $A \cap B = \emptyset$, $I \notin A \cup B$. Since A and B are computable, there exist enumerations $\nu : N \xrightarrow{\text{onto}} A$,$\zeta : N \xrightarrow{\text{onto}} A \cup B$ such that the natural embedding $A_\nu \subset (A \cup B)_\zeta$, $(A \cup B)_\zeta \subset \mathcal{L}^e_\pi$ are morphism of enumerated sets. Suppose the semilattice $\bar{\mathcal{L}}$ is obtained from the semilattice A by externally adjoining a largest element, and θ is an enumeration of $\bar{\mathcal{L}}$ for which $\bar{\mathcal{L}}_{\bar{\theta}}$ is an L-semilattice and the natural embedding $A_\nu \subset \bar{\mathcal{L}}_\theta$ is a K-morphism. Let C be the smallest ideal of $\mathcal{L}^e$ containing $A \cup B$. Then $I \notin C$ and there exists an enumeration $\mu : N \xrightarrow{\text{onto}} C$ for which the natural embedding $(A \cup B)_\zeta \subset C_\mu$, $C_\mu \subset \mathcal{L}^e_\pi$ are morphisms of enumerated sets. We collect the objects and morphisms in a single diagram:

$$\begin{tikzcd}
\bar{\mathcal{L}}_\theta \arrow[r, dashed, "e"] & \mathcal{L}^e_\pi \\
A_\nu \arrow[u, "u"] \arrow[ur, "v"] \arrow[r, "q"'] & C_\mu \arrow[u, "p"']
\end{tikzcd}$$

where u, v, p, q are natural embeddings. By Theorem 1', there exists $e \in K$ making the diagram commutative and such that $e(\overline{\mathcal{L}}) \cap C = A$. By considering $e(I_{\overline{\mathcal{L}}})$ we obtain everything we need.

COROLLARY 2. V'yugin's Theorem (see [14]).

Suppose $a \in \mathcal{L}^e$, $a \neq I$; and $\mathcal{L}_\mu$ is an L-semilattice. By a theorem of Lachlan [12], there exists an enumeration $\theta : N \xrightarrow{\text{onto}} \mathcal{L}_a$ turning $\mathcal{L}_a$ into an L-semilattice $(\mathcal{L}_a)_\theta$ and such that the natural embedding $(\mathcal{L}_a)_\theta \subset \mathcal{L}^e_\pi$ is a K-morphism. Assuming that the sets $\mathcal{L}_a$ and $\mathcal{L}$ are disjoint, we define an order $\leq$ on the set $\overline{\mathcal{L}} = \mathcal{L}_a \cup \mathcal{L}$ as follows: each element of $\mathcal{L}$ is larger than any element of $\mathcal{L}_a$, $x \in \mathcal{L}_a \,\&\, y \in \mathcal{L} \rightarrow x \leq y$, the restriction of $\leq$ to $\mathcal{L}_a$ is the original order on $\mathcal{L}_a$, and the restriction of $\leq$ to $\mathcal{L}$ is the original order on $\mathcal{L}$. We also define an enumeration $\overline{\mathcal{L}}$: $\nu(2x) = \theta(x)$, $\nu(2x+1) = \mu(x)$. Obviously, $\overline{\mathcal{L}}_\nu$ is an L-semilattice and the natural embedding $(\mathcal{L}_a)_\theta \subset \overline{\mathcal{L}}_\nu$ is a K-morphism. By Theorem 1, there exists $c \in K$ making the diagram

$$\begin{tikzcd}
\overline{\mathcal{L}}_\nu \arrow[r, dashed, "c"] & \mathcal{L}^e_\pi \\
(\mathcal{L}_a)_\theta \arrow[u, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

commutative, where p, q are natural embeddings. By considering $c(I_{\overline{\mathcal{L}}})$, we obtain everything we need.

COROLLARY 3. We have the isomorphisms $\mathcal{L}^e \cong_a \mathcal{L} \cong \mathcal{L}(S_n)$.

Proof. Suppose $\mathcal{M}_\zeta$ is an enumerated semilattice. The expression "$\mathcal{M}_\zeta$ satisfies Theorem 1 (Theorem 2)" has the following meaning: "the theorem obtained by replacing $\mathcal{L}_\varphi$ by $\mathcal{M}_\zeta$" in the statement of Theorem 1 (Theorem 2) is valid." Suppose $\mathcal{L}^1_\nu, \mathcal{L}^2_\mu$ are nontrivial (i.e., $\mathcal{L}^1$, $\mathcal{L}^2$ are not singletons) enumerated semilattices with largest and smallest elements satisfying Theorems 1 and 2. We will prove that $\mathcal{L}^1 \cong \mathcal{L}^2$. In order to avoid multilevel notation, some enumerated semilattices will be denoted by Gothic letters (with indices) without property distinguishing the semilattice and the enumeration. Let $a_0, a_1, \ldots$ be an enumeration, possibly with repetitions, of all elements of $\mathcal{L}^1$ different from $I_{\mathcal{L}^1}$, and let $b_0, b_1, \ldots$ be an enumeration, possibly with repetitions, of all elements of $\mathcal{L}^2$ different from $I_{\mathcal{L}^2}$. We will construct a sequence of L-semilattices $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ and K-morphisms $f_i : \mathfrak{A}_i \rightarrow \mathfrak{A}_{i+1}$, $g_i : \mathfrak{A}_i \rightarrow \mathcal{L}^1_\nu$, $h_i : \mathfrak{A}_i \longrightarrow \mathcal{L}^2_\mu$ such that $g_i = g_{i+1} \circ f_i$, $h_i = h_{i+1} \circ f_i$, $I \notin g_i(\mathfrak{A}_i)$, $I \notin h_i(\mathfrak{A}_i)$, $a_\kappa \in g_{2\kappa+1}(\mathfrak{A}_{2\kappa+1})$, $b_\kappa \in h_{2(\kappa+1)}(\mathfrak{A}_{2(\kappa+1)})$. Suppose $\mathfrak{A}_0$ is a one-element enumerated semilattice and g_0, h_0 are the uniquely defined K-morphisms $g_0 : \mathfrak{A}_0 \rightarrow \mathcal{L}^1_\nu$, $h_0 : \mathfrak{A}_0 \longrightarrow \mathcal{L}^2_\mu$. Assume that to step $n = 2\kappa$ we have constructed $\mathfrak{A}_i$, g_i, h_i , $i \leq n$, and f_j , $j < n$, satisfying the induction assumption. Suppose $\mathfrak{A}_n$ is $\mathcal{L}_\varphi$. Let $\mathcal{M} = g_n(\mathcal{L}) \cup \{a_\kappa\}$, $\zeta(0) = a_\kappa$, $\zeta(x+1) = g_n \zeta(x)$. Consider the enumerated set $\gamma_1 = \langle \mathcal{M}, \zeta : N \xrightarrow{\text{onto}} \mathcal{M} \rangle$. Obviously, the natural embedding $\gamma \subset \mathcal{L}_\nu$ is a morphism of enumerated sets and $I \notin \mathcal{M}$. By Theorem 2, there exists an L-semilattice $\mathfrak{A}_{n+1}$, a morphism of enumerated sets $a : \gamma \rightarrow \mathfrak{A}_{n+1}$ and a K-morphism $g_{n+1} : \mathfrak{A}_{n+1} \rightarrow \mathcal{L}^1_\nu$ such that $g_{n+1} \circ a$ is an embedding $\mathcal{M} \subset \mathcal{L}^1$ and $I \notin g_{n+1}$

$(\mathfrak{A}_{n+1})$. Let f_n be the composite mapping $\mathfrak{A}_n \xrightarrow{g_n} \mathcal{Y} \xrightarrow{a} \mathfrak{A}_{n+1}$. It is easy to see that f_n is in fact a K-morphism. Applying Theorem 1, we obtain a K-morphism $h_{n+1}: \mathfrak{A}_{n+1} \to \mathcal{L}^2_\mu$ such that $I \notin h_{n+1}(\mathfrak{A}_{n+1})$ and $h_n = h_{n+1} \circ f_n$. At an odd step $n=2K+1$ we proceed analogously and include b_K in the image of h_{n+1}. We now define $e: \mathcal{L}^1 \to \mathcal{L}^2$. Suppose $x \in \mathcal{L}^1$; if $x = I_{\mathcal{L}^1}$, then $e(x) = I_{\mathcal{L}^2}$, but if $x = a_K$, then $e(x) = h_{2K+1}(g^{-1}_{2K+1}(x))$. In view of our construction, e is an isomorphic embedding of the semilattice $\mathcal{L}^1$ onto the semilattice $\mathcal{L}^2$. Thus, Corollary 3 is proved.

LITERATURE CITED

1. Yu. L. Ershov, Theory of Enumerations [in Russian], Nauka, Moscow (1977).
2. A. I. Mal'tsev, Algorithms and Recursive Functions [in Russian], Nauka, Moscow (1965).
3. H. Rogers Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill (1967).
4. E. L. Post, "Recursively enumerable sets of positive integers and their decision problems, Bull. Am. Math. Soc., 50, 284-316 (1944).
5. I. Myhill, "Creative sets," Z. Math. Logic Grundl. Math., 1, No. 2, 97-108 (1955).
6. A. H. Lachlan, "A note on universal sets," J. Symb. Logic, 31, 573-574 (1966).
7. Yu. L. Ershov, "Hyperhypersimple m-degrees," Algebra Logika, 8, No. 5, 523-552 (1969).
8. S. D. Denisov, "On m-degrees of recursively enumerable sets," Algebra Logika, 9, No. 4, 422-427 (1970).
9. Yu. L. Ershov, "Inseparable pairs," Algebra Logika, 9, No. 6, 661-666 (1970).
10. A. H. Lachlan, "Initial segments of many-one degrees," Can. J. Math., 22, No. 1, 75-85 (1970).
11. A. H. Lachlan, "Two theorems on many-one degrees of recursively enumerable sets," Algebra Logika, 11, No. 2, 216-229 (1972).
12. A. H. Lachlan, "Recursively enumerable many-one degrees," Algebra Logika 11, No. 3, 326-358 (1972).
13. Yu. L. Ershov and I. A. Lavrov, "The upper semilattice $\mathcal{L}(\mathcal{Y})$," Algebra Logika, 12, No. 2, 167-189 (1973).
14. V. V. V'yugin, "Segments of recursively enumerable m-degrees," Algebra Logika, 13, No. 6, 635-654 (1974).
15. Yu. L. Ershov, "The upper semilattice of enumerations of a finite set," Algebra Logika, 14, No. 3, 258-284 (1975).
16. E. A. Palyutin, "Addendum to Yu. L. Ershov's article 'The upper semilattice of enumerations of a finite set'," Algebra Logika, 14, No. 3, 284-287 (1975).