On S-duality in Abelian Gauge Theory

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Abstract. $U(1)$ gauge theory on $\mathbf{R}^4$ is known to possess an electric-magnetic duality symmetry that inverts the coupling constant and extends to an action of $SL(2,\mathbf{Z})$. In this paper, the duality is studied on a general four-manifold and it is shown that the partition function is not a modular-invariant function but transforms as a modular form. This result plays an essential role in determining a new low-energy interaction that arises when $N = 2$ supersymmetric Yang-Mills theory is formulated on a four-manifold; the determination of this interaction gives a new test of the solution of the model and would enter in computations of the Donaldson invariants of four-manifolds with $b_2^+ \leq 1$. Certain other aspects of abelian duality, relevant to matters such as the dependence of Donaldson invariants on the second Stieffel-Whitney class, are also analyzed.

§1. Introduction

S-duality asserts that certain four-dimensional gauge theories are invariant under modular transformations acting on

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \tag{1.1}$$

with θ and g being a theta angle and gauge coupling constant. For the by now classical case of $N = 4$ supersymmetric Yang-Mills theory, this assertion was tested in [1] by actually computing the partition function of the theory (with gauge group $SO(3)$ or $SU(2)$) on certain four-manifolds X, in some cases with a topological twist. It was found that modular invariance did hold, in an appropriate sense: (i) modular transformations in general exchange the gauge group $SU(2)$ with the dual group $SO(3)$; (ii) the partition function is not a modular-invariant function but transforms as a modular form, with a weight proportional to the Euler characteristic of the four-manifold.

The first point was anticipated by Montonen and Olive [2] but the second perhaps requires comment. That the partition function transforms as a modular form rather than a modular function, with a modular weight as observed, can be interpreted to mean that in coupling an S-dual theory to gravity, to maintain S-duality, one requires certain non-minimal c-number couplings that involve the background gravitational field only, of the general form $\int_X \left(B(\tau)\mathrm{tr}\, R \wedge \widetilde{R} + C(\tau)\mathrm{tr}\, R \wedge R \right)$; here

$\operatorname{tr} R \wedge \widetilde{R}$ and $\operatorname{tr} R \wedge R$ are the densities whose integrals are proportional to the Euler characteristic and the signature. B and C are chosen so that e^B and e^C are modular forms (of weights chosen to cancel the anomaly); for $N = 4$ supersymmetric Yang-Mills theory, one can take $C = 0$ (as the modular weight depends only on the Euler characteristic).

This effect is perhaps not really surprising, but given our limited understanding of S-duality it appears difficult to explain it *a priori* in, say, the $N = 4$ theory; similarly, one does not know how to predict the coefficients of the anomaly in putatively S-dual theories in which computations on four-manifolds have not yet been performed, such as the $N = 2$, $N_f = 4$ theory with gauge group $SU(2)$ [8]. The first goal of the present paper is to explore this phenomenon in a much simpler example, namely free $U(1)$ gauge theory without charges, where everything can be understood rather explicitly. (The S-duality of this theory is an observation that goes back essentially to [3-5].) In this case, the modular anomaly is, as will become clear, quite analogous to the transformation law for the dilaton under $R \to 1/R$ symmetry in two dimensions, as described in [6-7].

One would suppose that in S-dual theories that contain dynamical gravity — string theory is of course the only known candidate — the modular anomaly is somehow canceled. Knowing how the anomaly works in field theory with gravity as a spectator may ultimately be helpful for understanding the case with dynamical gravity.

The shift from the group to the dual group also has an analog for the free abelian theory. In fact, even self-dual lattices play a special role rather as in the theory of chiral bosons in two dimensions.

The computation that we will perform is also relevant to various more complicated problems in which this $U(1)$ theory is approximately embedded. For instance, after computing the modular anomaly in section two, we will use it in section three as a necessary ingredient in order to compute a new effective interaction in $N = 2$ supersymmetric gauge theory with gauge group $SU(2)$ on a general four-manifold. The unique and consistent determination of this interaction is an interesting check on the framework of [9].

In section four, we discuss some further details of how duality works in the $N = 2$ theory on a general four-manifold. The results of section three are needed for integration over the u-plane to compute Donaldson invariants of four-manifolds with $b_2^+ = 1$, and the results of section four are needed for a precise derivation of the relation of the Donaldson invariants to the monopole invariants of four-manifolds described in [10]. Some of these themes will be further pursued elsewhere.

§2. Analysis Of The Abelian Theory

In what follows, X is a four-manifold and b_i, $i = 0, \ldots, 4$ will denote the ith Betti number, that is the dimension of the space of harmonic i-forms on X. As a harmonic

two-form in four dimensions can be decomposed as a sum of self-dual and anti-self-dual pieces, we can write $b_2 = b_2^+ + b_2^-$ with b_2^+ and b_2^- the dimensions of the spaces of self-dual and anti-self-dual harmonic two-forms. The Euler characteristic of X is $\chi = \sum_{i=0}^4 (-1)^i b_i$ and the signature is $\sigma = b_2^+ - b_2^-$.

We will say that a not necessarily holomorphic function F transforms as a modular form of weight (u, v) for a finite index subgroup Γ of $SL(2, \mathbf{Z})$ if for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{2.1}$$

one has

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^u (c\overline{\tau} + d)^v F(\tau). \tag{2.2}$$

Our main result concerning the partition function $Z(\tau)$ of the $U(1)$ Maxwell theory on a four-manifold is that

$$Z(-1/\tau) = \tau^u \overline{\tau}^v Z(\tau), \tag{2.3}$$

where $(u, v) = ((\chi + \sigma)/4, (\chi - \sigma)/4)$. Note that as $(\chi \pm \sigma)/2 = 1 - b_1 + b_2^\pm$ is in general integral but not necessarily divisible by two, the weights in (2.3) are half-integral in general. Since also $Z(\tau + 1) = Z(\tau)$ (or $Z(\tau + 2) = Z(\tau)$ if X is not a spin manifold, as explained momentarily), (2.3) implies that Z is modular of weight (u, v).

Now we proceed to the analysis. We consider a $U(1)$ gauge field A_m (a connection on a line bundle L), with field strength $F_{mn} = \partial_m A_n - \partial_n A_m$; we also set $F_{mn}^\pm = \frac{1}{2}\left(F_{mn} \pm \frac{1}{2}\epsilon_{mnpq} F^{pq}\right)$. On a four-manifold X of Euclidean signature, we take the classical action to be

$$\begin{aligned} I =& \frac{1}{8\pi} \int_X d^4 x \sqrt{g} \left(\frac{4\pi}{g^2} F_{mn} F^{mn} + \frac{i\theta}{2\pi} \frac{1}{2} \epsilon_{mnpq} F^{mn} F^{pq}\right) \\ =& \frac{i}{8\pi} \int_X d^4 x \sqrt{g} \left(\overline{\tau} F_{mn}^+ F^{+mn} - \tau F_{mn}^- F^{-mn}\right), \end{aligned} \tag{2.4}$$

with τ defined in (1.1).

Before proceeding, let us determine the periodicity in θ. We permit L to be an arbitrary line bundle, so the general constraint on periods of F is simply the Dirac quantization law; thus, if U is any closed two-surface in X, $\int_U F$ is an arbitrary integer multiple of 2π. If X is a spin manifold, the smallest possible non-zero value of $J = \int d^4 x \sqrt{g} \epsilon^{mnpq} F_{mn} F_{pq}$ is obtained by taking $X = U \times V$, with U and V being two-spheres, and picking the gauge field so that $\int_U F_{12} = \int_V F_{34} = 2\pi$ (with other components vanishing). Then one gets $J = 8(2\pi)^2$, so the θ-dependent part of the action is $i\theta$. This means that the theory is invariant under $\theta \to \theta + 2\pi$, that

is $\tau \to \tau + 1$. On the other hand, if X is not a spin manifold, one can find a line bundle on X such that $J = 4(2\pi)^2$, leading to invariance only under $\theta \to \theta + 4\pi$ or $\tau \to \tau + 2$.[1] We will discuss at the end of this section what modification of the theory would be needed to get full $SL(2, \mathbf{Z})$ invariance, including $\tau \to \tau + 1$, when X is not spin.

We will determine the modular weight of the partition function in two ways: first we simply calculate the partition function and see what modular weight it has; second we give an *a priori* explanation by manipulation of the path integral. Apart from being more conceptual, the second derivation could be extended to determine the modular transformation law of correlation functions.

2.1 The Computation. The partition function of the $U(1)$ theory is a product of several factors. There is a sum over the isomorphism class of the line bundle L. Because the gauge group is abelian, the L dependence comes entirely from the value of the classical action for a connection on L that minimizes it; the sum over L's will give a generalized theta function. For each L we must integrate over b_1 zero modes and evaluate a determinant for the non-zero modes.

In computing the determinant, one can ignore the θ term, which is a topological invariant. The rest of the kinetic energy is proportional to $1/g^2 \sim \operatorname{Im}\tau$. The τ dependence of the regularized determinant is then roughly a factor of $\operatorname{Im}\tau^{-1/2}$ for every non-zero eigenvalue of the kinetic operator of A. The zero modes do not give factors of $\operatorname{Im}\tau$; they are tangent to the space of classical minima, which is a torus of dimension $b_1(X)$ and has a volume independent of $\operatorname{Im}\tau$.

Let B_k be the dimension of the space of k-forms on X; of course, the B_k are infinite, but we can make them finite with a lattice regularization. The total number of modes of A is thus B_1. However, we only want to count modes modulo gauge transformations. As the gauge parameter is a zero-form, the number of modes of infinitesimal gauge transformations is B_0, but one (the constant mode) acts trivially on A, so the number of A modes modulo gauge transformations is $B_1 - B_0 + 1$. Removing also the zero modes of A, the number of non-zero modes mod gauge transformations is $B_1 - B_0 + 1 - b_1$ (since $b_0 = 1$, this is the same as $B_1 - B_0 + b_0 - b_1$). The τ dependence of the determinant is thus

$$\operatorname{Im}\tau^{\frac{1}{2}(b_1 - 1)} \cdot \operatorname{Im}\tau^{\frac{1}{2}(B_0 - B_1)}. \tag{2.5}$$

In a lattice regularization, one would eliminate the last, cutoff-dependent factor by simply including in the definition of the path integral a factor of $\operatorname{Im}\tau^{1/2}$ for every integration variable (every one-simplex on the lattice), and a factor of $\operatorname{Im}\tau^{-1/2}$ for every generator of a gauge transformation (every vertex or zero-simplex). Since

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1)   To state this in a less computational way, $J/16\pi^2$ measures $c_1(L)^2$, which for a spin manifold is an even integer (as the intersection form on the second cohomology group is even) while on a four-manifold that is not spin it is subject to no divisibility conditions.

the factors required are defined locally, this eliminates the cutoff-dependence of the theory while preserving locality. With this definition of the coupling to gravity, the τ dependence of the determinant is simply a factor

$$\operatorname{Im} \tau^{\frac{1}{2}(b_1 - 1)}. \tag{2.6}$$

Alternatively, the numbers B_k would be regularized in a Pauli-Villars regularization by replacing them with something like

$$\operatorname{Tr} e^{-\epsilon \nabla_k} \tag{2.7}$$

with ϵ a small parameter and ∇_k the Laplacian on k-forms. The small ϵ behavior of (2.7) can be worked out using the short time behavior of the heat kernel; the various terms can be written as the integrals over X of local densities. As these terms are *local*, dropping them and dropping B_0 and B_1 in (2.5) simply amounts to a specific choice of how the theory is coupled to gravity. The choice that leads to (2.6) is the most minimal one in that it corresponds, for instance, to the usual prescription in which factors of g in the determinant come only from zero modes. (Notice that, as $b_1 - 1$ cannot be computed as a local integral, no choice of the coupling to gravity will remove the factor obtained in (2.6).)

Now we come to the sum over line bundles L. The object $F/2\pi$ is a de Rham representative of the first Chern class $m = c_1(L)$. We will think of m as a point in $H^2(X, \mathbf{R})$ that lies in the lattice Λ consisting of points with integer periods. For a given L, the real part of the action is minimized for a connection such that the two-form F is harmonic. For F harmonic, the dual two-form $\widetilde{F}_{mn} = \frac{1}{2}\epsilon_{mnpq}F^{pq}$ is also harmonic. In terms of m, the map $F \to \widetilde{F}$ is a linear map $m \to *m$ with $*^2 = 1$; note that $*m$ takes values in $H^2(X, \mathbf{R})$, but not necessarily in Λ. If (m, n) is the intersection pairing on H^2, which is integral for lattice points m, n, then we have for F harmonic

$$
\begin{aligned}
(m, m) &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \epsilon^{mnpq} F_{mn} F_{pq} \\
(m, *m) &= \frac{1}{8\pi^2} \int d^4x \sqrt{g} F_{mn} F^{mn}.
\end{aligned}
\tag{2.8}
$$

For F harmonic, the action can therefore be written

$$I = \frac{4\pi^2}{g^2}(m, *m) + \frac{i\theta}{2}(m, m) = \frac{i\pi}{2}\left\{\tau\left((m, m) - (m, *m)\right) + \overline{\tau}\left((m, m) + (m, *m)\right)\right\}. \tag{2.9}$$

The sum over line bundles therefore gives a lattice sum

$$F(q) = \sum_{m \in \Lambda} q^{\frac{1}{4}(-(m,m)+(m,*m))} \, \overline{q}^{\frac{1}{4}((m,m)+(m,*m))} \tag{2.10}$$

with $q = e^{2\pi i \tau}$.

The function $F(q)$ is very similar to the generalized theta functions that appeared in the work of Narain [11] on toroidal compactification of two-dimensional conformal field theory. It transforms as a modular function of weight (b_2^-, b_2^+) (for a subgroup of $SL(2, \mathbf{Z})$, in general). b_2^+ and b_2^- enter as the number of positive and negative eigenvalues of the intersection form on H^2. For instance, if X is $\mathbf{CP}^2$ with its usual complex orientation, so $b_2^+ = 1$, $b_2^- = 0$, then F reduces to the complex conjugate of a standard theta function:

$$F = \sum_{n \in \mathbf{Z}} \bar{q}^{n^2/2}. \tag{2.11}$$

In particular, this F is modular of weight $(0, 1/2)$. Similarly, with opposite orientation, $\mathbf{CP}^2$ has $(b_2^+, b_2^-) = (0, 1)$ and gives for F a standard theta function of modular weight $(1/2, 0)$.

Including also the determinant of the non-zero modes, the Maxwell partition function up to a τ-independent multiplicative constant (which depends on the Riemannian metric of X) is

$$Z(\tau) = (\mathrm{Im}\, \tau)^{\frac{1}{2}(b_1 - 1)} F(\tau). \tag{2.12}$$

As

$$\mathrm{Im}\,(-1/\tau) = \frac{1}{\tau \bar{\tau}} \mathrm{Im}(\tau), \tag{2.13}$$

$\mathrm{Im}\,(\tau)$ is modular of weight $(-1, -1)$. So the Maxwell partition function is modular of weight

$$(u, v) = \frac{1}{2}(1 - b_1 + b_2^-, 1 - b_1 + b_2^+) = \frac{1}{4}(\chi - \sigma, \chi + \sigma). \tag{2.14}$$

In particular, as in [1], the modular weights are linear in χ and σ.

2.2 A Priori Explanation.

Now we proceed to a more *a priori* explanation of the above result, along lines similar to a familiar discussion of T-duality in two-dimensions [6, 12] (see also section 3.2 of [9] where a version of this argument is given, on flat $\mathbf{R}^4$, including supersymmetry).

First we rewrite the Maxwell theory in terms of some additional degrees of freedom. We introduce a two-form field G. We want to consider a theory which, beyond the usual Maxwell gauge invariance $A \to A - d\epsilon$, $G \to G$, has invariance under

$$\begin{aligned} A &\to A + B \\ G &\to G + dB \end{aligned} \tag{2.15}$$

where B is an arbitrary one-form or more generally a connection on an arbitrary line bundle M. Note that if A is a connection on a line bundle L, then $A + B$ should be interpreted as a connection on $L \otimes M$. Thus, while the ordinary Maxwell theory is well-defined for any given L, its extension to include the G field can only have the invariance (2.15) if we sum over all L's. Invariance under (2.15) means that one can shift the periods of G — that is the integrals of G over closed two-dimensional cycles $S \subset X$ — by multiples of 2π:

$$\int_S G \to \int_S G + 2\pi n. \qquad (2.16)$$

Note that the usual Maxwell gauge invariance is a special case of (2.15) obtained by setting $B = -d\epsilon$.

There is an obvious way to achieve invariance under (2.15): we set $\mathcal{F} = F - G$ and replace F everywhere in the Maxwell Lagrangian by $\mathcal{F}$. The resulting theory however is trivial and in particular not equivalent to Maxwell theory. To get something of interest, we introduce a dual connection V on a dual line bundle $\widetilde{L}$, with field strength $W_{mn} = \partial_m V_n - \partial_n V_m$. Like the curvature on any line bundle, W obeys a Dirac quantization law

$$\int_S W \in 2\pi \mathbf{Z}. \qquad (2.17)$$

This ensures that

$$\frac{1}{8\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} W_{mn} G_{pq} \qquad (2.18)$$

is invariant under (2.15) modulo 2π. We then take the Lagrangian to be

$$I = \frac{i}{8\pi} \int_X d^4x \sqrt{g} \left(\epsilon^{mnpq} W_{mn} G_{pq} + \overline{\tau} \mathcal{F}^+_{mn} \mathcal{F}^{+mn} - \tau \mathcal{F}^-_{mn} \mathcal{F}^{-mn} \right). \qquad (2.19)$$

The terms involving $\mathcal{F}$ are manifestly invariant under (2.15) (which acts trivially on V; V transforms nontrivially only under the dual gauge transformations $V \to V - d\alpha$), and the same is true mod $2\pi i$ of the first term by virtue of what has just been said.

We now proceed as follows. After showing that (2.19) is equivalent to the original Maxwell theory (provided that one sums over all L's in each case), we will show how to obtain a dual version by a different manipulation. We first perform both computations in a somewhat cavalier manner and then more carefully study the quantum integration measure to compute the modular weight.

For the first manipulation, the part of the path integral that involves V is

$$\int DV \exp \left(\frac{i}{8\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} W_{mn} G_{pq} \right). \qquad (2.20)$$

The integral over V consists of a discrete sum over dual line bundles $\widetilde{L}$ and for each $\widetilde{L}$ a continuous integral over connections on $\widetilde{L}$. The continuous part of the integral gives a delta function setting $dG = 0$. The sum over $\widetilde{L}$ then gives a further delta function setting the periods of G to be integral multiples of 2π. But the exotic gauge invariance (2.15) permits one (in a fashion that is unique up to an ordinary gauge transformation) to set $G = 0$ precisely if $dG = 0$ and $G/2\pi$ has integral periods. After integrating over V, we gauge G to zero, reducing the extended gauge invariance to ordinary gauge invariance, and reducing the Lagrangian (2.19) to the original Lagrangian (2.4) of the abelian gauge theory.

The other way to analyze (2.19) begins by noting that the gauge invariance (2.15) precisely enables one to fix a gauge with $A = 0$. In that gauge, one can then integrate out G, giving a Lagrangian containing V only. This Lagrangian inherits the dual gauge invariance $V \to V - d\alpha$ of (2.19), and is simply an abelian gauge theory of the general type we started with in (2.4), but with a different value of τ. The computation is very quick if one notes that in terms of the the self-dual and anti-self-dual projections $W^\pm$ and $G^\pm$ of W and G, the action is

$$\frac{i}{4\pi} \int_X d^4x \sqrt{g} \left(W^+ G^+ - W^- G^- \right) + \frac{i}{8\pi} \int_X d^4x \sqrt{g} \left(\overline{\tau}(G^+)^2 - \tau(G^-)^2 \right). \quad (2.21)$$

Integrating out $G^\pm$ at the classical level, one gets simply

$$\frac{i}{8\pi} \int_X d^4x \sqrt{g} \left(\left(\frac{-1}{\overline{\tau}} \right) (W^+)^2 - \left(\frac{-1}{\tau} \right) (W^-)^2 \right). \quad (2.22)$$

This is the original Lagrangian (2.4), but with $\tau \to -1/\tau$.

So far, we have integrated out various fields classically, without taking account of certain determinants that will give the modular anomaly. To see the anomaly, we will now go over some of the above steps in a more precise way. Let $I_\tau(A)$ be the Maxwell action (2.4) with coupling parameter τ. Then we define the partition function of the original theory to be

$$Z(\tau) = (\mathrm{Im}\,\tau)^{\frac{1}{2}(B_1 - B_0)} \frac{1}{\mathrm{Vol}(G)} \int DA\, e^{-I_\tau(A)}. \quad (2.23)$$

Here $\mathrm{Vol}(G)$ is the volume of the group of gauge transformations, and B_k is as before the dimension of the space of k-forms. Thus, in (2.23) we are simply including a factor of $\mathrm{Im}\,\tau^{1/2}$ for every integration variable and a factor of $\mathrm{Im}\,\tau^{-1/2}$ for every gauge generator, to cancel what would otherwise be the cutoff-dependence of the path integral. (We have already seen that the recipe in (2.23) leads to a cutoff-independent power of $\mathrm{Im}\,\tau$ in the final result for the partition function.) Of course, the partition function can equally well be written in dual variables

$$Z(\tau) = (\mathrm{Im}\,\tau)^{\frac{1}{2}(B_1 - B_0)} \frac{1}{\mathrm{Vol}(\widetilde{G})} \int DV\, e^{-I_\tau(V)}, \quad (2.24)$$

with $\widetilde{G}$ the dual gauge group.

The above derivation began with the fact that $Z(\tau)$ can alternatively be defined by

$$Z(\tau) = (\mathrm{Im}\,\tau)^{\frac{1}{2}(B_1 - B_0)} \frac{1}{\mathrm{Vol}(K) \times \mathrm{Vol}(\widetilde{G})} \int DA\,DG\,DV\,e^{-\hat{I}_\tau(A,G,V)} \qquad (2.25)$$

where $\hat{I}_\tau$ is the extended Lagrangian in (2.19) with coupling parameter τ, and K is the extended gauge group. Note that as the V-dependent part of $\hat{I}$ is independent of τ, one gets no factors of τ in integrating over V; the V integral gives a τ-independent delta function by means of which the G integral can be done without generating any powers of $\mathrm{Im}\,\tau$. That is why the same power of $\mathrm{Im}\,\tau$ appears in (2.25) as in (2.23).

On the other hand, suppose that we evaluate (2.25) by gauging A to zero and then integrating over G. One gets a factor of $\overline{\tau}^{-1/2}$ or $\tau^{-1/2}$ from the integral over any mode of G^+ or G^-, so altogether the τ-dependent factor that one obtains from the G integral is $\tau^{-\frac{1}{2}B_2^-}\overline{\tau}^{-\frac{1}{2}B_2^+}$ with now $B_2^\pm$ the dimensions of the spaces of self-dual and anti-self-dual two-forms. Now eliminating V and G in (2.25) actually gives (apart from a possible τ-independent factor)

$$Z(\tau) = (\mathrm{Im}\,\tau)^{\frac{1}{2}(B_1 - B_0)} \tau^{-\frac{1}{2}B_2^-}\overline{\tau}^{-\frac{1}{2}B_2^+} \frac{1}{\mathrm{Vol}(\widetilde{G})} \int DV\,e^{-I_{-1/\tau}(V)}. \qquad (2.26)$$

Comparing this to (2.24), we get

$$Z(\tau) = \tau^{-\frac{1}{2}(B_0 - B_1 + B_2^-)}\overline{\tau}^{-\frac{1}{2}(B_0 - B_1 + B_2^+)} Z(-1/\tau). \qquad (2.27)$$

With $B_0 - B_1 + B_2^\pm = b_0 - b_1 + b_2^\pm = (\chi \pm \sigma)/2$, this is equivalent to

$$Z(\tau) = \tau^{-\frac{1}{4}(\chi - \sigma)}\overline{\tau}^{-\frac{1}{4}(\chi + \sigma)} Z(-1/\tau), \qquad (2.28)$$

and thus as we found earlier, Z is modular (for $SL(2, \mathbf{Z})$ or a subgroup) with weights $((\chi - \sigma)/4, (\chi + \sigma)/4)$.

2.3 Full Modular Invariance? Finally, one might wonder how one must modify the construction if one wants to obtain $SL(2, \mathbf{Z})$ covariance (and not just covariance under a subgroup) even when X is not a spin manifold.

In fact, we will embed this question in a somewhat broader question of abelian gauge theory with gauge group $U(1)^n$ for arbitrary positive integer n. We introduce $n\,U(1)$ gauge fields A^I, $I = 1, \ldots, n$, with curvatures $F^I = dA^I$. Picking a positive quadratic form on the Lie algebra of $U(1)^n$ — which we represent by a symmetric positive definite matrix d_{IJ} — we take the Lagrangian to be

$$\begin{aligned}
&\frac{1}{8\pi} \sum_{I,J} d_{IJ} \int_X d^4x \sqrt{g} \left(\frac{4\pi}{g^2} F^I_{mn} F^{Jmn} + \frac{i\theta}{2\pi} \frac{1}{2} \epsilon_{mnpq} F^{Imn} F^{Jpq} \right) \\
&= \frac{i}{8\pi} \sum_{I,J} d_{IJ} \int_X d^4x \sqrt{g} \left(\overline{\tau} F^{I+}_{mn} F^{J+mn} - \tau F^{I-}_{mn} F^{J-mn} \right).
\end{aligned} \qquad (2.29)$$

In the space of F^I's there is an integral structure given by the fact that, for S a closed two-surface in X, the quantities

$$\int_S \frac{F^I}{2\pi} \tag{2.30}$$

are integers. Thus, d can be considered to define a quadratic form on a certain lattice Γ.

To implement duality, we extend the gauge-invariance as before, adding two-form fields G^I and replacing F^I by $\mathcal{F}^I = F^I - G^I$. Also, we add dual gauge fields V_I, with curvatures $W_I = dV_I$. We introduce an extended action which is the obvious generalization of (2.19):

$$I = \frac{i}{8\pi} \int_X d^4x \sqrt{g}$$
$$\left(\sum_I \epsilon^{mnpq} W_{Imn} G^I_{pq} + \sum_{I,J} d_{IJ} \left(\bar{\tau} \mathcal{F}^{I+}_{mn} \mathcal{F}^{J+mn} - \tau \mathcal{F}^-_{Imn} \mathcal{F}^{J-mn} \right) \right). \tag{2.31}$$

Obvious generalizations of the previous manipulations show, on the one hand, that after integrating out V and G, the theory defined by (2.31) is equivalent to the theory defined by (2.29), and on the other hand, that after picking the gauge $A = 0$ and integrating out G, (2.31) can be replaced by the dual Lagrangian

$$\frac{1}{8\pi} \sum_{I,J} d^{IJ} \frac{i}{8\pi} \int_X d^4x \sqrt{g} \left(\left(\frac{-1}{\bar{\tau}} \right) W^+_{Imn} W^{+mn}_J - \left(\frac{-1}{\tau} \right) W^-_{Imn} W^{-mn}_J \right). \tag{2.32}$$

Here d^{IJ} is the inverse matrix to d_{IJ}.

Now we can easily determine the conditions for modular covariance. The transformation $\tau \to -1/\tau$ maps (2.29) to (2.32), so one has covariance under this transformation if and only if the lattice Γ with quadratic form d is equivalent to the dual lattice with quadratic form d^{-1}, that is if and only if the quadratic form defined by d is self-dual. If this quadratic form is also *integral*, one also has invariance under $\tau \to \tau + 2$ for any X, or $\tau \to \tau + 1$ if X is spin, by an argument given earlier in our original discussion of $U(1)$ gauge theory. If one wants invariance under $\tau \to \tau + 1$ for *any* X, not necessarily spin, one must pick the quadratic form to be *even*, not just integral. This ensures that

$$\frac{1}{16\pi^2} \sum_{I,J} d_{IJ} \int_X d^4x \sqrt{g} \epsilon^{mnpq} F^I_{mn} F^J_{pq} \tag{2.33}$$

is divisible by two whether or not X is spin so that the Lagrangian (2.29) is invariant mod $2\pi i$ under $\theta \to \theta + 2\pi$, that is, under $\tau \to \tau + 1$. It is curious that even integral lattices are thus related to modularity in four dimensions as they are in two dimensions.

§3.   A New Effective Interaction On The u Plane

In [9], the low energy dynamics of the pure $N = 2$ supersymmetric gauge theory with gauge group $SU(2)$ was determined. A familiarity with that discussion will be assumed here, but a few points will be repeated to fix notation. The basic order parameter in [9] was $u = \text{Tr } \phi^2$ (ϕ being the scalar field related by supersymmetry to the gauge field); the variables in the low energy theory were u and its superpartners. In the low energy theory, it is natural to count dimensions in such a way that u has dimension zero and its fermionic partners dimension one-half. With this way of counting, as long as one is on flat $\mathbf{R}^4$, the supersymmetric interactions of lowest dimension have dimension two. All interactions of dimension two were determined in [9] in terms of functions associated with the family of elliptic curves

$$y^2 = (x^2 - 1)(x - u). \tag{3.1}$$

If one works on a curved four-manifold, local interactions of dimension zero — involving couplings of u to polynomials in the Riemann tensor and its derivatives — become possible. In the physical theory on a four-manifold, a rather complicated structure might be possible in general, but in the twisted topological theory the only possible such interactions are proportional to $\int_X f(u) \text{ tr } R \wedge \widetilde{R}$ or $\int_X g(u) \text{ tr } R \wedge R$ where $\text{tr } R \wedge \widetilde{R}$ and $\text{tr } R \wedge R$ are the densities whose integrals are proportional to χ and σ, respectively, and the functions f and g are holomorphic.[2] Such interactions produce in the path integral measure a factor which if u is constant simply takes the form

$$\exp\left(b(u)\chi + c(u)\sigma\right). \tag{3.2}$$

In the present section, we will determine the functions b and c. These interactions are closely related to similar interactions that appear in the untwisted physical theory on a four-manifold, but the precise relation will not be analyzed here.

The reasons for performing this computation and presenting the result here are as follows:

(1) The computation turns out to involve the result of the last section in an interesting way.

(2) The ability to get a unique and consistent result for b and c provides an interesting new check on the formalism of [9].

(3) The result is needed for integrating over the u plane in Donaldson theory (in order to obtain formulas for Donaldson invariants of four-manifolds of $b_2^+ \leq 1$), though this application will not be developed in the present paper.

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2)   BRST-invariant configurations have u constant; for constant u a function $\int T(u)\mathcal{O}\sqrt{g}d^4x$, with $\mathcal{O}$ a locally constructed function of the metric, is a topological invariant if and only if $\mathcal{O}$ is related to the Euler characteristic or the signature or is an irrelevant total derivative. Moreover, BRST invariance holds if and only if T is holomorphic.

3.1 Asymptotic Behavior. The obvious way to proceed (temporarily overlooking a subtlety that will presently appear) is as follows. Singularities and zeroes in the holomorphic function (3.2) at finite places on the u plane can only occur at $u = \pm 1$, where extra massless particles (monopoles and dyons) appear, giving rise to singularities in various physical quantities, including those computed in [9]. (Note that either a zero or a pole in the function (3.2) is associated with a singularity in b or c and so is possible only at $u = 1, -1$, or ∞.) The behavior of the unknown function can be obtained for $u \to \infty$ using asymptotic freedom and weak coupling, and for $u \to \pm 1$ using a knowledge of which particles are becoming massless at those points. It would appear that knowledge of the behavior at infinity and at the singularities would determine the sought-for holomorphic function uniquely except for the possibility of adding constants to b or c. Let us see how this program works out.

First we consider the behavior for large u. Let us recall that the $N = 2$ theory has an anomalous $U(1)_R$ symmetry which in a field of instanton number k, on a four-manifold of given χ and σ, has an anomaly

$$\Delta R = 8k - \frac{3}{2}(\chi + \sigma). \tag{3.3}$$

(That is, the operators with non-zero expectation value have R-charge equal to ΔR.) The term in this formula of interest here is the part involving the coupling to gravity, namely

$$-\frac{3}{2}(\chi + \sigma). \tag{3.4}$$

This has the following interpretation: the index theorem is such that for each generator of the gauge group there is an anomaly $-(\chi + \sigma)/2$; an extra factor of three arises because $SU(2)$ is three-dimensional.

For large u, $SU(2)$ is spontaneously broken to $U(1)$. The zero modes of the fermion fields in the u multiplet carry the anomaly $-(\chi+\sigma)/2$ of the one-dimensional unbroken part of the group. The remaining anomaly $-(\chi + \sigma)$ must be manifested in an interaction proportional to χ and σ obtained by integrating out the massive $SU(2)$ partners of the light fields. This interaction must have no derivatives (or it would not lead to violation of the symmetry when u is constant) so it is of the form (3.2). We can therefore determine the large u behavior of (3.2): as u has R-charge four, we need

$$e^{b\chi + c\sigma} \sim u^{(\chi+\sigma)/4} \text{ for } u \to \infty. \tag{3.5}$$

The behavior near $u = \pm 1$ can be determined similarly. The effective theory near $u = 1$ has an accidental low energy R symmetry. The anomaly in this symmetry is

$$-\frac{1}{2}(\chi + \sigma) + \frac{c_1(L)^2}{4} - \frac{\sigma}{4}. \tag{3.6}$$

Here $-(\chi + \sigma)/2$ is the contribution from the u-multiplet and is carried by the zero modes related to u by supersymmetry, and $c_1(L)^2/4 - \sigma/4$ is the contribution of the monopoles that become massless at $u = 1$. The gravitational part of the monopole contribution is $-\sigma/4$ and must show up in a singular behavior of the interaction $e^{b\chi + c\sigma}$ obtained by integrating out the light monopoles. (The $c_1(L)^2/4$ appears in the behavior of the effective gauge couplings near $u = 1$.) As $u - 1$ has charge two under the R symmetry near $u = 1$, the singular behavior is

$$e^{b\chi + c\sigma} \sim (u - 1)^{\sigma/8} \text{ for } u \to 1. \tag{3.7}$$

By a similar argument, the singular behavior near $u = -1$, obtained by integrating out light dyons that again violate the effective R-symmetry by $-\sigma/4$, must be

$$e^{b\chi + c\sigma} \sim (u + 1)^{\sigma/8} \text{ for } u \to -1. \tag{3.8}$$

We are left then looking for a holomorphic function with the singularities and zeroes given in (3.5), (3.7), and (3.8). For the terms involving σ there is an evident and unique formula:

$$e^{c\sigma} = (u^2 - 1)^{\sigma/8}. \tag{3.9}$$

(When this is multiplied by the partition function of the massless photon, allowing for the fact that as explained in section four the "dual line bundle" is really a Spin^c structure, the branch cuts that are present if σ is not divisible by eight disappear.) But for the terms involving χ we face a quandary: no holomorphic function $e^{b\chi}$ can have the asymptotic behavior given by (3.5) and the lack of singularities or zeroes indicated in the subsequent formulas.

3.2 Modular Anomaly. To unravel this puzzle, we have to recall in somewhat more depth the structure found in [9].

There is no completely canonical way to describe the low energy effective action of this theory. Over the u-plane punctured at $u = 1, -1$, there is a flat $SL(2, \mathbf{Z})$ bundle E, the fiber at a point u being the first cohomology group of the elliptic curve described in equation (3.1). Every local trivialization of E gives a way of writing an effective action; different trivializations give different formulas related by $SL(2, \mathbf{Z})$ transformations.

Periods of a certain differential form (described precisely in [9], section 6) give a certain holomorphic section (a_D, a) of u. Either a_D or a (or any integer linear combination) can be taken as the basic variable in the low energy description; they are related by supersymmetry to gauge fields A_D and A respectively. A_D and A are dual gauge fields related by the duality transformation of the previous section. The transformation from a to a_D as the basic variable is a special case of an $SL(2, \mathbf{Z})$ transformation, achieved by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a \\ -a_D \end{pmatrix}. \tag{3.10}$$

The gauge coupling constant is expressed in terms of a_D and a by

$$\tau = \frac{da_D}{da}. \tag{3.11}$$

It therefore transforms under (3.10) by

$$\tau \to -\frac{1}{\tau}. \tag{3.12}$$

As we have seen in the last section, this is how the coupling transforms when one changes from the description by A to the description by A_D.

Now let us discuss how the *quantum* low energy theory transforms under duality. In brief, we will find an anomaly that will precisely cancel the difficulty that we had above in determining the function $e^{b\chi}$. Since the dimension two part of the classical action is duality-invariant (this being part of the construction in [9]) and since the operators (such as u) whose correlations one wishes to compute are also completely duality invariant, the question of invariance of the quantum theory under duality amounts to the question of how the quantum integration measure transforms under duality and how the function $e^{b\chi}$ transforms. We really only need to consider the special modular transformation $\tau \to -1/\tau$, since $SL(2, \mathbf{Z})$ is generated by this transformation together with $\tau \to \tau + 1$, which acts trivially on the integration variables and thus produces no anomaly in the measure. To understand what happens under $\tau \to -1/\tau$, one must examine three things:

(1) The integration measure of a and $\bar{a}$ (or a_D and $\bar{a}_D$).

(2) The fermion integration measure.

(3) The integration measure of the gauge fields.

We will work out the behavior of the three measures under duality as follows.

(1) Because the kinetic energy of a and $\bar{a}$ is proportional to $\mathrm{Im}\,\tau$, the integration measure for those fields has the form

$$\mathrm{Im}\,\tau \; da \, d\bar{a}. \tag{3.13}$$

(This is true for every mode of $a, \bar{a}$, and so most importantly, it is true for the zero modes.) Setting $\tau_D = -1/\tau$, one has

$$\mathrm{Im}\,\tau \, da \, d\bar{a} = \mathrm{Im}\,\tau_D \; da_D \, d\bar{a}_D \tag{3.14}$$

so the integration measure for these fields is completely duality invariant.

(2) In contrast to the scalars, which have just been seen to have no modular anomaly, the fermions do have such an anomaly. The reason for this is that in [9],

duality acts on the fermions via a chiral transformation, which has an anomaly on a curved four-manifold.

The details can be worked out as follows. Like that of the scalars, the fermion kinetic energy is proportional to $\operatorname{Im}\tau$, so the integration measure for any normalized fermi mode β is

$$d\left(\sqrt{\operatorname{Im}\tau}\,\beta\right) = \frac{d\beta}{\sqrt{\operatorname{Im}\tau}}. \tag{3.15}$$

The fermi modes can be divided into modes of $R = 1$ and $R = -1$, which we will generically call α and $\overline{\alpha}$, respectively.[3] It will be evident from the structure of what follows that only zero modes need to be considered, the other fermion modes canceling in pairs. The duality transformation in [9] was such that under $\tau \to -1/\tau$, α transforms to $\alpha_D = \tau\alpha$ and $\overline{\alpha}$ transforms to $\overline{\alpha}_D = \overline{\tau}\,\overline{\alpha}$. Hence

$$\frac{d\alpha}{\sqrt{\operatorname{Im}\tau}} = \sqrt{\frac{\tau}{\overline{\tau}}}\,\frac{d\alpha_D}{\sqrt{\operatorname{Im}\tau_D}}$$

$$\frac{d\overline{\alpha}}{\sqrt{\operatorname{Im}\tau}} = \sqrt{\frac{\overline{\tau}}{\tau}}\,\frac{d\overline{\alpha}_D}{\sqrt{\operatorname{Im}\tau_D}}. \tag{3.16}$$

Thus if $d\mu^F$ is the fermion measure for $\alpha,\overline{\alpha}$ and $d\mu_D^F$ is the measure for $\alpha_D,\overline{\alpha}_D$, one has

$$d\mu^F = \left(\frac{\tau}{\overline{\tau}}\right)^{-(\chi+\sigma)/4} d\mu_D^F, \tag{3.17}$$

using the fact that the number of α zero modes minus the number of $\overline{\alpha}$ zero modes is $-(\chi + \sigma)/2$.

(3) The transformation law for the gauge measure $d\mu^G$ is the most subtle of the three and can be read from (2.28):

$$d\mu^G = \tau^{-\frac{1}{4}(\chi-\sigma)}\overline{\tau}^{-\frac{1}{4}(\chi+\sigma)} d\mu_D^G. \tag{3.18}$$

Actually, the formulation in (3.18) is imprecise as $d\mu^G$ and $d\mu_D^G$ are defined in different spaces. This statement simply means that the integral of μ^G, that is the partition function, differs from the integral of μ_D^G by the stated factor.

Multiplying the factors found in the above equations, the relation between the measure $d\mu$ of the theory using a as the basic variable and the measure $d\mu_D$ of the theory using a_D is

$$d\mu = \tau^{-\chi/2} d\mu_D. \tag{3.19}$$

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3)   In the twisted theory, α is a one-form and $\overline{\alpha}$ a linear combination of a zero-form and a self-dual two-form.

The fact that the $\bar{\tau}$ dependence cancels here is crucial in making it possible to cancel this anomaly by the holomorphic term considered below.

3.3 Final Determination. It is now clear that $e^{b\chi}$ should not be a function of u but rather should transform under duality as a holomorphic modular form of weight $-\chi/2$. The fact that the weight depends on χ and not on σ is the reason that above we found a straightforward determination of the function $e^{c\sigma}$ but not a straightforward determination of $e^{b\chi}$. With the modular anomaly understood, it is now easy to find the "function" $e^{b\chi}$:

$$e^{b\chi} = \left((u^2 - 1) \frac{d\tau}{du} \right)^{\chi/4}. \tag{3.20}$$

The first point of this formula is that if likewise

$$e^{b_D\chi} = \left((u^2 - 1) \frac{d\tau_D}{du} \right)^{\chi/4} \tag{3.21}$$

with $\tau_D = -1/\tau$, then

$$e^{b\chi} = \tau^{\chi/2} e^{b_D\chi}, \tag{3.22}$$

so including in the theory a factor of $e^{b\chi}$ cancels the modular anomaly. Also, for $u \to \infty$, $d\tau/du \sim 1/u$ according to [9], so (3.20) has the large u behavior required in (3.5).

We still need to check (3.7), but first we must interpret it correctly. (3.7) was based on a computation near $u = 1$, where the good description is in terms of a_D. (For u away from the punctures at $u = \pm 1$ and the singularity at ∞, one can use either a or a_D.) Therefore, (3.7) must be interpreted as a condition on the behavior of $e^{b_D\chi}$ near $u = 1$, and this condition is obeyed since according to [9] one has $d\tau_D/du \sim 1/(u - 1)$ near $u = 1$, ensuring that $e^{b_D\chi}$ has no singularity at $u = 1$. Likewise, after transforming to the appropriate local description, one finds the desired absence of singularity near $u = -1$. (The local description near $u = -1$ is [9] a third one using $a + a_D$ as the basic variable.)

The only point that remains is that the function $d\tau/du$ has neither zeroes nor poles for finite u away from $u = \pm 1$. This is so because the family of curves (3.1) is the modular curve of the subgroup $\Gamma(2)$ of $SL(2, \mathbf{Z})$ (consisting of matrices congruent to one modulo two), and there are no orbifold points in the moduli space.

Given that our candidate for $e^{b\chi}$ has the correct behavior near $u = 1, -1$, and ∞ and has no unwanted zeroes or poles, it must be the correct answer (up to a multiplicative constant), since any other candidate would be obtained by multiplying by an ordinary meromorphic function which if not constant would have unwanted zeroes and poles somewhere.

In sum, in any computation that involves an integration on the u-plane, the more obvious factors in the path integral must be supplemented by a new interaction that gives an additional factor

$$e^{b\chi + c\sigma} = \left((u^2 - 1)\frac{d\tau}{du} \right)^{\chi/4} (u^2 - 1)^{\sigma/8}. \tag{3.23}$$

This factor will enter in computations of Donaldson invariants for $b_2^+ \leq 1$.

§4. Abelian Duality Embedded In $SU(2)$ And $SO(3)$

In this section, we will explain a few subtleties about the duality transformation which — in $N = 2$ super Yang-Mills theory with gauge group $SU(2)$ or $SO(3)$ — relates the description in variables appropriate at large u to the description valid near $u = 1$. This will enable us to explain some assertions made in [10] and will serve as background for a further discussion which will appear elsewhere.

First we must discuss how the $U(1)$ duality considered in section two is embedded in $SU(2)$ or $SO(3)$. To begin with, we assume that the gauge group is $SU(2)$ with rank two gauge bundle F. At a generic point of the u plane, $SU(2)$ is broken down to $U(1)$, and F splits as $F = T \oplus T^{-1}$ with T a line bundle. The gauge field at low energies then reduces to a $U(1)$ gauge field C, which one can think of as a connection on T.

However, we want to be free to consider the case that the gauge group is actually $SO(3)$, with a rank three gauge bundle E which may have $w_2(E) \neq 0$. With the symmetry broken to $U(1)$, E decomposes at low energies as $\mathcal{O} \oplus L \oplus L^{-1}$, with $\mathcal{O}$ a trivial bundle and L a line bundle. The connection reduces at low energies to a $U(1)$ connection A on L. If $w_2(E) = 0$, the gauge group can be lifted to $SU(2)$ and the rank two bundle F exists; in that case $L = T^{\otimes 2}$ and $A = 2C$.

Near $u = \pm 1$, one has instead a description using a dual gauge field V and dual "line bundle" $\widetilde{L}$.[4]

There are two goals in this section:

(1) To explain how the description near $u = 1$ (or -1) depends on $w_2(E)$.
(2) To explain by what mechanism it turns out that the "line bundle" in the dual description near $u = 1$ is really not a line bundle but a Spin^c structure.

With the first goal in mind, it is clearly suitable to choose variables adapted to the possibility that $w_2(E) \neq 0$, so we will take the basic variable to be A rather than C. Setting $F = dA$, let us now describe the Dirac quantization condition on F

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4) Note that $\widetilde{L}$ was simply called L in [10] – the tilde was deleted as the "original" line bundle L never entered explicitly in that paper.

for general w_2. For simplicity in exposition, we suppose that there is no torsion in the cohomology of X so that we can pick a basis of two-dimensional closed surfaces U_α giving a basis of $H_2(X, \mathbf{Z})$. Then $w_2(E)$ can be described by the conditions

$$(w_2(E), U_\alpha) = c_\alpha \tag{4.1}$$

with each $c_\alpha = 0$ or 1.

The Dirac quantization condition asserts that

$$\int_{U_\alpha} F = 2\pi(2k_\alpha + c_\alpha), \tag{4.2}$$

with $k_\alpha \in \mathbf{Z}$. The idea behind this formula is that if $C = A/2$, then Dirac quantization for C fails precisely for those U_α for which $c_\alpha = 1$.

4.1 Dependence On w_2. We will now determine how the dual description near $u = 1$ (or -1) depends on $w_2(E)$. As in [9], we take the starting Lagrangian for A to be

$$I = \frac{i}{16\pi} \int_X d^4x \sqrt{g} \left(\overline{\tau}(F^+)^2 - \tau(F^-)^2 \right). \tag{4.3}$$

(One can think of this as (2.29) adapted to the root lattice of $SU(2)$, which is generated by a point of length squared two; that is, (2.29) turns into (4.3) if one sets $d = 2$ and renames the variable called A in (2.29) as $C = A/2$. That rescaling also turns the standard Dirac condition assumed in discussing (2.29) to the special case $c_\alpha = 0$ of (4.2).)

Now we introduce a dual $U(1)$ gauge field V, which is a connection on a dual line bundle $\widetilde{L}$. We take the curvature $W = dV$ to obey standard Dirac quantization, that is $\int_{U_\alpha} W$ is an arbitrary integer multiple of 2π without any refinement such as that in (4.2). We also introduce a two-form G and the extended gauge invariance

$$\begin{aligned} A &\to A + 2B \\ G &\to G + 2dB \end{aligned} \tag{4.4}$$

with B a connection on an arbitrary line bundle N. The reason for the factor of two in (4.4) is to preserve the structure of (4.2); that is, with this factor of two in the gauge transformation law, $\int_{U_\alpha} F$ is gauge invariant modulo 4π. Rather as in section two, we introduce the gauge invariant object $\mathcal{F} = F - G$ and the extended Lagrangian

$$I' = \frac{i}{16\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} W_{mn} G_{pq} + \frac{i}{16\pi} \int_X d^4x \sqrt{g} \left(\overline{\tau}(\mathcal{F}^+)^2 - \tau(\mathcal{F}^-)^2 \right) \tag{4.5}$$

This has been chosen so that the theory defined by I' is equivalent to the theory defined by I. To prove this, as in section two, one integrates over V, obtaining a delta function that sets G to zero up to a gauge transformation; one uses the fact that the periods of G are gauge-invariant modulo 4π.

To obtain a dual description, the first step in section two was to use the extended gauge invariance to set $A = 0$. Now we cannot do that (unless $w_2(E) = 0$) because of the factor of two in (4.4). The best that we can do is to select a fixed set of line bundles Q_α, with connections θ_α and curvatures g_α, such that

$$\int_{U_\alpha} g_\beta = 2\pi\delta_{\alpha\beta}, \tag{4.6}$$

and use the extended gauge invariance to set $A = \sum_\alpha c_\alpha\theta_\alpha$. Then, by shifting G by $G \to G + \sum_\alpha c_\alpha g_\alpha$, the extended Lagrangian turns into

$$\begin{aligned} I' =& \frac{i}{16\pi}\int_X d^4x\sqrt{g}\epsilon^{mnpq}W_{mn}(G_{pq} - c_\alpha g_{\alpha\,pq}) \\ & + \frac{i}{16\pi}\int_X d^4x\sqrt{g}\left(\overline{\tau}(G^+)^2 - \tau(G^-)^2\right). \end{aligned} \tag{4.7}$$

Integrating out G now gives

$$\begin{aligned} I'' =& \frac{i}{16\pi}\int_X d^4x\sqrt{g}\left(\left(\frac{-1}{\overline{\tau}}\right)(W^+)^2 - \left(\frac{-1}{\tau}\right)(W^-)^2\right) \\ & - \frac{i}{16\pi}\sum_\alpha c_\alpha\int_X d^4x\sqrt{g}\epsilon^{mnpq}W_{mn}g_{\alpha\,pq}. \end{aligned} \tag{4.8}$$

The c_α thus appear only in the last term, which is a topological invariant, and moreover is always an integral multiple of πi so that its exponential is ± 1. In fact, the exponential of the last term is

$$(-1)^{(c_1(\widetilde{L}), w_2(E))}. \tag{4.9}$$

This factor determines the dependence of the dual Lagrangian on $w_2(E)$. It must be compared to the factor called $(-1)^{x' \cdot z}$ in eqn. (2.17) of [10], which was claimed to determine the dependence of the Donaldson invariants on $w_2(E)$. In [10], z was defined as $w_2(E)$, and the definition of x' was such that if $w_2(X) = 0$, then $x' = -c_1(\widetilde{L})$.[5] So for spin manifolds, $(-1)^{x' \cdot z}$ coincides with (4.9).

To generalize this discussion to manifolds that are not spin, there are additional subtleties to which we now turn.

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5) In fact, x was defined in [10] by $x = -2c_1(\widetilde{L})$, and x' by $2x' = x + w_2(X)$, so if $w_2(X) = 0$, $x' = -c_1(\widetilde{L})$ and $(-1)^{x' \cdot z}$ coincides with the formula in (4.9).

4.2 Appearance Of Spin^c **Structures.** The low energy theory near $u = 1$ has light magnetic monopoles. In the topologically twisted version which leads to the considerations in [10], the "monopoles," if $w_2(X) = 0$, are sections of $S_+ \otimes \widetilde{L}$, where S_+ is the positive chirality spin bundle and $\widetilde{L}$ is the dual line bundle. If $w_2(X) \neq 0$, then S_+ does not exist and it is clear, pragmatically, that the monopoles must be sections of a Spin^c bundle that we can informally write as $S_+ \otimes \widetilde{L}$ but is no longer really defined as a tensor product. (A precise description of this situation is that $\widetilde{L}^{\otimes 2}$ is an ordinary line bundle such that $x = -c_1(\widetilde{L}^{\otimes 2})$ is congruent modulo two to $w_2(X)$; and $S_+ \otimes \widetilde{L}$ is a Spin^c bundle with an isomorphism $\wedge^2(S_+ \otimes \widetilde{L}) \cong \widetilde{L}^{\otimes 2}$.)

But how do Spin^c structures arise at $u = 1$ starting with the underlying $SU(2)$ gauge theory on the u plane? Our next goal is to explain how the duality transformation that leads from the variables appropriate at large u to the variables appropriate near $u = 1$ can in fact generate a Spin^c structure near $u = 1$. So far the dual object V has always been a connection on a line bundle $\widetilde{L}$; we want to see how instead a Spin^c structure can arise under duality.

Let us return to the starting point, and add to (4.3) an additional interaction

$$\frac{i}{32\pi} \sum_\alpha e^\alpha \int_X d^4x \sqrt{g} \epsilon^{mnpq} F_{mn} g_{\alpha\,pq} \tag{4.10}$$

with integers e^α. The origin of this interaction will be explained at the end of this section.

To carry out a duality transformation, we, as always, introduce the two-form G, and replace F by $\mathcal{F} = F - G$. Then the extended Lagrangian becomes

$$I' = \frac{i}{16\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} W_{mn} G_{pq} + \frac{i}{16\pi} \int_X d^4x \sqrt{g} \left(\overline{\tau}(\mathcal{F}^+)^2 - \tau(\mathcal{F}^-)^2 \right) $$
$$+ \frac{i}{32\pi} \sum_\alpha e^\alpha \int_X d^4x \sqrt{g} \epsilon^{mnpq} \mathcal{F}_{mn} g_{\alpha\,pq}. \tag{4.11}$$

First we consider the case that $w_2(E) = c_\alpha = 0$. This will enable us to see the appearance of Spin^c structures without extraneous complications. After gauging A to zero using the extended gauge invariance, the extended Lagrangian becomes

$$I' = \frac{i}{16\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} W_{mn} G_{pq} + \frac{i}{16\pi} \int_X d^4x \sqrt{g} \left(\overline{\tau}(G^+)^2 - \tau(G^-)^2 \right) $$
$$- \frac{i}{32\pi} \sum_\alpha e^\alpha \int_X d^4x \sqrt{g} \epsilon^{mnpq} G_{mn} g_{\alpha\,pq}. \tag{4.12}$$

We can eliminate the last term if we replace V by

$$\widetilde{V} = V - \frac{1}{2} \sum_\alpha e^\alpha \theta_\alpha. \tag{4.13}$$

Proceeding with the rest of the derivation, we will arrive at the same dual Lagrangian in terms of $\widetilde{V}$ that we have previously had in terms of V. This does not mean that the e^α play no material role. Because of the $1/2$ in (4.13), $\widetilde{V}$ is not a connection on a line bundle in the usual sense (unless the e^α are all even) so we have obtained the same Lagrangian, but for a different kind of object.

In fact, as we will see at the end of this section, the peculiar interaction (4.10) is actually present in this theory with very special e^α. To be precise, $\sum_\alpha e^\alpha g_\alpha/2\pi$ represents $w_2(X)$ modulo two:

$$w_2(X) = \sum_\alpha e^\alpha \left[\frac{g_\alpha}{2\pi}\right] \text{ mod } 2, \tag{4.14}$$

with $[g/2\pi]$ the cohomology class of the differential form $g/2\pi$. Although the $1/2$ in (4.13) means that there is not really a "dual line bundle" $\widetilde{L}$ with connection $\widetilde{V}$, the fact that the obstruction is $w_2(X)$ means that there is a Spin^c structure $S_+ \otimes \widetilde{L}$ (the central part of whose curvature is $\widetilde{W} = d\widetilde{V}$). In addition, $\widetilde{L}^{\otimes 2}$ exists as an ordinary line bundle, so $x = -c_1(\widetilde{L}^{\otimes 2})$ makes sense as an integral cohomology class. Thus explaining the origin of (4.10) will also explain the appearance of Spin^c structures in the dual theory near $u = 1$.

4.3 Combining The Two Effects.
So far we have studied two effects: we introduced a non-zero $w_2(E)$ by shifting the underlying Dirac quantization law as in (4.2), and we allowed for the possibility that $w_2(X) \neq 0$ by adding a new interaction (4.10). It is also possible — and natural — to combine the two effects. We return to the extended Lagrangian (4.11) but use the general Dirac quantization law of (4.2). We pick the gauge $A = \sum_\alpha c_\alpha \theta_\alpha$ and shift G by $G \to G + \sum_\alpha c_\alpha g_\alpha$. Instead of (4.12) we then get

$$\begin{aligned}
I' = &\frac{i}{16\pi} \int_X d^4 x \sqrt{g} \epsilon^{mnpq} W_{mn}(G_{pq} + \sum_\alpha c_\alpha g_{\alpha\,pq}) \\
&+ \frac{i}{16\pi} \int_X d^4 x \sqrt{g} \left(\overline{\tau}(G^+)^2 - \tau(G^-)^2\right) \\
&- \frac{i}{32\pi} \sum_\alpha e^\alpha \int_X d^4 x \sqrt{g} \epsilon^{mnpq} G_{mn} g_{\alpha\,pq}.
\end{aligned} \tag{4.15}$$

And we remove the last term by replacing V with

$$\widetilde{V} = V - \frac{1}{2} \sum_\alpha e^\alpha \theta_\alpha, \tag{4.16}$$

so that the dual description involves a Spin^c structure rather than a line bundle.

The dependence on $w_2(E)$ can now be found as follows. If we let $\widetilde{W} = d\widetilde{V}$, then the terms in (4.15) that depend on the c_α, that is on $w_2(E)$, are

$$\Delta L = \frac{i}{16\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} \left(\widetilde{W}_{mn} + \frac{1}{2} \sum_\beta e^\beta g_{\beta\,mn} \right) \cdot \sum_\alpha c_\alpha g_{\alpha\,pq}. \qquad (4.17)$$

The dependence of the Donaldson invariants on $w_2(E)$ will then be determined by a factor $e^{-\Delta L}$. We claim that this factor coincides with the one given in [10], or in other words that

$$e^{-\Delta L} = (-1)^{x' \cdot z} \qquad (4.18)$$

in the notation of [10], eqn. (2.17). To justify this claim, note that the differential form $(2\widetilde{W} + \sum_\alpha e^\alpha g_\alpha)/2\pi$ represents the cohomology class $c_1(\widetilde{L}^{\otimes 2}) - w_2(X)$ (with a particular integral lift of $w_2(X)$, namely $-\sum e^\alpha g_\alpha/2\pi$), so can be identified with $-2x'$ in the notation of [10]. With also $\sum_\alpha c_\alpha g_\alpha/2\pi$ as an integral lift of $z = w_2(E)$, (4.18) follows from (4.17).

Thus modulo the assumption that on the u-plane there is an interaction (4.10), we have accounted via duality for the dependence of the Donaldson invariants on $w_2(E)$ as well as explaining the origin of Spin^c structures. It remains to explain the origin of (4.10).

4.4 A Curious Minus Sign.
What remains is to explain the presence in the effective action on the u-plane of the term that played a crucial role above, namely

$$W = \frac{i}{32\pi} \int_X d^4x \sqrt{g} \epsilon^{mnpq} F_{mn} H_{pq}, \qquad (4.19)$$

where $H/2\pi$ (written above as $\sum_\alpha e^\alpha g_\alpha/2\pi$) represents in de Rham cohomology an integral lift of $w_2(X)$. W only depends on the isomorphism class of the line bundle L.

We want to study the theory with $SO(3)$ bundles E of a fixed value of $w_2(E)$. This means that we consider only L such that $c_1(L) = w_2(E)$ modulo two. So picking a fixed line bundle U with $c_1(U) = w_2(E)$ modulo two, we write $L = U \otimes T^{\otimes 2}$ where now T is arbitrary.

The interaction (4.19) gives in the path integral a factor e^{-W}, which is

$$e^{-W} = \exp(c_1(T) \cdot w_2(X)) \cdot P, \qquad (4.20)$$

where the factor P is independent of T. Our goal will be to explain the T-dependent factor in (4.20), to which the above derivations were sensitive. The T-independent factors influence only the overall (instanton-number independent) sign convention for the Donaldson invariants. As regards the T-independent factors, (4.19) and P

are not the whole story (they hardly could be as P is not necessarily real); we will find below an additional T-independent interaction not affecting anything we have said hitherto.

Since what is at issue in (4.20) is only a T-dependent minus sign, we have to be rather precise about the treatment of fermions. For complex fermions there are subtleties in defining the phase of the fermion measure (these subtleties are related, among other things, to the Adler-Bell-Jackiw anomaly and its generalizations); even for real fermions, which we meet in the present problem, there is a subtlety with the sign. The subtlety arises from the fact that given orthonormal fermi modes $\psi_1, \ldots, \psi_n$, the relation $d\psi_i \, d\psi_j = -d\psi_j \, d\psi_i$ implies that the sign of the measure $d\psi_1 \, d\psi_2 \ldots d\psi_n$ depends on an ordering of the fermions up to an even permutation.

In our problem, since the instanton number of the $SU(2)$ gauge theory, which is $k = -c_1(L)^2/4$, depends on T, the question of determining the T-dependence of the effective action on the u-plane only makes sense once one has given for all values of the instanton number the sign of the fermion measure in the underlying $SU(2)$ theory. For the physical, untwisted $SU(2)$ theory on $\mathbf{R}^4$, one usually uses cluster decomposition to constrain the k-dependence of the phase of the path integral measure. On a four-manifold, some additional issues arise and were analyzed by Donaldson (see [13], p. 281) with arguments some of which will be adapted below.

I will here explain how to fix the sign of the measure in the microscopic theory and deduce the interaction (4.19) in the macroscopic theory for the case that X admits an almost complex structure. (This is a fairly mild condition, whose import is analyzed in [13], p. 11, and is satisfied by all simply-connected four-manifolds with b_2^+ odd, a class that includes all those with non-trivial Donaldson invariants.) We start with the fact that the fermion fields in the untwisted $N = 2$ theory are a pair of gluinos α^i, $i = 1, 2$ of positive chirality and $U(1)_R$ charge 1, and conjugate fields $\overline{\alpha}_j$, $j = 1, 2$ of negative chirality and charge -1. (α and $\overline{\alpha}$ have values in the adjoint representation of the gauge group.) The kinetic energy has the general form

$$\int_X d^4x \sqrt{g} \ \overline{\alpha}_i D\alpha^i + \ldots \qquad (4.21)$$

where D is a Dirac operator and the omitted terms involve Yukawa couplings of a scalar field to α^2 or $\overline{\alpha}^2$ (as opposed to $\overline{\alpha} \cdot \alpha$ in (4.21)). As the Dirac operator is elliptic and first order and the Yukawa terms are zeroth order, it will suffice to define the sign of the fermion measure (or equivalently the sign of the fermion determinant) for the case that the scalar fields are zero and the $\alpha^2, \overline{\alpha}^2$ terms are absent; the dependence on the scalars then follows by continuity. Therefore, we can ignore the mixing between α and $\overline{\alpha}$ that the Yukawa couplings would cause.

Now, to construct the "twisted" topological field theory, one couples the i index to the positive spin bundle S_+ of X, so that $\alpha, \overline{\alpha}$ are now interpreted as spinors with values in S_+. (If X is not a spin manifold, the twist is needed to formulate any theory on X, since on such an X, ordinary spinors would not exist, but spinors with values

in S_+, which can be reinterpreted as differential forms, still do exist. In particular, the interpretation as differential forms shows that the fermions have a natural real structure in the twisted theory.) An almost complex structure on a spin manifold is equivalent to a decomposition of S_+ as $S_+ = K^{1/2} \oplus K^{-1/2}$ with $K^{\pm 1/2}$ being line bundles, along with a choice of K (or K^{-1}) as the "canonical line bundle." Alternatively, without assuming that X is spin, an almost complex structure is equivalent to a Spin^c bundle (which one might informally write as $S_+ \otimes K^{1/2}$ even though the factors do not exist separately) which has a decomposition as $K \oplus \mathcal{O}$, $\mathcal{O}$ being a trivial line bundle and K a line bundle known as the canonical line bundle of the almost complex structure.

On an almost complex manifold, we write $\alpha = \alpha^+ \oplus \alpha^-$, with $\alpha^\pm$ the components of α in $S_+ \otimes K^{\pm 1/2}$; and similarly $\overline{\alpha} = \overline{\alpha}^+ \oplus \overline{\alpha}^-$. If X is Kahler, the kinetic energy (4.21) (with Yukawa couplings suppressed) has a decomposition

$$\int_X d^4 x \sqrt{g} \left(\overline{\alpha}^- D\alpha^+ + \overline{\alpha}^+ D\alpha^- \right). \tag{4.22}$$

If X is not Kahler, the kinetic energy also contains $\overline{\alpha}^+ \alpha^+$ and $\overline{\alpha}^- \alpha^-$ mixing terms. These are of zeroth order, so just as in our discussion of the Yukawa terms, they can be ignored in the sense that if one defines the sign of the fermion determinant for the Lagrangian (4.22), the effect of the mixing terms can be determined by continuity.

With the various kinds of mixing terms suppressed, the eigenmodes of D^2 have definite chirality and charge. So if α_I^+, for example, are an orthonormal basis of eigenmodes of α^+, we can make an expansion

$$\alpha^+ = \sum_I w_I^+ \, \alpha_I^+ \tag{4.23}$$

with anticommuting c-number coefficients w_I. Complex conjugation produces eigenmodes $\alpha_I^- = \overline{\alpha_I^+}$ of α^-, for which we write a similar expansion:

$$\alpha^- = \sum_I w_I^- \, \alpha_I^-. \tag{4.24}$$

Now we can formally fix the sign of the quantum integration measure for α by writing simply

$$\prod_I dw_I^+ \, dw_I^-. \tag{4.25}$$

Similarly, one can expand $\overline{\alpha}^\pm$ in orthonormal eigenmodes of D^2

$$\overline{\alpha}^+ = \sum_J \overline{w}_J^+ \, \overline{\alpha}_J^+$$

$$\overline{\alpha}^- = \sum_I \overline{w}_J^- \, \overline{\alpha}_J^-, \tag{4.26}$$

and take the measure for α^- to be formally

$$\prod_I d\overline{w}_I^+ \, d\overline{w}_I^-. \tag{4.27}$$

So far we have given a prescription for the fermion measure

$$\mu = \prod_I dw_I^+ \, dw_I^- \prod_J d\overline{w}_J^+ \, d\overline{w}_J^- \tag{4.28}$$

which would be quite well-defined in the case of a finite set of fermi variables, since we have given a definite ordering of the integration variables up to an even permutation. With infinitely many variables, one needs the further observation that (with the exception of the zero modes) the action of D maps modes of α to modes of $\overline{\alpha}$, preserving the eigenvalue of D^2. Therefore, except for finitely many zero modes, the fermi modes come in groups of *four*, namely modes of α^+, α^-, $\overline{\alpha}^+$, and $\overline{\alpha}^-$, permuted by complex conjugation and multiplication by D. Every such group of four gives a *positive* factor in the path integral, because for any real λ the integral

$$\int dw^+ \, dw^- \, d\overline{w}^+ \, d\overline{w}^- \, \exp(\overline{w}^- \lambda w^+ + \overline{w}^+ \lambda w^-) \tag{4.29}$$

is positive. Thus the sign of the measure on the space of zero modes (after integrating out the non-zero modes) is simply given by the ordering (up to an even permutation) of the zero modes in (4.28).

The point of this is that we have made sense of the sign of the quantum measure for *all* values of the instanton number k, with only a k-independent auxiliary choice (of almost complex structure on X). Moreover, this has been done in a way compatible with locality and cluster decomposition, at least formally, since the ordering of the different types of fields given in (4.28) can be carried out locally. Our choice agrees with the prescription introduced by Donaldson, and leads to a theory that behaves well under duality (since it will give the interaction (4.19), which has been seen to lead to a good behavior under duality).

Now that we have defined the theory at a microscopic level, it makes sense to reduce to slowly varying configurations that can be described by the low energy effective action on the u plane, and to ask whether a factor (4.20) appears in the T-dependence of the effective low energy theory.

In the low energy theory at a generic value of u, the $SU(2)$ gauge group is spontaneously broken to $U(1)$. Correspondingly, the gluinos split up as "neutral" components (valued in the Lie algebra of the unbroken $U(1)$) as well as components of charge ± 1. The neutral components are massless and so are included in the low energy theory; the recipe (4.28), restricted to those components, gives a natural (and T-independent) sign of the integration measure for the light fermions.

Of more interest are the massive fermions. They can be integrated out to give an effective theory for the light degrees of freedom. The effective action is complex in general (since it includes the effective theta angle induced by the massive fermions), but the integral over the massive fermions is real if the Higgs field is such that the fermion mass is real. The claim we wish to make is that if the fermion mass is positive, the sign of the integral over the massive fermions is precisely given in (4.20).

The massive fermions now carry *two* $U(1)$ charges, associated respectively with the almost complex structure and the unbroken gauge $U(1)$; we call these the internal and gauge charges. So we write the expansion coefficients of the massive fermions as $w^{\pm\pm}$, where the first sign refers to the internal charge and the second to the gauge charge. The measure defined in (4.28) orders each pair of fermions with the field of positive internal charge first, so when written out in more detail with both charges included, it takes the form

$$\mu = \prod_I dw_I^{++} \, dw_I^{--} \prod_J dw_J^{+-} \, dw_J^{-+} \prod_K d\overline{w}_K^{++} \, d\overline{w}_K^{--} \prod_L d\overline{w}_L^{+-} \, d\overline{w}_L^{-+}. \quad (4.30)$$

We could have imagined an alternative measure with the field of positive gauge charge written first:

$$\widetilde{\mu} = \prod_I dw_I^{++} \, dw_I^{--} \prod_J dw_J^{-+} \, dw_J^{+-} \prod_K d\overline{w}_K^{++} \, d\overline{w}_K^{--} \prod_L d\overline{w}_L^{-+} \, d\overline{w}_L^{+-}. \quad (4.31)$$

If N_{+-} and $\overline{N}_{+-}$ are the number of modes of positive and negative chirality, respectively, with charges $+-$, then the relation between the two measures is

$$\mu = (-1)^{N_{+-}+\overline{N}_{+-}}\widetilde{\mu} = (-1)^{N_{+-}-\overline{N}_{+-}}\widetilde{\mu}. \quad (4.32)$$

Now, N_{+-} and $\overline{N}_{+-}$ are infinite, of course, but the difference $N_{+-} - \overline{N}_{+-}$ is naturally interpreted [6] as the index of the Dirac operator acting on a spinor of charges $+-$, that is, on sections of $S_+ \otimes K^{1/2} \otimes L$. Let us, as at the outset of this discussion, write $L = U \otimes T^2$ where U is a fixed line bundle in the allowed class and T is arbitrary. If we denote the Dirac index for spinors with values in $K^{1/2} \otimes L$ as $\mathrm{ind}(T)$, then from the index theorem

$$\mathrm{ind}(T) = c_1(K) \cdot c_1(T) + 2c_1(T)^2 + \text{constant}. \quad (4.33)$$

Hence

$$\mu = \text{constant} \cdot (-1)^{c_1(K) \cdot c_1(T)}\widetilde{\mu}, \quad (4.34)$$

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6)   Just as above, if one turns off the Yukawa couplings and terms coming from non-integrability of the almost complex structure, the action of D gives a natural pairing of non-zero modes, whose effects cancel, so only the zero modes really have to be counted.

with the "constant" being independent of T.

Now in fact, as we will argue momentarily, the massive fermion integral with measure $\widetilde{\mu}$ is positive for $u > 0$. With measure μ it therefore has the sign given in (4.34). The T-dependent part of this is a factor $(-1)^{c_1(T) \cdot c_1(K)}$ in the effective path integral for the light degrees of freedom. Since, for any almost complex structure with canonical bundle K, the first Chern class $c_1(K)$ reduces modulo two to $w_2(X)$, this factor has precisely the T-dependence claimed in (4.20). This is the sought-for result.[7]

The argument that shows that the fermion determinant is positive for u positive is actually closely related to a standard argument about vector-like gauge theories such as QED. Let us change notation slightly and refer to the fermions of gauge charge 1 and -1 as $\overline{\psi}_\beta$ and ψ_β as in QED (β keeps track of labels other than gauge charge, such as chirality). The measure $\widetilde{\mu}$, with every mode of $\overline{\psi}$ paired with the conjugate mode of ψ (this measure is often written formally $\widetilde{\mu} = \prod_{x,\beta} d\overline{\psi}_\beta(x) \, d\psi_\beta(x)$ or just $\widetilde{\mu} = D\overline{\psi} \, D\psi$) is the conventional measure of a vector-like fermion. After integrating over the non-zero modes of the Dirac operator (which give a positive contribution because of pairing of modes of opposite chirality) the contribution of the zero modes becomes

$$\int D\overline{\psi} \, D\psi \; e^{\overline{\psi} M \psi}, \qquad (4.35)$$

with everything truncated to the space of zero modes. The integral (4.35) is positive if the mass matrix M is positive-definite; indeed, upon diagonalizing M, the integral factors as a product of integrals

$$\int d\overline{w} \, dw \; \exp(m\overline{w}w), \qquad (4.36)$$

and each of these is positive if $m > 0$.

Just such a situation prevails in the twisted $N = 2$ model for $u > 0$. For instance, the positive chirality fermions are vectors ψ_m in the twisted model, and the mass term is $a \, \overline{\psi}_m \, \psi_m$ with a the Higgs field; this is certainly positive if a is positive, which leads to u positive. The negative chirality fermions are likewise differential forms (a zero-form and a self-dual two-form) in the twisted model, with a mass term that is similarly positive for positive a. This then implies that with measure $\widetilde{\mu}$, the integral over the massive fermions is positive for u positive, and therefore that with the measure μ that arises naturally from the microscopic theory, the integral has the sign needed in (4.20).

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7)   For $u > 0$, the effective theta angle as computed in [9] vanishes, so the sign must be attributed to the interaction (4.19) that appears in going to a general four-manifold.

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