SUBRINGS OF FINITELY PRESENTED ASSOCIATIVE RINGS

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In the Dnestr Notebook [5] L. A. Bokut' raised the following question: "Is an arbitrary recursively presented associative (Lie) algebra over a prime field embeddable in a finitely presented associative (Lie) algebra?" We will show that for associative rings and algebras this question has an affirmative answer.

Suppose that K is either a commutative associative finitely generated ring with unity or a finitely generated extension of a prime field. We will show that any associative K-algebra with a recursively enumerable set of defining relations can be embedded in a finitely presented associative K-algebra.

In this paper the expression "K-algebra" means "associative K-operator ring." The unity of an algebra, if there is one, is not fixed in the signature. We denote the set of all positive integers by N. For a K-algebra A and elements $a_1,\dots,a_n \in A$, we denote by $A[a_1,\dots,a_n]$ the subalgebra of A generated by $a_1,\dots,a_n$. For K-algebra A and B, the direct sum of these K-algebras is denoted by $A \times B$, and $A \subseteq B$ signifies that A is a subalgebra of B. For a set of symbols X, we denote by $L[X]$ the set of all nonempty associative words in the alphabet X, and by $K[X]$ the free K-algebra with set X of free generators. The elements of $K[X]$ are linear combinations of the form $\sum_{w \in L[X]} \alpha_w w$, where the $\alpha_w \in K$ and almost all are equal to zero.

LEMMA 1. Suppose A is an arbitrary K-algebra $a_1,\dots,a_n, b_1,\dots,b_n \in K$, and $\varphi: A \to A$ is a mapping such that $\varphi(a_i) = b_i$ $(i=1,\dots,n)$, and the following conditions are satisfied:

1) φ is an endomorphism of A as a K-module;

2) the restriction of φ to the subalgebra $A[a_1,\dots,a_n]$ is a homomorphism into A.

Then in some K-algebra containing A as a subalgebra the following system of equations in the unknowns $x, y, z, \beta_1,\dots,\beta_n$ is solvable:

$$
\begin{aligned}
x a_i z y &= b_i, \\
a_i z &= z \beta_i, \\
x z \beta_i \beta_j &= b_i x z \beta_j \qquad (i, j = 1, \dots, n).
\end{aligned}
$$

Proof. Suppose $A = K[X]/I$, where I is an ideal of the free algebra $K[X]$. It is convenient to assume that preimages of the elements a_i, b_i $(i=1,\dots,n)$ under the canonical homomorphism $K[X] \to A$ are chosen to be distinct letters of X, which we also denote by a_i, b_i $(i=1,\dots,n)$. Furthermore, we assume that for each word $w \in L[X]$ there is chosen in $L[X]$ a word w' such that the equality $\varphi(w+I) = w'+I$ holds in $K[X]/I$. We again denote

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this w' by $\varphi(w)$. Finally, in view of condition 2) of the lemma we may assume that we always have

$$\varphi(a_{i_1}a_{i_2}\dots a_{i_k}) = b_{i_1}b_{i_2}\dots b_{i_k}.$$

Suppose $X_1 = X \cup \{x, y, z, \beta_1, \dots, \beta_n\}$ and I_1 is the ideal of the free K-algebra $K[X_1]$, generated by the set

$$\begin{gathered} I \cup \{xwz\beta y - \varphi[w\beta(\bar{a})] / w \in L[X] \text{ or } w = \emptyset, \\ \beta \in L[\beta_1, \dots, \beta_n] \text{ or } \beta = \emptyset,\ w\beta \neq \emptyset\} \cup \\ \cup \{a_i z - z\beta_i \,/\, i = 1, \dots, n\} \cup \\ \cup \{xz\beta_i\beta_j - b_i xz\beta_j \,/\, i, j = 1, \dots, n\}. \end{gathered}$$

Here and below, $\beta(\bar{a})$ denotes the result of replacing each other β in the word β_i by a_i $(i = 1, \dots, n)$. To prove the lemma it suffices to show that $I_1 \cap K[X] = I$.

Suppose $W \in I_1 \cap K[X]$. We will prove that $W \in I$. The element W can be written as a linear combination, with coefficients in K , of elements of the form

$$\begin{array}{cc} v_1 U v_2, & w_1(xwz\beta y - \varphi[w\beta(\bar{a})])w_2, \\ w_1(a_i z - z\beta_i)w_2, & w_1(xz\beta_i\beta_j - b_i xz b_j)w_2, \end{array} \tag{1}$$

where $U \in I$. v_1, v_2, w_1, w_2 are certain words in $L[X_1]$ or are empty words. We may assume that v_1 does not end, and that v_2 does not begin, with a letter of X .

Let us imagine x and y to be left and right parentheses. In a word $w \in L[X_1]$ with properly arranged parentheses, these are naturally divided into pairs $\langle x, y \rangle$, a left parenthesis x and its corresponding right parenthesis y . By the depth of occurrence of some such pair $\langle x, y \rangle$ in w we mean the difference between the number of parentheses y and the number of the parentheses x to the left of the x in the considered pair $\langle x, y \rangle$ in w . If w has a pair of parentheses with depth of occurrence s but no pair with depth of occurrence $s+1$, then the number $s+1$ is called the rank of w . If w has no parentheses, its rank is zero.

It is easy to see that if in an expression for W we group together the monomials with properly arranged parentheses, then we again obtain a linear combination of the elements (1). Since W itself is an element of $K[X]$, the monomials with improperly arranged parentheses cancel. We may therefore assume that W is a linear combination of the elements (1), where parentheses are properly arranged in all monomials.

Let s be the largest number such that an expression for W contains words of rank s . If $s = 0$, then $W \in I$. Suppose $s > 0$. We will show that W has a representation of the same form in which all words have rank less than s . The proof of the lemma then follows by induction.

A word $w \in L[X_1]$ with properly arranged parentheses is called proper if its rank is either less than s , or is equal to s and for any pair $\langle x, y \rangle$ of a depth $s-1$ the part of the word w from x to y for these x, y has the form

$$x u z \beta y,$$

where $u \in L[X]$, or $u = \emptyset$, $\beta \in L[\beta_1, \ldots, \beta_n]$, or $\beta = \emptyset$, $u\beta \neq \emptyset$.

It is easy to see that for each element of the form (1), all words of which it is a linear combination are proper or improper simultaneously. If in the representation for W we collect the improper words and cancel them, we again obtain a representation for W (since $W \in K[X]$). In view of what was said, this representation is a linear combination of elements of the form (1).

In each word of rank W in this representation of s, for each pair $\langle x, y \rangle$ of depth $s-1$ we replace the subwords $xuz\beta y$ for these x, y by $\varphi[u\beta(\bar{a})]$. We again obtain a representation of W.

We will show that the new representation is, as before, a linear combination of elements of the form (1). Since the ranks of words in the new representation are less than s, this will prove the lemma.

Consider a summand $v_1 U v_2$, where $U \in I$. Suppose $U = \sum_i \alpha_i w_i$, where $\alpha_i \in K$. Clearly, the ranks of all of the words $v_1 w_i v_2$ are the same. If their common rank is equal to s, then under the replacement described above the words $w_i \in L[X]$ are affected only when $v_1 \equiv v_1' x$, $v_2 \equiv z\beta y v_2'$. In the remaining cases, we obviously obtain a sum of the same form. Then after the replacement we have

$$\sum_i \alpha_i v_1'' \varphi[w_i \beta(\bar{a})] v_2'' = v_1'' \left(\sum_i \alpha_i \varphi[w_i \beta(\bar{a})]\right) v_2''.$$

But $\sum_i \alpha_i w_i \beta(\bar{a}) \in I$, and since φ is an endomorphism of A as a K-module, it follows that $\sum_i \alpha_i \varphi[w_i \beta(\bar{a})] \in I$.

Consider a summand $w_1 (xwz\beta y - \varphi[w\beta(\bar{a})]) w_2$. If in the word $w_1 \varphi[w\beta(\bar{a})] w_2$ the subword $\varphi[w\beta(\bar{a})]$ occurred within a pair of depth $s-1$, then clearly in the word $w_1 xwz\beta y w_2$ the pair $\langle x, y \rangle$ would have depth s, which is impossible. Therefore, obviously, after the replacement the expression under consideration either vanishes or keeps the same form.

Consider a summand $w_1 (a_i z - z\beta_i) w_2$. It suffices to look at the case where $w_1 \equiv w_1' x w$, $w_2 \equiv \beta y w_2'$ and this pair $\langle x, y \rangle$ has depth $s-1$. Then after the replacement we obtain

$$w_1'' \varphi[w a_i \beta(\bar{a})] w_2'' - w_1'' \varphi[w(\beta_i \beta)(\bar{a})] w_2''.$$

This expression vanishes, since $(\beta_i \beta)(\bar{a}) \equiv a_i \beta(\bar{a})$.

Finally, consider a summand $w_1 (xz\beta_i \beta_j - b_i xz\beta_j) w_2$. Again, it suffices to look at the case where $w_2 \equiv \beta y w_2'$ and the considered pair $\langle x, y \rangle$ has depth $s-1$. After the replacement we obtain

$$w_1' \varphi[(\beta_i \beta_j)(\bar{a})] w_2'' - w_1' b_i \varphi[(\beta_j \beta)(\bar{a})] w_2''.$$

This expression also vanishes, since by choice of $\varphi(w)$ we have

$$\varphi[(\beta_i \beta_j)(\bar{a})] \equiv b_i b_j b_{i_1} \ldots b_{i_k} \equiv b_i \varphi[(\beta_j \beta)(\bar{a})],$$

if $\beta = \beta_{i_1} \cdots \beta_{i_k}$.

The lemma is proved.

Remark 1 (in some sense a converse to Lemma 1). Let $Hom_n(\bar{a}, \bar{b}, \bar{\theta})$, where $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$, $\bar{\theta} = (\beta_1, \ldots, \beta_n, x, y, z)$, denote the system of equalities in the statement of Lemma 1. We will show that if $Hom_n(\bar{a}, \bar{b}, \bar{\theta})$ is satisfied in some K-algebra A for certain elements $\bar{a}$, $\bar{b}$, $\bar{\theta}$, then the mapping $\varphi: A[a_1, \ldots, a_n] \to A$, defined by the rule $\varphi(a) = xazy$ for $a \in A[a_1, \ldots, a_n]$, is a K-algebra homomorphism for which $\varphi(a_i) = b_i$ $(i = 1, \ldots, n)$. It suffices to observe that

$$\varphi(a_{i_1} a_{i_2} \ldots a_{i_k}) = x a_{i_1} a_{i_2} \ldots a_{i_k} zy = xz\beta_{i_1}\beta_{i_2}\ldots\beta_{i_k} y = b_{i_1} b_{i_2} \ldots b_{i_{k-1}} x a_{i_k} zy = b_{i_1} b_{i_2} b_{i_k}.$$

Remark 2. In the sequel, we will apply Lemma 1 in the following situation. Suppose A, B are K-algebras, A is the subalgebra of B, generated by elements $a_1, \ldots, a_n \in B$, $\varphi: A \to B$ is a K-algebra homomorphism for which $\varphi(a_i) = b_i$ $(i = 1, \ldots, n)$, and A as a K-module is a direct summand of B, i.e., the K-module B contains a K-submodule C such that $A \dotplus C = B$. Then there exists a K-algebra containing B as a subalgebra in which the system $Hom_n(\bar{a}, \bar{b}, \bar{\theta})$ is solvable.

The assertion that for associative rings $A \subseteq B$, where A is a direct summand of the additive group of B, an additive homomorphism $\varphi: A \to B$ is defined in an extension of B by a rule $\varphi(a) = xay$, is due to Taitslin [1].

LEMMA 2. Suppose $f(i,j)$ is a recursive function defined for all $i, j = 1, 2, \ldots$ $(i \neq j)$ such that $f(i,j) = f(j,i)$. Suppose that $Y \subseteq N^2$ is some recursively enumerable set and A is a K-algebra with generators x, y, z and defining relations

$$\{xy^i z + xy^j z = xy^{f(i,j)} z \,/\, i \neq j\ ;\ i, j = 1, 2, \ldots\} \cup \{xy^i z = xy^j z \,/ < i, j > \in Y\}.$$

Then there exists a K-algebra B with the following properties:

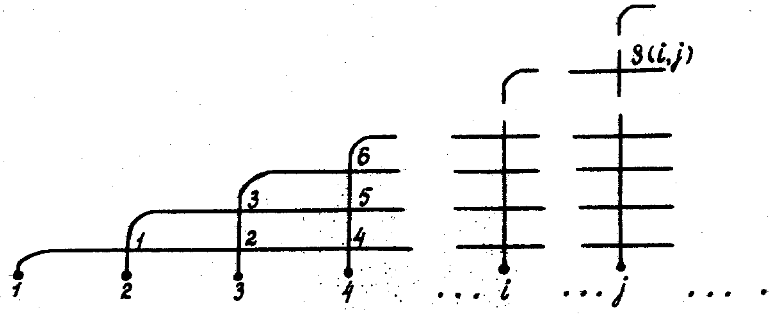
a) A is a subalgebra of B and, as a K-module, is a direct summand of the K-module B;

b) B has a finite number of generators and a recursively enumerable set of defining relations, one of which has the form $\alpha + \beta = \gamma$ and the others are word equalities in the generator symbols.

Proof. For each $i = 1, 2, \ldots$ we define a function $n_i : N \to N$, and for $i \neq j$; $i, j = 1, 2, \ldots$, we define $S(i,j) \in N$ as follows:

$$\begin{aligned} &\text{if } 1 \le i < j, &&\text{then } n_j(i) = \tfrac{1}{2}(j-1)(j-2) + i; \\ &\text{if } 1 \le j \le i, &&\text{then } n_j(i) = n_{i+1}(j); \\ &\text{if } 1 \le i < j, &&\text{then } S(i,j) = S(j,i) = n_j(i). \end{aligned}$$

The definition of these functions is illustrated by the following diagram.



Here from the natural numbers $1, 2, 3, \ldots$ there emanate lines, each of which intersects each other line in exactly one point. The points of intersection are enumerated by the natural numbers. These numbers define the functions n_i and s. That is to say, the numbers on the line emanating from i are $n_i(1), n_i(2), \ldots$. For example, $2, 3, 6, 9$ are the values $n_3(1), n_3(2), n_3(3), n_3(4)$ respectively. The number appearing at the intersection of the line emanating from i and the line emanating from j is $s(i,j)$. Thus, for any $n \in N$ there exists a unique pair $1 \leq i < j$ such that $n = s(i,j) = s(j,i)$. Furthermore, $n = n_j(i) = n_i(j-1)$ and the functions n_i are one-to-one.

As generators of the desired K-algebra B we take the symbols $x, y, z, u, \alpha, \beta, \gamma$. As the set of defining relations we take the set of equalities

$$\{xy^iz = xy^jz \,/\, \langle i,j\rangle \in Y\} \cup \{xy^iz = xu^{n_i(j)}\varepsilon_{ij}z \,/\, i,j = 1,2,\ldots,\} \cup$$
$$\cup \{xy^{f(i,j)}z = xu^{s(i,j)}\gamma z \,/\, i \neq j\,;\ i,j = 1,2,\ldots\} \cup \{\alpha + \beta = \gamma\}.$$

Here ε_{ij} is equal to α if $i+j$ is even, and is equal to β if $i+j$ is odd.

Note first that the defining relations of B imply those of A. Indeed, if $i<j$, then it follows from the relations of B that $xy^iz + xy^jz = xu^{n_i(j-1)}\varepsilon_{ij-1}z + xu^{n_j(i)}\varepsilon_{ij}z$ $= xu^{s(i,j)}(\varepsilon_{ij} + \varepsilon_{ij-1})z = xu^{s(i,j)}\gamma z = xy^{f(i,j)}z$.

Now consider the ideal I_1 of the free K-algebra $K[x, y, z, u, \alpha, \beta, \gamma]$ generated by the relations of B. Any element of this ideal can be written as a linear combination of elements of the form

$$w_1(xy^iz + xy^jz - xy^{f(i,j)}z)w_2,\quad w_1(xy^iz - xy^jz)w_2 \,/\langle i,j\rangle \in Y/,$$
$$w_1(xy^iz - xu^{n_i(j)}\varepsilon_{ij}z)w_2,\quad w_1(xy^{f(i,j)}z - xu^{s(i,j)}\gamma z)w_2 \quad (i \neq j), \tag{1}$$
$$w_1(\alpha + \beta - \gamma)w_2,$$

where w_1, w_2 are words in $x, y, z, u, \alpha, \beta, \gamma$, possibly empty.

By a reduction of a word $w \in L[x, y, z, u, \alpha, \beta, \gamma]$ we mean the simultaneous replacement in it:

of all subwords of the form $xu^n\alpha z$ by xy^iz, where i is such that $n = n_i(j)$ and $i+j$ is given.

of all subwords of the form $xu^n\beta z$ by $xy^i z$, where i is such that $n=n_i(j)$ and $i+j$ is odd;

of all subwords of the form $xu^n\gamma z$ by $xy^{f(i,j)}z$, where $i+j$ are such that $n=s(i,j)$.

A word w without subwords of the indicated form is called reduced. Let us now assume that some element $W\in K[x,y,z,u,\alpha,\beta,\gamma]$ is a linear combination of reduced words (in particular, belongs to $K[x,y,z]$) and at the same time belongs to I_1. Then W can be represented as a linear combination of elements of the forms (1). Reducing all words in such a representation, we obviously again obtain an expression for W in which there occur terms (1) of only the first, second, and perhaps the last forms. If $W\in K[x,y,z]$, then, cancelling the words in this representation in which the letters u,α,β,γ occur, we obtain that $W\in I$, the ideal generated in $K[x,y,z]$ by the defining relations of A. This proves that A is a subalgebra of B. Now let R be the submodule of the K-module B generated in B by the images under the canonical homomorphism of all reduced words in $x,y,z,u,\alpha,\beta,\gamma$, in which there must necessarily be occurrences of u,α,β,γ. Since any element of $K[x,y,z,u,\alpha,\beta,\gamma]$ modulo I is equal to a linear combination of reduced words, the K-module B is the sum of A and R. This sum is direct. Indeed, suppose $W_1+W_2\in I_1$, where $W_1\in K[x,y,z]$, and W_2 is a linear combination of reduced words in $x,y,z,u,\alpha,\beta,\gamma$ in which u,α,β,γ occur. Then, by what has been proved, W_1+W_2 can be represented as a linear combination of expressions of the first, second, and last forms in (1). Collecting the words in x,y,z and all the others separately, we obtain $W_1\in I$, $W_2\in I_1$.

The lemma is proved.

A small modification of an assertion proved by Mal'tsev [2] is the following

LEMMA 3. Suppose A is an arbitrary, at most countable K-algebra and $A=\{a_1,a_2,\ldots\}$ is an arbitrary enumeration of its elements. Then there exist a K-algebra B containing A as a subalgebra and elements $a,b,c\in B$, such that $a_i=ab^ic$ $(i=1,2,\ldots)$.

Proof. Consider $A_1=A\times K[x]$. There exists an endomorphism $\varphi: A_1\longrightarrow A_1$ of A_1 as a K-module under which $\varphi(x^n)=a_n$ $(n=1,2,\ldots)$. By Lemma 1, there exist a K-algebra $B\supseteq A_1$ and elements $u,v\in B$ such that $\varphi(a)=uav$ for all $a\in A_1$. Clearly, B is the desired algebra.

THEOREM. Suppose K is a commutative associative ring with unity or a field that is finitely generated over its prime subfield. Suppose A is an arbitrary associative K-algebra with a recursively enumerable set of defining relations. Then there exists a finitely generated associative K-algebra with a finite set of defining relations in which A is contained as a subalgebra.

Proof. Suppose $A=\{a_1,a_2,\ldots,\}$ is an enumeration of the elements of A in which each element of A is repeated at least twice. By Lemma 3, there exists a K-algebra B with three generators a,b,c such that $A\subseteq B$ and $a_i=ab^ic$ $(i=1,2,\ldots)$. Since in the ring or field K the set of all true equalities in the generators of K is recursively enumerable, we may assume that B is also recursively presented, i.e., the set of all equalities in the elements a,b,c that are true in B is recursively enumerable. Moreover, we may assume that the original algebra A has a finite set of generators, say $a_{i_1},\dots,a_{i_k}$.

Let $Y=\{<i,j> \mid ab^i c = ab^j c\}$. Then $Y\subseteq N^2$ is a recursively enumerable set. Let $f(i,j)$ be a recursive function, defined for all $i,j\in N$ with $i\neq j$, such that $f(i,j)=f(j,i)$ and

$$ab^i c + ab^j c = ab^{f(i,j)}c \text{ in } B.$$

Consider the K-algebra S with generators x,y,z and defining relations

$$\{xy^i z + xy^j z = xy^{f(i,j)}z \mid i\neq j;\ i,j=1,2,\dots\}\cup\{xy^i z = xy^j z \mid <i,j>\in Y\}.$$

By Lemma 2, there exists a K-algebra $S_1\supseteq S$ in which the K-module S is a direct summand and which has an enumerable set of defining relations

$$\Sigma\,(x,y,z,u,\alpha,\beta,\gamma)\cup\{\alpha+\beta=\gamma\},\tag{1}$$

where Σ contains only word equalities in $x,y,z,u,\alpha,\beta,\gamma$.

There exists $\varphi: B\times S_1 \longrightarrow B\times S_1$ such that $\varphi(x)=a$, $\varphi(y)=b$, $\varphi(z)=c$, φ is an endomorphism of $B\times S_1$ as a K-module, and the restriction of φ to S is a K-algebra homomorphism. By Lemma 1, there is K-algebra $S_2\supseteq B\times S_1$ in which we can solve the system of equations

$$\mathrm{Hom}_3\,(x,y,z,a,b,c,\bar\theta_1).\tag{2}$$

Now consider the semigroup G with generators $x',y',z',u',\alpha',\beta',\gamma'$ and set of defining relations $\Sigma(x',y',z',u',\alpha',\beta',\gamma')$. By a theorem of Murskii [3], there exists a semigroup $G_1\supseteq G$ with a finite set of generators and a finite set of defining relations

$$\Sigma_1\,(x',y',z',u',\alpha',\beta',\gamma',\bar\theta_2).\tag{3}$$

Let $K[G]$ denote the semigroup K-algebra of the semigroup G. Clearly, $K[G]\subseteq K[G_1]$ and $K[G]$, as a K-module, is a direct summand of $K[G_1]$. By Lemma 1, there is a K-algebra $S_3\supseteq S_2\times K[G_1]$ in which we can solve the system

$$\mathrm{Hom}_7\,(x',y',z',u',\alpha',\beta',\gamma',x,y,z,u,\alpha,\beta,\gamma,\bar\theta_3).\tag{4}$$

Suppose $g: N^2\longrightarrow N$ is a recursive function such that $ab^i c\;\; ab^j c = ab^{g(i,j)}c$ in the K-algebra B. If H is the semigroup with generators p, q, r and defining relations $pq^i r\;\; pq^j r = pq^{g(i,j)}r$ $(i,j\in N)$, then, by the theorem of Murskii [3], there exists a semigroup $H_1\supseteq H$ with a finite set of defining relations

$$\Sigma_2\,(p,q,r,\bar\theta_3).\tag{5}$$

Again, by Lemma 1, there exists a K-algebra $S_4\supseteq S_3\times K[H_1]$ in which we can solve the system

$$\mathrm{Hom}_3\,(p,q,r,a,b,c,\bar\theta_4).\tag{6}$$

Finally, suppose $\alpha_1,\dots,\alpha_\ell\in K$ are generators of the ring or field K. Recall that $a_{i_1},\dots,a_{i_k}$ generate A. There exist $j_{ts}\in N$ such that

$$\alpha_t ab^{i_s}c = ab^{j_{ts}}c \qquad (t=1,\dots,\ell;\ s=1,\dots,k).\tag{7}$$

Now consider the K-algebra defined by the set of relations Σ^*, the union of all of the system (1)-(7). It is easy to see that this K-algebra on the elements $a_i = ab^i c$ $(i=1,2,\ldots)$ satisfies all equalities of the K-algebra A and only these. This proves the theorem.

COROLLARY. If K is a commutative associative finitely generated ring with unity or a field that is finitely generated over its prime subfield, then there exists a 2-generator finitely presented associative K-algebra in which any associative K-algebra with a recursively enumerable set of defining relations can be isomorphically embedded. In particular, when $K=Z$, there exists a 2-generator finitely presented associative ring A in which any associative ring with a recursively enumerable set of defining relations can be isomorphically embedded. Among the defining relations of A only one has the form $w_1+w_2=w_3$, and the others are word equalities in the generators.

The proof is analogous to that of Higman [4] for groups, owing to the fact that any countable K-algebra can be embedded in a 2-generator algebra (see [2]) and in the case $K=Z$ the equalities (7) are superfluous.

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