

Superstrings and Manifolds of Exceptional Holonomy

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The condition of having an $N = 1$ spacetime supersymmetry for a heterotic string leads to 4 distinct possibilities for compactifications, namely compactifications down to 6, 4, 3 and 2 dimensions. Compactifications to 6 and 4 dimensions have been studied extensively before (corresponding to $K3$ and a Calabi-Yau three-fold respectively). Here we complete the study of the other two cases corresponding to compactification down to 3 on a 7-dimensional manifold of G_2 holonomy and compactification down to 2 on an 8-dimensional manifold of $Spin(7)$ holonomy. We study the extended chiral algebra and find the space of exactly marginal deformations. It turns out that the role that $U(1)$ current plays in the $N = 2$ superconformal theories is played by the tri-critical Ising model in the case of G_2 and the Ising model in the case of $Spin(7)$ manifolds. Certain generalizations of mirror symmetry are found for these two cases. We also discuss the topological twisting in each case.

§1. Introduction

Supersymmetric sigma models in two dimensions have been the source of many interesting ideas in the interplay between quantum field theories and geometry and the topology of manifolds. In the context of superstring theories, viewing strings moving on a manifold leads to the use of sigma models as the building blocks of string vacua. To be a string vacuum, the sigma model must lead to a conformal theory in two dimensions. Moreover to lead to spacetime supersymmetry, which is the only class of superstrings we know which are perturbatively stable, the manifold should admit covariantly constant spinors which can be used to define the supersymmetry transformation. It turns out that having a covariantly constant spinor already guarantees conformal invariance to one loop order in the sigma model perturbation theory and perhaps to all orders (with appropriate adjustments of the metric), so the study of manifolds admitting covariantly constant spinors seems like an important question for string theory. In general if we have an n -dimensional Riemannian manifold the holonomy group is in $SO(n)$; however having a covariantly constant spinor, the holonomy group is smaller and it is (a subgroup of) the little group which leaves a spinor of $SO(n)$ invariant.

Since superstrings live in 10 dimensions and we in 4, the most important physical case to study is the 6-dimensional manifolds with covariantly constant spinors. If we require only one spacetime supersymmetry, this means we need only one covariantly constant spinor (for a fixed chirality) and this leads to the manifolds of $SU(3)$ holonomy, i.e. the Calabi-Yau manifolds [1]. These manifolds have been investigated a great deal with interesting physical results. Among these one could mention that many classically singular Calabi-Yau manifolds lead to non-singular sigma models. Also there is a mirror phenomenon which means that strings on two inequivalent manifolds can lead to the same sigma model. Moreover there is a topological ring in these theories known as the chiral ring [2] which captures the deformation structure of the Calabi-Yau manifold.

In fact the sigma models based on Calabi-Yau manifolds have been studied for all dimensions and not just 6, and many important aspects of the theory behave uniformly well in all dimensions. Also, one could consider odd dimensional manifolds with a minimal number of covariantly constant spinors by considering the product of Calabi-Yau with a circle. However, if one is interested in the minimum number of supersymmetries allowed, this class misses two special cases (for a review see [3]): First of all, in manifolds of 7 dimensions the minimum number of supersymmetries is given by a manifold of G_2 holonomy which has only one covariantly constant spinor as opposed to a manifold of $SU(3)$ holonomy times a circle which has 2. Furthermore in dimension 8 an $SU(4)$ holonomy manifold leads to 2 covariantly constant spinors, whereas for an 8-dimensional manifold of $Spin(7)$ holonomy there will be only 1 covariantly constant spinor. The possible existence of these two special cases had been known for a long time [4]. It is very amusing that these two special cases can be used in physical models simply because the dimensions where they occur are less than 10, which means that if we were to compactify superstrings down to 3 or 2-dimensional Minkowski space and ask which manifolds would lead to a minimal number of nonvanishing supersymmetries (1 for heterotic strings and 2 for type II strings) we would have to study sigma models on 7d manifolds of G_2 holonomy and 8d manifolds of $Spin(7)$ holonomy. The study of these two classes of sigma models is the subject of the present paper.

The organization of this paper is as follows: In Section 2 we will review basic facts about superconformal theories in general and manifolds with covariantly constant spinors in particular. This includes a quick review of aspects of $N=2$ superconformal theories and their relation to the geometry of Calabi-Yau manifolds. This review is a good exercise for setting the stage for the two special cases of manifolds of G_2 and $Spin(7)$ holonomy and the associated conformal theories. In this section we also review some geometrical facts about manifolds of G_2 and $Spin(7)$ holonomy that we will need in the rest of the paper.

In Section 3 using the geometrical data at hand, we construct the extended chiral algebra associated to G_2 and $Spin(7)$ manifolds. It turns out that the role played by the $U(1)$ piece of the $N = 2$ algebra in the context of Calabi-Yau com-

pactification is now played by the tri-critical Ising model for G_2 manifolds and the Ising model for $Spin(7)$ manifolds! The algebra and its construction is very similar in both cases and will be discussed in parallel. The existence of these two minimal models as an integral part of the theory is crucial. In particular it allows us to identify the space of marginal operators which preserve both the superconformal symmetry as well as the G_2 and $Spin(7)$ structure of the algebra and prove their exact marginality to all orders in conformal perturbation theory. Moreover we find the identification of these deformations with the geometrical facts explained in Section 2.

In Section 4 we discuss concrete orbifold examples of these manifolds constructed very recently by Joyce. These examples are illuminating as far as the structure of the algebra we found in the previous section. Moreover we find a special kind of mirror phenomenon takes place, in that we find inequivalent orbifold resolutions (having different Betti numbers) found by Joyce correspond to the same underlying conformal theory up to deformation in moduli space. Moreover we find that whenever there are discrete torsions which lead to different conformal theories, there are also inequivalent geometrical resolutions.

In Section 5 we construct a topological twisting for these cases. Again amazing facts about the tri-critical Ising model and Ising model are crucial for making this twist possible. In Appendix 1 and Appendix 2 we collect some relevant facts about the structure of the extended chiral algebras that we have encountered for these exceptional manifolds.

§2. Superconformal Sigma Models and Special Holonomy Manifolds

In this section we review some general aspects of 2d supersymmetric sigma models and their interplay with geometry of the target manifolds. The most basic observation in this regard is that if we consider the Hilbert space on a circle with periodic boundary conditions (the Ramond sector) of a 2d supersymmetric sigma model with an n -dimensional target space M , or for that matter the 1d supersymmetric sigma model on M , there is an identity for Witten's index [5]

$$\mathrm{Tr}(-1)^F \exp(-\beta H) = \chi(M) = \sum_{i=0}^n (-1)^i b_i = n_+ - n_-,$$

where $\chi(M)$ is the Euler characteristic of M and b_i are the Betti numbers of M and n_+ and n_- denote the total number of even and odd dimensional cohomologies respectively. The basic idea is that only the ground states with $H = 0$ contribute to the index (as the $H > 0$ come in pairs with opposite $(-1)^F$) and that in a suitable limit, the ground states are related to the harmonic forms on M , and $(-1)^F$, up to an overall sign ambiguity, can be identified with the parity of the degree of the harmonic forms. Actually there is more information [6]: It is possible to show

that the number of ground states in the theory are exactly equal to the number of harmonic forms. In other words all the non-cohomological perturbative ground states are lifted up by non-perturbative effects but the other ground states, which are in equal number to the cohomology elements are exact non-perturbative ground states and are not paired to become massive even if they have opposite $(-1)^F$, i.e.,

$$\text{Tr exp}(-\beta H) \Big|_{\beta \rightarrow \infty} = \sum_{i=0}^n b_i = n_+ + n_-.$$

Note that even though the number of ground states is equal to the number of cohomology elements of M , there is no canonical correspondence. In particular it is not in general possible to determine the Betti numbers individually. From the two physical computations above, we can only deduce n_+ and n_- . Actually even that we cannot deduce unambiguously because, as mentioned before, the sign of $(-1)^F$ cannot be canonically fixed. Therefore from the physical Hilbert space we can only deduce n_+ and n_- up to the exchange $n_+ \leftrightarrow n_-$. So much is true for general supersymmetric sigma models. If there are further restrictions on M , we can deduce more from the physical theory. For example if M is a Kähler manifold, the fermions are complex and so there is a $U(1)$ conserved charge corresponding to the fermion number F which, acting on the ground state, can be identified with the number of holomorphic forms p minus the number of antiholomorphic forms q of the harmonic form

$$F = p - q.$$

So in this way, by decomposing the ground state to eigenstates of F , we can compute the number of cohomology elements with a given value of $p - q$. There is also the chiral (or axial) fermion number F_A which is non-perturbatively conserved only when the first chern class $c_1(M) = 0^1$, i.e., when M is a Calabi-Yau manifold. F_A can be identified when acting on ground states with

$$F_A = p + q - d,$$

where $d = n/2$ is the complex dimension of M . So we can compute

$$p - \frac{d}{2} = \frac{1}{2}(F_A + F) = F_L,$$

$$q - \frac{d}{2} = \frac{1}{2}(F_A - F) = F_R.$$

Just as before there is a relative ambiguity in the identification of the sign of $F_{L,R}$ which means that we can determine the hodge numbers $h^{p,q}$ only up to the ambiguity

$$h^{p,q} \leftrightarrow h^{d-p,q}.$$

1) For the 1d supersymmetric sigma models, since there are no instantons to ruin the conservation, the F_A is always conserved.

This apparent deficiency in the supersymmetric sigma model in capturing the geometry of Calabi-Yau was conjectured [2], [7] to be related to the beautiful possibility that CY manifolds may come in pairs which lead to the same sigma model but for which the hodge numbers are mirror to each other. There is by now a large body of evidence supporting this conjecture [8].

There are more interesting relations with the geometry of CY if we consider operator products of some special operators in the theory. It turns out that there is a natural ring of operators in the theory known as the *chiral ring* [2] which are in one-to-one correspondence with the cohomology elements and which are most easily defined by using the fact that there exists a metric on CY which gives rise to a 2d conformal field theory. One can also define this ring by purely topological means [9], and it turns out that (at least in one version) it is related to a *quantum deformed* cohomology ring of the manifold which has the information about the holomorphic maps from CP^1 to M encoded in it.

Calabi-Yau 3-folds are interesting for string theory as mentioned in the introduction precisely because they have covariantly constant spinors and they have a minimal number of them leading to $N = 1$ spacetime supersymmetry when we compactify heterotic strings on them. However let us ask a question which would be a very natural question in the context of superstring or heterotic string compactifications: If we compactify the heterotic strings to any lower dimension, in which dimensions can we obtain the minimal non-vanishing number of spacetime supersymmetries? The answer to this question is rather interesting (for a simple derivation see [3]): To have one spacetime supersymmetry, we need the minimum allowed covariantly constant spinors (1 or 2 depending on the dimension). This is possible only if we compactify from 10 down to 6, 4, 3 or 2 on manifolds with holonomy $Sp(1)(= SU(2))$, $SU(3)$, G_2 and $Spin(7)$, with dimensions 4, 6, 7, 8 respectively. Moreover in the case of 4 dimensions, there is a unique manifold $K3$ which has $Sp(1)$ holonomy. The six-dimensional case is a three-fold Calabi-Yau and possibilities for this have also been studied extensively. Here we begin the completion of this systematic classification by studying sigma models on manifolds with G_2 and $Spin(7)$ holonomy. It is amusing to note that the $K3$ and CY three-folds have generalization to higher dimensions with manifolds of $Sp(n)$ and $SU(n)$ holonomy respectively, but the G_2 and $Spin(7)$ case are unique structures with no generalizations to higher dimensions.

Before we go on to describe some general properties of manifolds with G_2 and $Spin(7)$ holonomy, motivated by the success of the mirror conjecture for CY target spaces, let us make a conjecture which will prove helpful in clarifying the observations we shall make later:

Generalized Mirror Conjecture: The degree of ambiguity left by being unable to decipher all the topological aspects of the target manifold using the algebraic formulation of quantum field theories is precisely explained by having topologically

inequivalent manifolds allowed by the ambiguity to lead to the same quantum field theory up to deformation in the moduli of the quantum field theory.

What we mean by this conjecture is the following: Suppose we are given all the correlation functions of a quantum field theory corresponding to some unknown sigma model. Furthermore suppose we wish to reconstruct as much as possible the target manifold from this data. The statement of the conjecture is motivated by the belief that if we cannot fully reconstruct the target manifold from this data there is a good reason. In other words the number of inequivalent target manifolds that cannot be distinguished using the quantum field theory data are precisely the number of inequivalent target manifolds which give rise to the same quantum field theory.

We call such manifolds mirror of one another in the generalized sense of mirror symmetry.

We shall see in later sections the first non-Calabi-Yau examples which support the above conjecture in the case of manifolds of G_2 and $Spin(7)$ holonomy.

Let us begin discussing some facts about manifolds of G_2 and $Spin(7)$ holonomy. Until very recently the only known examples of manifolds of G_2 holonomy and $Spin(7)$ holonomy were non-compact manifolds [10]. The situation has dramatically changed recently due to the work of Joyce [11] who constructed the first compact examples of manifolds with G_2 and $Spin(7)$ holonomy, which we denote by M^7 and M^8 respectively. Just as in the Calabi-Yau case where the fact that the manifold has $SU(n)$ holonomy leads to the existence of a unique non-vanishing holomorphic covariantly constant n -form (and of course its conjugate), in these two exceptional cases a similar thing happens (see [12] Chapters 11 and 12): In the case of G_2 manifolds, there is a canonical 3-form ϕ and its dual which is a 4-form $*\phi$ which are covariantly constant and in the case of $Spin(7)$ there is a self-dual 4-form Ω . They can be locally written as follows: we choose a local vielbein so that the metric is $\sum e_i \otimes e_i$ where e_i are one-forms and for the G_2 case, i runs from 1 to 7 and for the $Spin(7)$ case i runs from 1 to 8. Then these forms can be written as

$$\begin{aligned} \phi &= e_1 \wedge e_2 \wedge e_7 + e_1 \wedge e_3 \wedge e_6 + e_1 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_5 \\ &\quad - e_2 \wedge e_4 \wedge e_6 + e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7, \end{aligned} \tag{2.1}$$

$$\begin{aligned} *\phi &= e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_1 \wedge e_2 \wedge e_5 \wedge e_6 \\ &\quad - e_1 \wedge e_3 \wedge e_5 \wedge e_7 + e_1 \wedge e_4 \wedge e_6 \wedge e_7 \\ &\quad + e_2 \wedge e_3 \wedge e_6 \wedge e_7 + e_2 \wedge e_4 \wedge e_5 \wedge e_7 + e_3 \wedge e_4 \wedge e_5 \wedge e_6, \end{aligned} \tag{2.2}$$

$$\Omega = e_8 \wedge \phi - *\phi. \tag{2.3}$$

These can be understood as follows: In the case of G_2 if we view e_i as forming the fundamental representation of $O(7)$, the fact that the holonomy is in the G_2 subgroup of $O(7)$ means that in the three-fold tensor product of this representation there is a totally anti-symmetric singlet of G_2 which is identified with ϕ . Similarly

$*\phi$ is invariant under G_2 . In the case of 8-dimensional $Spin(7)$ manifolds, e_i form the fundamental representation of $O(8)$. If we view the embedding of $Spin(7)$ in $O(8)$ such that the 8-dimensional spinor representation of $O(8)$ transforms as the $7 \oplus 1$ of $Spin(7)$, and thus the 8-dimensional vector representation of $O(8)$ transforms as an 8-dimensional spinor representation of $Spin(7)$, then in the four-fold totally antisymmetric product of this latter representation, there is a unique singlet of $Spin(7)$ denoted by Ω above. Metric $\sum e_i \otimes e_i$ can be uniquely reconstructed from ϕ and Ω .

Moreover it is true [13], [11] that the dimension of moduli space of deformation of manifolds of G_2 holonomy is $b_3(M^7)$ and the dimension of the moduli space of deformation of manifolds of $Spin(7)$ holonomy is $b_4^-(M^8) + 1$, where b_4^\pm denote the self-dual/anti-self-dual dimensions of $H^4(M^8)$. The simplest class of examples considered by Joyce involve toroidal orbifolds. In the case of G_2 , the minimal example is obtained by modding out $T^7/(Z_2)^3$ where each Z_2 has for eigenvalues of holonomy $(-1, -1, -1, -1, 1, 1, 1)$, but they sit in $SO(7)$ in such a way that they cannot be embedded in an $SU(3)$ subgroup of it, but can be embedded into a G_2 subgroup of it. Moreover it is clear from the above discussion that this group will preserve ϕ and $*\phi$. Moreover for simplicity of analysis, Joyce considers some of these Z_2 's to be accompanied with translations of the T^7 , and shows that the singular orbifold can be desingularized. For the case of $Spin(7)$ holonomy the simplest examples he constructs involve again desingularizing a toroidal orbifold. In this case he considers $T^8/(Z_2)^4$, where each Z_2 has eigenvalues of holonomy $(-1, -1, -1, -1, 1, 1, 1, 1)$, but again in such a way that the full group does not sit in $SU(4)$ but does sit in a $Spin(7)$ subgroup of $O(8)$. In both the G_2 case and the $Spin(7)$ case he finds that there are in general many inequivalent ways of desingularizing the manifold, which we will be able to explain physically in Section 4 as a consequence of the generalized mirror conjecture stated above. In fact it is crucial to note that the dimension of the moduli space of the conformal theory is actually *bigger* than that predicted geometrically. The reason for this is that the possibility of using the anti-symmetric two-form to add a phase to the action has no geometrical analog. Therefore we have

$$\dim. \text{moduli}_{physical} = \dim. \text{moduli}_{geometrical} + b_2.$$

In particular for the G_2 case the dimension of sigma model moduli is $b_2 + b_3$ and not b_3 and for the case of $Spin(7)$ the dimension of sigma model moduli is $b_4^- + b_2 + 1$. Let us also briefly talk about the structure of the Betti numbers of these two cases: In both cases we are dealing with manifolds with $b_1 = 0$ in order to obtain the minimum number of covariantly constant spinors. In the case of G_2 holonomy, therefore there are two independent Betti numbers to compute b_2 and b_4 , since by duality $b_3 = b_4$ and $b_5 = b_2$. As discussed before, physically we can a priori only compute the dimension of even or odd cohomologies $b_2 + b_4 = b_5 + b_3$. So a priori physically we can expect to deduce only one geometrical index in this case, namely $b_2 + b_4$ which is also equal to $b_2 + b_3$ which is the dimension of moduli space.

In the case of $Spin(7)$ holonomy manifolds there are a priori four topological numbers one can hope to compute b_2, b_3, b_4^\pm and the rest are obtained by duality. However the fact that there is a unique zero mode for the Dirac operator implies using the index theorem that [14]

$$b_3 + b_4^+ - b_2 - 2b_4^- - 1 = 24. \quad (2.4)$$

So geometrically there are only three independent numbers in this case. Physically to begin with we have the number of even and odd cohomologies from which we can deduce $b_2 + b_4 + b_6 = 2b_2 + b_4$ and $b_3 + b_5 = 2b_3$. We should also in addition expect to compute $b_4^- + b_2 + 1$ by finding the dimension of exactly marginal deformations. However the relation (2.4) implies that there is a linear relation between these numbers which would mean that there are only two independent physical numbers one could hope to compute, as opposed to three in the geometrical case. The validity of (2.4) for sigma model should follow from modular invariance type arguments in relation to sigma models [15].

§3. Extended Symmetry Algebra, Consequences and Deformations

In this section we will unravel the extended symmetry algebras which underlie sigma models with $N = 1$ superconformal symmetry on manifolds of G_2 and $Spin(7)$ holonomy. The idea for obtaining these symmetry algebras is familiar from the study of Calabi-Yau manifolds, where one appends to the $N = 1$ superconformal algebra, a $U(1)$ current to obtain the $N = 2$ algebra, and the spectral flow operator, to guarantee integrality of $U(1)$ charges. For the case of Calabi-Yau three-folds this has been studied in [16]. We study the representation theory of this extended algebra. Consequences of this symmetry allows us to gain insight into the structure of the theory and in particular construct the space of exactly marginal deformations (the moduli).

Perhaps to make some aspects of the algebra that we obtain a little less mysterious, it would be helpful to see a priori what we should expect to play the role that $U(1)$ plays for sigma models on manifolds of $SU(n)$ holonomy². If we start with a sigma model on a Kähler manifold we have a priori a $U(n)$ symmetry. Having a holonomy in $SU(n)$ means that part of the symmetry is broken but we are left with an unbroken $U(1) = U(n)/SU(n)$. Similarly in the case of 7-dimensional manifolds of G_2 holonomy, a priori we have $SO(7)$ symmetry (more precisely $SO(7)$ current algebra at level 1). The holonomy of the manifold being in G_2 means that we are left with the residual symmetry $SO(7)/G_2$, which in geometrical terms is no longer a group, however, from the viewpoint of conformal theory it is a coset model. Computing its central charge we see that since $SO(7)$ at level 1 has central charge $7/2$

2) This line of thought was developed following a suggestion of E. Martinec.

and G_2 at level 1 has central charge $14/5$, the central charge of the residual system is

$$\frac{7}{2} - \frac{14}{5} = \frac{7}{10},$$

which is thus a tri-critical Ising model [17]! Similarly for the case of $Spin(7)$ manifolds one considers $SO(8)/Spin(7)$ which gives a central charge

$$4 - 7/2 = 1/2,$$

which is just the Ising model. Below we shall recover these facts directly as well as find out that these symmetries mix in a very interesting way with the $N = 1$ superconformal algebra to obtain the extended symmetry algebra of our models.

3.1 G_2 . As far as the algebraic structure is concerned we start from the flat 7 dimensional space, and construct the chiral operators which we expect to exist even after we perturb the metric to obtain a non-trivial G_2 holonomy. We of course expect to have the energy momentum tensor T and its superpartner G . Moreover the fact that a three-form ϕ exists even after the perturbation suggests that one can add to $N = 1$ superconformal generators $T = T_b + T_f = \frac{1}{2} \sum_1^7 : J^i J^i :$ $-\frac{1}{2} \sum_1^7 : \psi^i \partial \psi^i ,$ $G = \sum_1^7 : J^i \psi^i :$ a new spin $3/2$ operator

$$\begin{aligned} \Phi &= \psi^1 \psi^2 \psi^5 + \psi^1 \psi^3 \psi^6 + \psi^1 \psi^4 \psi^7 - \psi^2 \psi^3 \psi^7 \\ &\quad + \psi^2 \psi^4 \psi^6 - \psi^3 \psi^4 \psi^5 + \psi^5 \psi^6 \psi^7 \\ &= f_{ijk} \psi^i \psi^j \psi^k, \end{aligned} \tag{3.1}$$

with the coefficients f_{ijk} defined by the G_2 invariant three-form ϕ from the previous section. We use the notation $J^i = \partial x^i$ with x^i being a bosonic sigma model coordinate. $N = 1$ generators are invariant under the rotation group $SO(7)$ and Φ is invariant only under the G_2 subgroup of $SO(7)$. If we compute the operator expansion of new generator Φ with itself we obtain

$$\Phi(z)\Phi(w) = -\frac{7}{(z-w)^3} + \frac{6}{z-w} X(w), \tag{3.2}$$

where operator X has spin 2

$$\begin{aligned} X &= -\psi^1 \psi^2 \psi^3 \psi^4 + \psi^1 \psi^2 \psi^6 \psi^7 - \psi^1 \psi^3 \psi^5 \psi^7 + \psi^1 \psi^4 \psi^5 \psi^6 - \\ &\quad \psi^2 \psi^3 \psi^5 \psi^6 - \psi^2 \psi^4 \psi^5 \psi^7 - \psi^3 \psi^4 \psi^6 \psi^7 - 1/2 : \partial \psi^i \psi^i : = - * \Phi + T_f, \end{aligned} \tag{3.3}$$

and is a linear combination of “dual” operator $*\Phi$ (defined by dual form $*\phi$) and a fermionic stress-tensor. Next step is to compute operator expansion of the operators X and Φ . We obtain

$$\Phi(z)X(w) = -\frac{15}{2} \frac{1}{(z-w)^2} \Phi(w) - \frac{5}{2} \frac{1}{z-w} \partial \Phi(w), \tag{3.4}$$

$$X(z)X(w) = \frac{35}{4} \frac{1}{(z-w)^4} - \frac{10}{(z-w)^2} X(w) - \frac{5}{z-w} \partial X(w). \tag{3.5}$$

This is not the end of story because now we need to deal with superpartners of new generators with respect to the original $N = 1$ algebra. This introduces two new operators of spins 2 and $\frac{5}{2}$ into the game; we will denote them by K and M respectively:

$$G(z)\Phi(w) = \frac{1}{z-w}K(w), \quad (3.6)$$

$$G(z)X(w) = -\frac{1}{2} \frac{1}{(z-w)^2}G(w) + \frac{1}{z-w}M(w). \quad (3.7)$$

New operators in the right-hand side have the following free field representation:

$$\begin{aligned} K = & J^1\psi^2\psi^5 + J^1\psi^3\psi^6 + J^1\psi^4\psi^7 - J^2\psi^1\psi^5 - J^2\psi^3\psi^7 + \\ & J^2\psi^4\psi^6 - J^3\psi^1\psi^6 + J^3\psi^2\psi^7 - J^3\psi^4\psi^5 - J^4\psi^1\psi^7 - J^4\psi^2\psi^6 + \\ & J^4\psi^3\psi^5 + J^5\psi^1\psi^2 - J^5\psi^3\psi^4 + J^5\psi^6\psi^7 + J^6\psi^1\psi^3 + J^6\psi^2\psi^4 - \\ & J^6\psi^5\psi^7 + J^7\psi^1\psi^4 - J^7\psi^2\psi^3 + J^7\psi^5\psi^6, \end{aligned} \quad (3.8)$$

$$\begin{aligned} M = & -J^1\psi^2\psi^3\psi^4 + J^1\psi^2\psi^6\psi^7 - J^1\psi^3\psi^5\psi^7 + J^1\psi^4\psi^5\psi^6 + J^2\psi^1\psi^3\psi^4 - \\ & J^2\psi^1\psi^6\psi^7 - J^2\psi^3\psi^5\psi^6 - J^2\psi^4\psi^5\psi^7 - J^3\psi^1\psi^2\psi^4 + J^3\psi^1\psi^5\psi^7 + \\ & J^3\psi^2\psi^5\psi^6 - J^3\psi^4\psi^6\psi^7 + J^4\psi^1\psi^2\psi^3 - J^4\psi^1\psi^5\psi^6 + J^4\psi^2\psi^5\psi^7 + \\ & J^4\psi^3\psi^6\psi^7 - J^5\psi^1\psi^3\psi^7 + J^5\psi^1\psi^4\psi^6 - J^5\psi^2\psi^3\psi^6 - J^5\psi^2\psi^4\psi^7 + \\ & J^6\psi^1\psi^2\psi^7 - J^6\psi^1\psi^4\psi^5 + J^6\psi^2\psi^3\psi^5 - J^6\psi^3\psi^4\psi^7 - J^7\psi^1\psi^2\psi^6 + \\ & J^7\psi^1\psi^3\psi^5 + J^7\psi^2\psi^4\psi^5 + J^7\psi^3\psi^4\psi^6 + 1/2J^i\partial\psi^i - 1/2\partial J^i\psi^i. \end{aligned} \quad (3.9)$$

A nontrivial fact deeply related to “ G_2 structure” is that operator expansion algebra formed by these six operators T , G , Φ , X , K and M closes. (The results of further computation is presented in Appendix 1 together with commutation relations written in mode expansion.) Thus, we have demonstrated that there is an extended chiral algebra which contains quadratic combinations in the right-hand side and thus reminds (just reminds) one of W -algebra.³

After extended chiral algebra is derived we can forget about the free field picture recalling that the perturbation will destroy the fact that the theory is free, but assume the existence of the algebra beyond free realization and study the corresponding conformal field theory. As a first step, we have to find the spectrum of low lying states and in particular the spectra of Ramond ground states which carry the geometrical information about the manifold. In this study it is extremely useful to note that our extended algebra contains two (non-commutative) $N = 1$ superconformal subalgebras: 1. Original $N = 1$ generated by G and T , and 2. $N = 1$

3) The existence of extended symmetry for $N = 1$ sigma model on G_2 manifold in classical approximation was previously mentioned in [18]; we also have been informed by M. Rocek and J. de Boer that recently they also have found extended symmetry in above sigma model.

superconformal algebra generated by $G_I = \frac{i}{\sqrt{15}}\Phi$ and $T_I = -\frac{1}{5}X$. Moreover, the latter is a very interesting one — it has a Virasoro central charge $\frac{7}{10}$ as predicted in the beginning of this section and is the tri-critical Ising model which is the only bosonic minimal model in the list of $N = 1$ superconformal minimal models [19]. In addition a simple observation that

$$T_I(z)T_r(w) = 0, \quad T = T_I + T_r \tag{3.10}$$

allows us to classify the highest weight representations of our algebra using two numbers: Tri-critical Ising highest weight and eigenvalue of the zero mode of the remaining stress-tensor T_r .

Now, at the beginning we consider only chiral sector (left-movers say). The theory is supersymmetric and thus we have two sectors — Neveu-Schwarz and Ramond. We shall see below that the $(-1)^F$ for the full theory can be identified with the $(-1)^{F_I}$ which is the Z_2 symmetry of the tri-critical Ising model viewed as an $N = 1$ superconformal system. From the observation that total stress-tensor can be written as a sum of two commutative Virasoro generators where one is tri-critical Ising, we conclude that unitary highest weight representations should have following tri-critical Ising dimensions:

$$NS : \quad [0]_{Vir}, \quad \left[\frac{1}{10}\right]_{Vir}, \quad \left[\frac{6}{10}\right]_{Vir}, \quad \left[\frac{3}{2}\right]_{Vir}; \tag{3.11}$$

or in $N = 1$ terms

$$NS : \quad [0], \quad \left[\frac{1}{10}\right] \tag{3.12}$$

and

$$R : \quad \left[\frac{7}{16}\right], \quad \left[\frac{3}{80}\right]. \tag{3.13}$$

Supersymmetry requires that Ramond vacuum for whole theory has dimension $\frac{d}{16} = \frac{7}{16}$, and this leads to the following unitary highest weight representations of extended chiral algebra in the Ramond ground state (we use the notation $[\Delta_I, \Delta_r]$ for operators that correspond to Virasoro highest weights $|\Delta_I, \Delta_r\rangle$ with first dimension being the dimension of tri-critical Ising part and the second the dimension of the remaining Virasoro algebra T_r):

$$R : \quad \left|\frac{7}{16}, 0\right\rangle, \quad \left|\frac{3}{80}, \frac{2}{5}\right\rangle. \tag{3.14}$$

It is one of the most remarkable facts for this theory that there exists a ground state in the Ramond sector which is entirely constructed out of the tri-critical Ising sector, namely the $|\frac{7}{16}, 0\rangle$ state. It is as if the tri-critical Ising model “knows” about the fact that the dimension of the manifold of interest is 7. As we will see this

is crucially related to having an $N = 1$ spacetime supersymmetry as well as the possibility of twisting the theory. In many ways the operator corresponding to this ground state plays the same role as the spectral flow operator in $N = 2$ theories which is also entirely built out of the $U(1)$ piece of $N = 2$. To have one spacetime supersymmetry, we would be interested in the realization of this algebra which has exactly one Ramond ground state of the form $|\frac{7}{16}, 0\rangle$ (we shall make this statement a little bit more precise when we talk about putting left- and right-movers together). In this regard it is crucial to note that in the tri-critical Ising model we have unique fusion rules for the operator $[\frac{7}{16}]$

$$[\frac{7}{16}][\frac{7}{16}] = [0]_{Vir} + [\frac{3}{2}]_{Vir} = [0], \quad (3.15)$$

$$[\frac{7}{16}][\frac{3}{80}] = [\frac{1}{10}]_{Vir} + [\frac{6}{10}]_{Vir} = [\frac{1}{10}]. \quad (3.16)$$

The existence of this operator in the Ramond sector allows us to *predict* the existence of certain states in the NS sector. This follows from the fact that it sits entirely in the tri-critical Ising part of the theory and its OPE with other fields depend only on the tri-critical Ising content of other state and thus by considering the OPE of the operator corresponding to $|\frac{7}{16}, 0\rangle$ state with the other states in the Ramond sector we end up with certain special NS states. From (3.15) we conclude that Ising spin field $[\frac{7}{16}]$ maps Ramond ground state $|\frac{7}{16}, 0\rangle$ to NS vacuum $|0, 0\rangle$ and vice versa. More importantly when we consider the OPE of the $|\frac{7}{16}, 0\rangle$ state with $|\frac{3}{80}, \frac{2}{5}\rangle$ we end up with a primary state in the NS sector of the form $|\frac{1}{10}, \frac{2}{5}\rangle$, which has total dimension $\frac{1}{2}$ and is primary. This procedure can be repeated in the opposite direction: tri-critical Ising model spin field $[\frac{7}{16}]$ maps primary field of NS sector $[\frac{1}{10}, \frac{2}{5}]$ to an R ground state $|\frac{3}{80}, \frac{2}{5}\rangle$. This leads to the prediction of existence of the following special states in NS sector:

$$NS : \quad |0, 0\rangle, \quad \left| \frac{1}{10}, \frac{2}{5} \right\rangle. \quad (3.17)$$

Note in particular that since the T_r part of the theory is un-modified as we go from the R sector to the NS sector. It is again quite remarkable that the state in the NS sector corresponding to $|\frac{1}{10}, \frac{2}{5}\rangle$ is a primary field of dimension $1/2$ and so $G_{-1/2}$ acting on it is of dimension 1, preserving $N=1$ supersymmetry and thus a candidate for exactly marginal perturbation in the theory. We will use the extended chiral algebra below to show that indeed they lead to exactly marginal directions. Again the fact that this state has dimension $1/2$ is a consequence of a miraculous relation between the dimension of tri-critical Ising model states. If one traces back one finds that it comes from the fact that

$$\frac{7}{16} - \frac{3}{80} + \frac{1}{10} = \frac{1}{2}.$$

In the above discussion we assumed that Z_2 fermion number assignment on any state is equal to the Z_2 grading for its tri-critical part alone which in particular implies that in the NS sector of the full theory only NS dimensions of tri-critical model show up and similarly in the R sector. Let us now discuss how this comes about. Our chiral algebra has three bosonic T, X, K and three fermionic G, Φ, M generators. We have the following tri-critical Z_2 assignments: $[0]^+, [\frac{1}{10}]^-, [\frac{6}{10}]^+, [\frac{3}{2}]^-$. To prove that $(-1)^F = (-1)^{F_I}$ it suffices to derive tri-critical Ising dimensions of our generators and see if the two Z_2 assignments agree. Here we have to use relations presented in Appendix 2; we have

$$L_{-2}|0, 0\rangle = |2, 0\rangle^+ + |0, 2\rangle^+, \quad X_{-2}|0, 0\rangle = |2, 0\rangle^+,$$

$$K_{-2}|0, 0\rangle = \left| \frac{6}{10}, \frac{14}{10} \right\rangle^+, \tag{3.18}$$

$$G_{-3/2}|0, 0\rangle = \left| \frac{1}{10}, \frac{14}{10} \right\rangle^-, \quad M_{-5/2}|0, 0\rangle = a \left| \frac{1}{10}, \frac{24}{10} \right\rangle^- + b \left| \frac{1}{10}, 1, \frac{14}{10} \right\rangle^-. \tag{3.19}$$

We see that in the assignment in above expressions $(-1)^F = (-1)^{F_I}$ and thus we can use tri-critical gradings for the whole theory.

Now we are ready to discuss the non-chiral, left-right sector. This will also lead to a better understanding of the correspondence with geometry. We claim that only states in (R,R) ground state are

$$(R, R) : \quad \left| \left(\frac{7}{16}, 0 \right)_L ; \left(\frac{7}{16}, 0 \right)_R ; \pm \right\rangle, \quad \left| \left(\frac{3}{80}, \frac{2}{5} \right)_L ; \left(\frac{3}{80}, \frac{2}{5} \right)_R ; \pm \right\rangle. \tag{3.20}$$

where the significance of \pm will be explained momentarily. We had two other possibilities of left-right combinations: $\left| \left(\frac{7}{16}, 0 \right)_L ; \left(\frac{3}{80}, \frac{2}{5} \right)_R ; \pm \right\rangle$ and the same with exchange of L with R . The reason we didn't put these states in the list (3.20) is simple. If we use fusion rules (3.15) and (3.16) we see that primary field corresponding to first ground state in (3.20) acting on these additional states will lead (according to tri-critical Ising model fusion rules) to the highest weight state $\left| (0, 0)_L ; \left(\frac{1}{10}, \frac{2}{5} \right)_R \right\rangle$ in the Neveu-Schwarz sector. But this operator has total dimension $\frac{1}{2}$ and is chiral, so we get an additional chiral operator of half-integer spin in the theory which is not present in our original extended chiral algebra. This means that these additional states aren't present in the case of the generic theory (which is assumed to have only chiral operators described in the beginning of this section). The \pm signs next to the states are a reflection of the fact that since acting on the ground states we have $\{\Phi_0, \bar{\Phi}_0\} = 0, \Phi_0^2 = \bar{\Phi}_0^2 = \frac{6}{15}$, they form a 2-dimensional representation. The \pm sign therefore reflects states with 2 different $(-1)^F$ assignments. Thus, Ramond ground states are coming in pairs — $\Phi_0 \left| \left(\frac{7}{16}, 0 \right)_L ; \left(\frac{7}{16}, 0 \right)_R ; + \right\rangle = \left| \left(\frac{7}{16}, 0 \right)_L ; \left(\frac{7}{16}, 0 \right)_R ; - \right\rangle$, $\Phi_0 \left| \left(\frac{3}{80}, \frac{2}{5} \right)_L ; \left(\frac{3}{80}, \frac{2}{5} \right)_R ; + \right\rangle = \left| \left(\frac{3}{80}, \frac{2}{5} \right)_L ; \left(\frac{3}{80}, \frac{2}{5} \right)_R ; - \right\rangle$. Now we can better describe the relation of Ramond ground states with the cohomology of the manifold: The fact that states come in pairs is a consequence of the fact that in odd dimension the

dual of every cohomology state is another cohomology state with different degree mod 2. So the Ramond + states correspond to even cohomology elements and – to the odd ones. Concentrating on the even cohomology elements in principle we could have $b_0 = 1, b_2, b_4$ as the elements (note that having no extra supersymmetry leads to having $b_6 = b_1 = 0$ which is correlated with the fact that we assume the $|(\frac{7}{16}, 0)_L; (\frac{7}{16}, 0)_R, +\rangle$ is unique). We see that we can only compute one extra number, and not two, which is the number of ground states involving the $\frac{3}{80}$ tri-critical piece for both left- and right-movers which we identify with $b_2 + b_4$.

Let us discuss the special NS states taking into account both the left- and right-moving degrees of freedom. Acting on all + Ramond ground states with the state $|(\frac{7}{16}, 0)_L; (\frac{7}{16}, 0)_R, +\rangle$ leads to (NS, NS) states

$$(NS, NS) : \quad |(0, 0)_L; (0, 0)_R\rangle, \quad \left| \left(\frac{1}{10}, \frac{2}{5} \right)_L; \left(\frac{1}{10}, \frac{2}{5} \right)_R \right\rangle, \tag{3.21}$$

where the number of $|(\frac{1}{10}, \frac{2}{5})_L; (\frac{1}{10}, \frac{2}{5})_R\rangle$ states are the same as the states $|(\frac{3}{80}, \frac{2}{5})_L; (\frac{3}{80}, \frac{2}{5})_R\rangle$ which is equal to $b_2 + b_4$.⁴ Moreover as we will argue later in this section, each of all such NS operators are exactly marginal operators preserving the G_2 structure. This agrees with the geometrical facts discussed in Section 2 in that the dimension of conformal moduli space is thus $b_2 + b_4 = b_2 + b_3$.

Before we address the question of marginal deformations of our conformal field theory let us discuss the relation of the above construction to 10-dimensional Superstring Theory compactified down to 3 dimensions. It is easy to show that if corresponding compact 7-dimensional manifold is a G_2 -manifold we will have $N = 2$ supersymmetry for type II strings and $N = 1$ supersymmetry for heterotic strings in 3 dimensions. Let us construct the corresponding supersymmetry generators using all the information that we already obtained. We have

$$J_{L,R} = e^{-\frac{\phi_{gh}}{2}} S_3^\alpha \sigma_{\frac{7}{16}}^{L,R}. \tag{3.22}$$

Here ϕ_{gh} is a bosonized 10-dimensional ghost field, S_3^α are 3-dimensional spin fields and σ is tri-critical Ising model spin field that we had already discussed many times.⁵ First we notice that J has dimension 1; dimension of 10-dimensional ghost part doesn't depends on compactification and always is equal to $\frac{3}{8}$, dimension of 3d spin field is $3 \cdot \frac{1}{16} = \frac{3}{16}$ and dimension of sigma by definition is $\frac{7}{16}$, and all add up to 1. If we remember that σ has a unique OPE with vacuum $[0]$ in the right-hand side we can consider J as a chiral operator and this explains subscript L, R in (3.22). Now we can define 3d supersymmetry generators: $Q_{L,R} = \oint J_{L,R}$ and standard computation leads to supersymmetry algebra. Also one finds that one of

4) Also in principle we will get other higher dimension states such as $|(\frac{3}{2}, 0)_L; (\frac{3}{2}, 0)_R\rangle$ or $|(\frac{6}{10}, \frac{2}{5})_L; (\frac{6}{10}, \frac{2}{5})_R\rangle$.

5) This is a standard ansatz for target space supersymmetry current, see [20].

the supersymmetry transforms of $(\frac{3}{80}, \frac{2}{5})_{L,R}$ which is accompanied by spacetime spinor field and ghost degrees of freedom is simply the state $(\frac{1}{10}, \frac{2}{5})_{L,R}$.

Now we would like to consider marginal deformations of our theory. As mentioned before we will show that marginal deformations are given by perturbation with dimension 1 operators of the form $G_{-1/2}^L G_{-1/2}^R [(1/10, 2/5)_L; (1/10, 2/5)_R]$; the dimension of this moduli space is $b^2 + b^3$. In addition to showing that they preserve $N = 1$ superconformal symmetry we need to show that they do not have any tri-critical piece in them, which would otherwise destroy the existence of the extended algebra in question. This follows because the full algebra was generated by the $N = 1$ algebra together with the supersymmetry operator Φ of the tri-critical model. We will first show this fact by studying the content of above operator with respect to tri-critical Ising model. For this we have to apply the operator X_0 . We have (it is enough to consider only chiral sector):

$$\begin{aligned} X_0 G_{-1/2} \left| \frac{1}{10}, \frac{2}{5} \right\rangle &= G_{-1/2} X_0 \left| \frac{1}{10}, \frac{2}{5} \right\rangle + [X_0, G_{-1/2}] \left| \frac{1}{10}, \frac{2}{5} \right\rangle \\ &= \left(-\frac{1}{2} G_{-1/2} - M_{-1/2} \right) \left| \frac{1}{10}, \frac{2}{5} \right\rangle = P. \end{aligned} \tag{3.23}$$

It turns out that the right-hand side of this equation is identically zero in our theory: $P = 0$. One can check that this state has zero norm and P is a null vector:

$$\begin{aligned} |P|^2 &= \left\langle \frac{1}{10}, \frac{2}{5} \right| (G_{1/2} - M_{1/2}) \left(\frac{1}{2} G_{-1/2} + M_{-1/2} \right) \left| \frac{1}{10}, \frac{2}{5} \right\rangle \\ &= \left\langle \frac{1}{10}, \frac{2}{5} \right| (2L_0 - 2X_0 + 8X_0 L_0) \left| \frac{1}{10}, \frac{2}{5} \right\rangle = 0. \end{aligned} \tag{3.24}$$

Here we used the fact that $|\frac{1}{10}, \frac{2}{5}\rangle$ is a highest weight representation of the whole extended chiral algebra, the property $M_n^+ = \frac{1}{2}G_{-n} - M_{-n}$ the commutation relations given in Appendix 1 and $2L_0|\frac{1}{10}, \frac{2}{5}\rangle = -2X_0|\frac{1}{10}, \frac{2}{5}\rangle = |\frac{1}{10}, \frac{2}{5}\rangle$. So we conclude that our deformation is of the type $[(0, 1)_L; (0, 1)_R]$. All we are left to show is that the deformation preserves conformal invariance.

For simplicity we will denote our perturbation by $G_{-1/2}^L A(z, \bar{z})$ (we will work with the chiral part below and thus will suppress \bar{z} dependence and $G_{-1/2}^R$). The following proof is based on two facts:

1. Dixon [7] has shown using just $N=1$ superconformal algebra that perturbation with dimension 1 operator of the form $G_{-1/2} A$ is marginal if

$$F = \langle G_{-1/2} A(z_1) G_{-1/2} A(z_2) G_{-1/2} A(z_3) G_{-1/2} A(z_4) \dots G_{-1/2} A(z_n) \rangle \tag{3.25}$$

is a total derivative with respect to coordinates $z_i, i > 3$. Perturbation is truly marginal if n -point correlation function (3.25) integrated over all points, except the first three, is zero (first three points are fixed by $SL(2, C)$ invariance on sphere)

and Dixon has shown that in $N = 1$ super conformal theory the integrand can be regulated in such a fashion that if it is a total derivative there are no contact term contributions.⁶

2. As we have seen above $A(z)$ has a null vector

$$\left(\frac{1}{2}G_{-1/2} + M_{-1/2}\right)A(z) = 0; \tag{3.26}$$

In addition we need several relations between the generators of the extended algebra acting on $A(z)$ (which is a highest weight vector and thus is killed by positive energy modes of all generators):

$$M_{1/2}G_{-1/2}A(z) = -2X_0A(z) = A(z), \tag{3.27}$$

$$M_{-1/2}G_{-1/2}A(z) = \left(-X_{-1} + \frac{1}{2}L_{-1}\right)A(z), \tag{3.28}$$

$$M_{-3/2}G_{-1/2}A(z) = -L_{-1}X_{-1}A(z). \tag{3.29}$$

In fact, one can show that it is enough to prove that

$$I_0 = \langle G_{-1/2}A(z_1)A(z_2)A(z_3)G_{-1/2}A(z_4) \dots G_{-1/2}A(z_n) \rangle \tag{3.30}$$

is a total derivative $\frac{\partial}{\partial z_i}$, $i > 3$, of something. For this we need to write $G_{-1/2}A(z_1) = \oint_{z_1} G(z)dzA(z_1)$ in (3.25) and deform the contour. If we remember that the vacuum is annihilated by $G_{+1/2}$ and $G_{-1/2}$ (and also by M_k , $k = 3/2, 1/2, -1/2, -3/2$; this we will need later) we will have no contribution from infinity and:

$$\begin{aligned} F = & - \langle A(z_1)L_{-1}A(z_2)G_{-1/2}A(z_3)G_{-1/2}A(z_4) \dots G_{-1/2}A(z_n) \rangle - \\ & \langle A(z_1)G_{-1/2}A(z_2)L_{-1}A(z_3)G_{-1/2}A(z_4) \dots G_{-1/2}A(z_n) \rangle - \dots = \\ & - \frac{\partial}{\partial z_2} \langle A(z_1)A(z_2)G_{-1/2}A(z_3)G_{-1/2}A(z_4) \dots G_{-1/2}A(z_n) \rangle - \\ & \frac{\partial}{\partial z_3} \langle A(z_1)G_{-1/2}A(z_2)A(z_3)G_{-1/2}A(z_4) \dots G_{-1/2}A(z_n) \rangle - \dots, \end{aligned} \tag{3.31}$$

where we dropped the terms that are total derivatives with respect to z_i , $i > 3$: $G_{-1/2}^2 = L_{-1} = \partial_z$. Thus, if we can show that I_0 is zero modulo $\frac{\partial}{\partial z_i}$, $i > 3$ of something, we will have the proof of marginality: $\int F = 0$. Below we will deal with the object $I = \int d^2z_4 \dots d^2z_n I_0$ and ignore total derivatives inside the integral referring the reader to the regularization used by Dixon.

6) If there is a total derivative in holomorphic variable by symmetry we get total derivative both in holomorphic and antiholomorphic coordinates $\partial_{z_i}\partial_{\bar{z}_j}$ and this is crucial in showing that there are no contact term contributions.

Our main strategy is to use the null vector condition (3.26) and contour deformation argument first for $G_{-1/2}A(z_1)$ in I and then the same argument but now replacing $G_{-1/2}A(z_1)$ by $-2M_{-1/2}A(z_1)$. First we insert $\oint_{\infty}(w - z_l)G(w)$ with contour around infinity in the correlator $\langle A(z_1)A(z_2)A(z_3)(\int G_{-1/2}A(z))^{n-3} \rangle$ and place the zero z_l at z_3 and z_2 . After the contour deformation we get

$$\begin{aligned} (z_1 - z_3) \left\langle G_{-1/2}A(z_1)A(z_2)A(z_3) \left(\int G_{-1/2}A(z) \right)^{n-3} \right\rangle + \\ (z_2 - z_3) \left\langle A(z_1)G_{-1/2}A(z_2)A(z_3) \left(\int G_{-1/2}A(z) \right)^{n-3} \right\rangle = 0, \end{aligned} \tag{3.32}$$

$$\begin{aligned} (z_1 - z_2) \left\langle G_{-1/2}A(z_1)A(z_2)A(z_3) \left(\int G_{-1/2}A(z) \right)^{n-3} \right\rangle + \\ (z_3 - z_2) \left\langle A(z_1)A(z_2)G_{-1/2}A(z_3) \left(\int G_{-1/2}A(z) \right)^{n-3} \right\rangle = 0. \end{aligned} \tag{3.33}$$

Here we used the mode expansion

$$\oint_{z_k} (z - z_l)G(z)B(z_k) = ((z_k - z_l)G_{-1/2} + G_{1/2})B(z_k) \tag{3.34}$$

for any B . The total derivative term that was ignored has an insertion of

$$\int d^2 z_4 [(z_4 - z_l)L_{-1} + 2L_0]A(z_4) = \int d^2 z_4 \frac{\partial}{\partial z_4} (z_4 - z_l)A(z_4) \tag{3.35}$$

and this identity holds only if A has dimension $\frac{1}{2}$: $2L_0A = A$.

A similar formula can be written for M , which has dimension $5/2$, and thus we need to insert $\oint_{\infty} v(z)M(z)$ with v now having three zeros. Placing zeros at points z_1, z_2, z_3 we get

$$\begin{aligned} \oint_{z_k} (z - z_1)(z - z_2)(z - z_3)M(z)B(z_k) \\ = [M_{3/2} + (z_k - z_1 + z_k - z_2 + z_k - z_3)M_{1/2} \\ + ((z_k - z_1)(z_k - z_2) + (z_k - z_1)(z_k - z_3) + (z_k - z_2)(z_k - z_3))M_{-1/2} \\ + (z_k - z_1)(z_k - z_2)(z_k - z_3)M_{-3/2}]B(z_k). \end{aligned} \tag{3.36}$$

Now we consider correlation function:

$$\left\langle \oint_{\infty} (w - z_1)(w - z_2)(w - z_3)M(w)A(z_1)A(z_2)A(z_3) \left(\int G_{-1/2}A(z) \right)^{n-3} \right\rangle = 0, \tag{3.37}$$

with contour around infinity. Again, because all modes of M that enter in (3.36) kill the vacuum, the right-hand side of (3.37) is zero; we could deform the contour and obtain the identity:

$$\begin{aligned}
 & (z_1 - z_2)(z_1 - z_3) \left\langle M_{-1/2} A(z_1) A(z_2) A(z_3) \left(\int G_{-1/2} A(z) \right)^{n-3} \right\rangle \\
 & + (z_2 - z_1)(z_2 - z_3) \left\langle A(z_1) M_{-1/2} A(z_2) A(z_3) \left(\int G_{-1/2} A(z) \right)^{n-3} \right\rangle \\
 & + (z_3 - z_1)(z_3 - z_2) \left\langle A(z_1) A(z_2) M_{-1/2} A(z_3) \left(\int G_{-1/2} A(z) \right)^{n-3} \right\rangle \\
 & + (n-3) \left\langle A(z_1) A(z_2) A(z_3) \int d^2 z_4 (z_4 - z_1 + z_4 - z_2 + z_4 - z_3) M_{1/2} G_{-1/2} A(z_4) \right. \\
 & \quad \left. \left(\int G_{-1/2} A(z) \right)^{n-4} \right\rangle + (n-3) \left\langle A(z_1) A(z_2) A(z_3) \int d^2 z_4 [(z_4 - z_1)(z_4 - z_2) \right. \\
 & \quad \left. + (z_4 - z_1)(z_4 - z_3) + (z_4 - z_2)(z_4 - z_3)] M_{-1/2} G_{-1/2} A(z_4) \left(\int G_{-1/2} A(z) \right)^{n-4} \right\rangle \\
 & + (n-3) \left\langle A(z_1) A(z_2) A(z_3) \int d^2 z_4 (z_4 - z_1)(z_4 - z_2)(z_4 - z_3) \right. \\
 & \quad \left. M_{-3/2} G_{-1/2} A(z_4) \left(\int G_{-1/2} A(z) \right)^{n-4} \right\rangle = 0.
 \end{aligned}
 \tag{3.38}$$

Now we use relations (3.27), (3.28) and (3.29), and simply find that the last three terms combined lead to the integral of total derivative in z_4 . More concretely, we write $L_{-1} = \partial$ and integrating by part in the last term of (3.38) using (3.29) we cancel contribution of X_{-1} from (3.28) in the previous term; similarly, after integration by parts, the second term from (3.28) kills the contribution of X_0 from (3.27).⁷ Thus, we drop these terms and replace $M_{-1/2}$ by $-\frac{1}{2}G_{-1/2}$. Combined with the identities (3.32) and (3.33) we see that $-\frac{3}{2}I = 0$. This leads to the proof of the statement that our perturbation is truly marginal. It is very satisfying that we used many different aspects of the extended chiral algebra for this proof.

3.2 $Spin(7)$. We will follow the ideas described above for the case of G_2 holonomy and first discuss extended symmetry for the sigma models on $Spin(7)$ manifolds. This story is completely parallel to the previous case and so we will be brief.

First we describe symmetry algebra in free field representation. As before, we take $Spin(7)$ 4-form and replace e by target space fermions; thus we get a spin 2 operator — \tilde{X} :

$$\tilde{X} = \psi^8 \Phi - X + 1/2 \partial \psi^8 \psi^8.
 \tag{3.39}$$

7) The terms that have been ignored here are total derivatives only if $2L_0 A = -2X_0 A = A$, and this condition is exactly satisfied by our choice of A .

Pleasantly we find that the operator $T_I = \frac{1}{8}\tilde{X}$ forms a Virasoro algebra with central charge $\frac{1}{2}$ and this means that the tri-critical Ising model that we had in the previous case is replaced by the ordinary, bosonic Ising model as predicted at the beginning of this section. As before, we have to check the operator expansion with original $N = 1$ generators and we immediately find that \tilde{X} has a superpartner — \tilde{M} :

$$G(z)\tilde{X}(w) = 1/2(z - w)^2G(w) + 1/(z - w)\tilde{M}(w), \tag{3.40}$$

with

$$\tilde{M} = J^8\Phi - \psi^8K - M + 1/2\partial J^8\psi^8 - 1/2J^8\partial\psi^8. \tag{3.41}$$

This operator has dimension $\frac{5}{2}$ and will play the role of the operator M . It turns out that these four operators, G, T, \tilde{X} and \tilde{M} , form a closed operator expansion algebra, which again is a quadratic W -type algebra. Corresponding formulas together with mode expansion are given in Appendix 1. From this extended symmetry algebra it follows that one can again decompose original stress-tensor as a sum of two commutative Virasoro generators

$$T = T_I + T_r, \tag{3.42}$$

and we can classify our states again by two numbers: Ising model highest weight and the eigenvalue of the zero mode of T_r : $|\Delta_I, \Delta_r\rangle$.

In chiral (left-mover) sector above the observation immediately leads to the following content:

$$|0, \Delta_r\rangle, \quad \left|\frac{1}{2}, \Delta_r\right\rangle, \quad \left|\frac{1}{16}, \Delta_r\right\rangle. \tag{3.43}$$

This means that in the Ramond sector, where we have to have dimension of ground state equal to $\frac{8}{16} = \frac{1}{2}$, (this follows from the requirement of supersymmetry — dimension of the Ramond ground state has to be equal to $\frac{c}{24}$) we should have the following highest weight states:

$$R : \quad \left|\frac{1}{2}, 0\right\rangle, \quad \left|0, \frac{1}{2}\right\rangle, \quad \left|\frac{1}{16}, \frac{7}{16}\right\rangle. \tag{3.44}$$

Amazingly enough there is again a unique state in the ground state built purely from the Ising piece, which is the $|\frac{1}{2}, 0\rangle$ state. This will now play an identical role to that of spin operator of tri-critical Ising model $[\frac{7}{16}]$ that mapped Ramond ground state to NS sector and vice versa; the specific property this operator had was that it had unique fusion rules with itself and other operator from Ramond ground state. In the $Spin(7)$ model this operator is replaced by the Ising model energy operator $\epsilon = [\frac{1}{2}]$; it has unique fusion rules and maps the Ramond ground state to a certain special NS highest weight states and vice versa:

$$NS : \quad |0, 0\rangle, \quad \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad \left|\frac{1}{16}, \frac{7}{16}\right\rangle. \tag{3.45}$$

Here we are using Ising model fusion rules: $[\epsilon][\epsilon] = [0]$, $[\epsilon][\sigma] = [\sigma]$, $[\sigma][\sigma] = [0] + [\epsilon]$, $[\sigma] = [\frac{1}{16}]$. The operator $(\frac{1}{16}, \frac{7}{16})$ has total dimension $\frac{1}{2}$ and clearly is a candidate for marginal deformation after acting by $G_{-1/2}$ on it. Again the fact that the dimension of this operator is $\frac{1}{2}$ is magical and related to the existence of spacetime supersymmetry.

In the Ising sector we have Z_2 symmetry: $\sigma \rightarrow -\sigma, 1, \epsilon \rightarrow 1, \epsilon$. We would like to show that corresponding $(-1)^{F_I}$ is again identified with total $(-1)^F$. As in the G_2 case we have to compute Ising content of the generators of the chiral algebra. We have

$$L_{-2}|0, 0\rangle = |2, 0\rangle^+ + |0, 2\rangle^+, \tilde{X}_{-2}|0, 0\rangle = |2, 0\rangle^+, \tag{3.46}$$

$$G_{-3/2}|0, 0\rangle = \left| \frac{1}{16}, \frac{23}{16} \right\rangle^-, \tilde{M}_{-5/2}|0, 0\rangle = a \left| \frac{1}{16} + 1, \frac{23}{16} \right\rangle^- + b \left| \frac{1}{16}, \frac{39}{16} \right\rangle^-, \tag{3.47}$$

and we had used the commutation relations from Appendix 1. Now we see that $(-1)^{F_I} = (-1)^F$. Thus we use the Ising model fermion number assignment.

Let us now discuss non-chiral sector putting left and right sectors together. We claim that the content of RR ground state is given by the following combinations:

$$RR : \quad \left| \left(\frac{1}{2}, 0 \right)_L ; \left(\frac{1}{2}, 0 \right)_R \right\rangle, \quad \left| \left(0, \frac{1}{2} \right)_L ; \left(0, \frac{1}{2} \right)_R \right\rangle, \\ \left| \left(0, \frac{1}{2} \right)_L ; \left(\frac{1}{16}, \frac{7}{16} \right)_R \right\rangle, \quad \left| \left(\frac{1}{16}, \frac{7}{16} \right)_L ; \left(0, \frac{1}{2} \right)_R \right\rangle, \quad \left| \left(\frac{1}{16}, \frac{7}{16} \right)_L ; \left(\frac{1}{16}, \frac{7}{16} \right)_R \right\rangle. \tag{3.48}$$

Other possible combinations can be ruled out by similar arguments as in the G_2 case — using Ising model fusion rules they lead to existence of chiral half-integer spin operators that are not present in extended chiral algebra and thus such combinations can't appear in the ground state of a generic model.

We now wish to connect the above states as much as possible with the cohomology of the manifold. As far as even degrees are concerned they come from the first, second and last state which all have $(-1)^F = +1$. Moreover we will connect all the NS versions of the last state with exactly marginal deformations, and so as discussed in Section 2 there are $1 + b_2 + b_4^-$ of them. Moreover the condition of having exactly one supersymmetry means that the first state is unique. So the second states are as many as $b_6 + b_4^+$. The second and third state correspond to odd cohomology elements and each one are in number equal to $b_3 = b_5$.

Using the unique analog of spectral flow the above content of (R,R) ground state after mapping to (NS,NS) sector due to Ising model energy operator leads to following special states

$$(NS, NS) : \quad |(0, 0)_L ; (0, 0)_R\rangle, \quad \left| \left(\frac{1}{2}, \frac{1}{2} \right)_L ; \left(\frac{1}{2}, \frac{1}{2} \right)_R \right\rangle, \\ \left| \left(\frac{1}{2}, \frac{1}{2} \right)_L ; \left(\frac{1}{16}, \frac{7}{16} \right)_R \right\rangle, \quad \left| \left(\frac{1}{16}, \frac{7}{16} \right)_L ; \left(\frac{1}{2}, \frac{1}{2} \right)_R \right\rangle, \quad \left| \left(\frac{1}{16}, \frac{7}{16} \right)_L ; \left(\frac{1}{16}, \frac{7}{16} \right)_R \right\rangle. \tag{3.49}$$

As we already mentioned operator $G_{-1/2}^L G_{-1/2}^R [(\frac{1}{16}, \frac{7}{16})_L; (\frac{1}{16}, \frac{7}{16})_R]$ is a candidate for marginal perturbation. Again we wish to show that the Ising structure is not affected by this perturbation. In other words we will show that this operator has zero dimension in Ising part. To demonstrate this fact we have to show that it is annihilated by \tilde{X}_0 (again we will keep only chiral part in this computation):

$$\begin{aligned} \tilde{X}_0 G_{-1/2} \left| \frac{1}{16}, \frac{7}{16} \right\rangle &= G_{-1/2} \tilde{X}_0 \left| \frac{1}{16}, \frac{7}{16} \right\rangle + [\tilde{X}_0, G_{-1/2}] \left| \frac{1}{16}, \frac{7}{16} \right\rangle \\ &= \left(\frac{1}{2} G_{-1/2} - \tilde{M}_{-1/2} \right) \left| \frac{1}{16}, \frac{7}{16} \right\rangle = \tilde{P}. \end{aligned} \tag{3.50}$$

\tilde{P} is a null vector, $\tilde{P} = 0$, similar to the one in G_2 case (3.26). We have for norm:

$$\begin{aligned} |\tilde{P}|^2 &= \left\langle \frac{1}{16}, \frac{7}{16} \left| (G_{1/2} + \tilde{M}_{1/2}) \left(\frac{1}{2} G_{-1/2} - \tilde{M}_{-1/2} \right) \right| \frac{1}{16}, \frac{7}{16} \right\rangle \\ &= \left\langle \frac{1}{16}, \frac{7}{16} \left| (2L_0 - 8\tilde{X}_0 + 12\tilde{X}_0 L_0) \right| \frac{1}{16}, \frac{7}{16} \right\rangle = 0. \end{aligned} \tag{3.51}$$

We used commutation relations given in Appendix 1 and relations: $\tilde{M}_n^+ = -\frac{1}{2} G_{-n} - \tilde{M}_{-n}$, $2L_0 \left| \frac{1}{16}, \frac{7}{16} \right\rangle = 2\tilde{X}_0 \left| \frac{1}{16}, \frac{7}{16} \right\rangle = \left| \frac{1}{16}, \frac{7}{16} \right\rangle$. So, we see that $G_{-1/2} \left[\frac{1}{16}, \frac{7}{16} \right]$ is of the type $(0, 1)$ and if it is truly marginal it will preserve also extended $Spin(7)$ symmetry. In addition we got a very important null vector that will allow us to prove exact marginality as in the case of G_2 .

In fact, the only information from extended chiral algebra we had used in the G_2 case to prove exact marginality was null vector condition (relation between $G_{-1/2}A$ and $M_{-1/2}A$) and commutation relation (3.27), (3.28), (3.29). Null vector condition $\tilde{P} = 0$ is practically the same (relative coefficient in \tilde{P} doesn't play a key role) and analog of (3.27), (3.28), (3.29) can be derived from the expressions in Appendix 1:

$$\tilde{M}_{1/2} G_{-1/2} A = -2\tilde{X}_0 A = -A, \tag{3.52}$$

$$\tilde{M}_{-1/2} G_{-1/2} A = \left(-\frac{1}{2} L_{-1} - \tilde{X}_{-1} \right) A, \tag{3.53}$$

$$\tilde{M}_{-3/2} G_{-1/2} A = -L_{-1} \tilde{X}_{-1} A; \tag{3.54}$$

we use the notation $A = G_{-1/2}^R [(\frac{1}{16}, \frac{7}{16})_L; (\frac{1}{16}, \frac{7}{16})_R]$. Now the argument presented in the case of G_2 can be repeated identically with the same conclusion — our perturbation is truly marginal to all orders.

§4. Examples of Joyce

Here we will study some of the examples constructed by Joyce [11]. We will review his description of some of his models. It will be clear from the construction that

the story is easily generalizable using the standard methods familiar from orbifold constructions [21]. Let us discuss a G_2 example first (Example 4 of II in [11]): Consider T^7 modded out by Z_2^3 where the generators of the Z_2 's we denote by α , β , γ . Let us represent each of them by a pair of row vectors: the holonomy part of these elements, which are simultaneously diagonal, by a row of 7 (± 1)'s and they are accompanied by shifts acting as translation on the torus which again is written by another row vector. We take each of the 7 coordinates x_i of T^7 to have period 1. Then

$$\begin{aligned}\alpha &= [(-1, -1, -1, -1, 1, 1, 1); (0, 0, 0, 0, 0, 0, 0)], \\ \beta &= [(-1, -1, 1, 1, -1, -1, 1); (0, \frac{1}{2}, 0, 0, 0, 0, 0)], \\ \gamma &= [(-1, 1, -1, 1, -1, 1, -1); (\frac{1}{2}, 0, 0, 0, 0, 0, 0)].\end{aligned}$$

Note that the above holonomies preserve ϕ defined in (2.1), and that they do not sit in an $SU(3)$ group as there is no invariant direction. If we look at the untwisted Ramond sector, which can be identified with the cohomology elements of the torus, we see that of the cohomologies of the torus we project out all except for the H^0, H^7 which are one-dimensional and 7 in H^3 and 7 in H^4 . The 7 invariant elements precisely correspond to the 7 monomials in the definition of the forms ϕ (2.1) and $*\phi$ (2.2). It is straightforward to construct the 7 twisted sectors. However since we are interested in the topological aspects, let us concentrate on the sectors which give rise to new ground states in the Ramond sector. For this to happen there should be fixed points for the group action. It is easily seen that out of the 7 non-trivial elements only three have fixed points, namely α , β and γ . The fixed point set of α consists of 2^4 three tori, each of which has 8 cohomology elements (1, 3, 3, 1). To get the final answer we have to project to the invariant subsector under the action of the full group. β and γ act freely on this set and leave us with 4 invariant combinations of the 16 T^3 's. So finally we have 4 copies of (1, 3, 3, 1) added to the Ramond ground state from this sector. Similarly one can easily see that from the β sector after projection we get 4 copies of T^3 . As far as the structure discussed in the previous section is concerned, we can only say that we get a contribution to the $b_2 + b_4 = 4$ and to the $b_3 + b_5 = 4$ from each of the total of 8 tori coming from the α and β sector. The γ sector projected to its invariant fixed point set gives 8 copies of T^3/Z_2 , where the Z_2 acts in the neighborhood of each of these T^3 's by

$$(y_1, y_2, y_3, z_1, z_2) \rightarrow \left(\frac{1}{2} + y_1, -y_2, -y_3, z_1, -z_2\right),$$

where y_i denote the coordinates of the fixed T^3 and the z_i denote in complex notation the orthogonal direction (which by the action of γ goes to minus itself). Of the cohomologies of each of these T^3 's from the above Z_2 action only two elements survive, two in odd and two in even cohomology; so we get from the total of 8 tori the addition of 16 to $b_2 + b_4$ and addition of 16 to $b_3 + b_5$ from the γ sector. If we

put the contributions of all the sectors together we find

$$b_0 = 1; \quad b_2 + b_4 = 55; \quad b_3 + b_5 = 55; \quad b_7 = 1.$$

As noted in the previous section we must thus have a 55-dimensional moduli space: 7 of the moduli come from the untwisted sector and correspond to the 7 radii of T^7 . The other 48 come from blow up modes in the twisted sectors. As proven in the previous section all these deformations are exactly marginal. Just to give a better feeling for how the algebra discussed in the previous section fit with the geometry, let us describe the untwisted moduli. The primary superconformal field of dimension $1/2$ which correspond to the untwisted moduli are nothing but the ψ^i for $i = 1, \dots, 7$. From equation (3.3) we see that ψ^i has under X_0 the eigenvalue $-1/2$ which implies that for the tri-critical part of the energy momentum tensor, it has eigenvalue $-X_0/5 = 1/10$ as predicted by the analysis in the previous section. Note that we see the crucial role played by the normal ordered terms in the definition of X , which is responsible for giving ψ^i a tri-critical dimension of $1/10$ rather than zero. Also note that when we take $G_{-1/2}\psi^i = \partial X^i = J^i$, it is easy to see that it thus commutes with X_0 . This in particular means that the tri-critical dimension of it is 0, again a fact proven in full generality in the previous section.

Note that as emphasized in the previous section, physically we cannot identify b_2 and b_4 separately. Amazingly enough this structure is reflected mathematically and gives a first non-trivial example of our generalized mirror conjecture: there are inequivalent ways the singularities can be resolved to give manifolds with different Betti numbers, but in all these cases $b_2 + b_4$ is the same. More precisely Joyce found that depending on how he desingularizes the manifold

$$b_2 = 8 + l; \quad b_4 = 47 - l,$$

where l runs from 0 to 8. These different ways of resolving the singularity have to do with the fact that, when one desingularizes the fixed tori of γ action, there are different ways the Z_2 that we have to mod out acts: more precisely, the desingularization can take place using the Eguchi-Hanson space which is $T^*(CP^1)$ (as the orthogonal direction is locally R^4/Z_2). But the Z_2 written above can act in two different ways on the resolved space. If we let z be the coordinate of CP^1 , then the involution acting on $T^*(CP^1)$ can come from $z \rightarrow -z$ or $z \rightarrow \bar{z}$. In the first case we get a contribution to $\Delta b_2 = 1$ and $\Delta b_3 = 1$, and in the other case we get $\Delta b_2 = 0$ and $\Delta b_3 = 2$. In either case in the limit of shrinking down the sphere we get the Z_2 action above after appropriate redefinition of coordinates. It turns out that even though there are 2 ways of doing the desingularization for each of the eight tori, there are only 9 inequivalent Betti numbers which are listed above. But we know physically (from the conformal theory perturbations discussed in the previous section) that the moduli space is smooth near the orbifold point and so at most the difference between these answers has to do with turning on different

marginal operators. Thus we see that topologically distinct manifolds, as allowed from the ambiguity of decoding b_2 and b_4 , give rise to the same conformal theory (up to moduli deformation) as suggested by the generalized mirror conjecture⁸.

Actually there is one subtlety which needs to be considered: we have assumed that there is a unique orbifold theory. However there is the possibility of turning on discrete torsion [25] and thus we could have inequivalent orbifold theories. In the above example we could for example turn on a discrete torsion between two of the Z_2 's. However in the case of Z_2 torsions this does not lead to a new theory (and in the case of Calabi-Yau gives a simple example of mirror symmetry). However if instead of Z_2 's we had Z_n 's, the story would have been different. Indeed in that case we expect inequivalent theories at the orbifold points related to each other by turning on a discrete torsion. In such a case one would also expect that geometrically there should exist inequivalent ways of resolving the singularity — but here one would not expect them to preserve $b_2 + b_4$ because the underlying conformal theories are different. This prediction has been confirmed by a local model for the $Z_n \times Z_n$ singularity replacing the $Z_2 \times Z_2$ above [24]. In that case one finds that there are $n - 1$ different choices of resolution which lead to $\Delta b_2 = 0$, $\Delta b_4 = 2$ and one choice where $\Delta b_2 = \Delta b_4 = n - 1$. It is easy to check in the conformal theory computation that the turning on of the $n - 1$ different possibilities for discrete torsion lead to the first answer and no discrete torsion leads to the second answer, thus confirming the correspondence between conformal theory and the geometry of G_2 holonomy manifold.

There are other classes of examples of G_2 holonomy manifolds constructed by Joyce. One particularly general construction he suggested is to start from a Calabi-Yau three-fold M which has a real involution (an involution which locally looks like $z \rightarrow z^*$). This would be the case for example if one considers algebraic varieties with real coefficients in the defining equations. Then one may obtain a G_2 holonomy orbifold by considering

$$\frac{M \times S^1}{Z_2},$$

where Z_2 sends $M \rightarrow M^*$ and is a reflection on the circle. It is clear that the holonomy of this Z_2 ($4(-1)$'s and $2(+1)$'s) preserves supersymmetry and thus leads to a G_2 holonomy manifold. There are orbifold singularities which as we know physically are harmless. It is tempting to speculate that, using this construction, one can interpolate between Calabi-Yau mirrors by going through points on the moduli space of the G_2 holonomy manifold where b_2 and b_3 change but their sum does not change.

8) One may be tempted to identify this with the flop phenomenon for which distinct manifolds (albeit with the same hodge numbers) are part of the same moduli space of conformal theory [22], [23]. However, even though there are analogs of flop phenomenon for G_2 manifolds, this is not one of them [24].

The examples of (7) holonomy manifolds proceeds very similar to the above, and so we just summarize the main features. Again one starts with an 8-dimensional torus and mods out by some isometries, the simplest of which is Z_2^4 and resolves the singularities to obtain a smooth 8-dimensional manifold of $Spin(7)$ holonomy. Again one sees that there are inequivalent ways to desingularize manifolds but all have the property that they lead to the same sum for the even cohomology elements and for the odd cohomology elements, as predicted from the conformal theory view point. These examples therefore provide further evidence for the generalized mirror conjecture.

§5. Topological Twist

In previous sections we have shown that G_2 and $Spin(7)$ compactifications are very similar to $N = 2$ superconformal theories corresponding to $SU(n)$ or $N = 4$ corresponding to $Sp(n)$ holonomy. In particular they both lead to $N = 1$ spacetime supersymmetry upon heterotic compactification. In $N = 2$ (and similarly in the $N = 4$ [26]) there is a topological side to the story, which is deeply connected to spacetime supersymmetry in the compactified theory. Basically the spectral flow operator, which is the same operator used to construct spacetime supersymmetry operator is responsible for the twisting. Twisting is basically the same as insertions of $2g - 2$ of these operators at genus g . The spectral flow operator is constructed entirely out of the $U(1)$ piece of the $N = 2$ theory and since the spectral flow operator can be written as

$$\sigma = \exp(i\rho/2) \quad J = \partial\rho,$$

the twisting becomes equivalent to modifying the stress-tensor by

$$T \rightarrow T + \frac{\partial^2 \rho}{2},$$

where J is the $U(1)$ current of $N = 2$. With this change in the energy momentum tensor the central charge of the theory becomes zero. Once one does this twisting the chiral fields which are related by spectral flow operator to the ground states of the Ramond sector become dimension 0 and form a nice closed ring known as the chiral ring [2]. Given the similarities to $N = 2$ we would like to explore analogous construction for G_2 and $Spin(7)$. In the $N = 2$ case the main modification in the theory was in the $U(1)$ piece of the theory. Therefore also here we expect the main modifications to be in the tri-critical Ising piece for the G_2 and in the Ising piece for the $Spin(7)$ case.

Let us concentrate on the sphere. As noted above abstractly, on the sphere one can define twisted correlation functions by insertion of two spin fields ($\sigma_{\frac{7}{16}}$ in G_2 case and $\sigma_{\frac{1}{2}}$ in $Spin(7)$ case) in the NS sector:

$$\langle V_1(z_1, \bar{z}_1) \dots V_n(z_n, \bar{z}_n) \rangle_{twisted} = \langle \sigma(0) V_1(z_1, \bar{z}_1) \dots V_n(z_n, \bar{z}_n) \sigma(\infty) \rangle_{untwisted}. \tag{5.1}$$

Let us check this idea by bosonizing the Ising sector. First we discuss G_2 . Bosonized tri-critical Ising supercurrent and stress-tensor have the form

$$\Phi = e^{\frac{3i}{\sqrt{5}}\varphi}, \quad (5.2)$$

$$X = (\partial\varphi)^2 + \frac{1}{4\sqrt{5}}\partial^2\varphi. \quad (5.3)$$

At the same time we can write down the chiral primaries in terms of boson φ :

$$[0] = I, \quad (5.4)$$

$$\left[\frac{1}{10}\right] = e^{\frac{i}{\sqrt{5}}\varphi}, \quad (5.5)$$

$$\left[\frac{6}{10}\right] = e^{\frac{2i}{\sqrt{5}}\varphi}, \quad (5.6)$$

$$\left[\frac{7}{16}\right] = e^{\frac{-5i}{4\sqrt{5}}\varphi}, \quad (5.7)$$

$$\left[\frac{3}{80}\right] = e^{\frac{-i}{4\sqrt{5}}\varphi}. \quad (5.8)$$

Background charge is $-2\alpha_0 = -\frac{1}{2\sqrt{5}}$ and one can check that central charge is correct $c = 1 - 24\alpha_0^2 = \frac{7}{10}$. Insertion of spin fields according to (5.1) and (5.7) is equivalent to a change in background charge $-2\alpha_0 \rightarrow -2\tilde{\alpha}_0 = -\frac{3}{\sqrt{5}}$, and thus the new stress-tensor that replaces X is $X_{tw} = (\partial\varphi)^2 - \frac{3}{2\sqrt{5}}\partial^2\varphi$ with central charge $\tilde{c}_{tw} = 1 - 24\tilde{\alpha}_0^2 = -\frac{98}{10}$. If we compute total central charge (we don't touch the remaining sector T_r by our twist) since the central charge of T_r is equal to $21/2 - 7/10 = 98/10$ and we have not changed it by the twisting we get: $c_{twist} = -98/10 + 98/10 = 0$. This is indeed remarkable! It is the strongest hint for the existence of a topological theory. Obviously, before twisting we have a minimal model and correct vertex operators are given by above formulas dressed by screening operators (see [27], [28], [29]); screening charges are: $\alpha_+ = \frac{5}{2\sqrt{5}}, \alpha_- = -\frac{2}{\sqrt{5}}$. At the same time after twisting we get a model which is not a minimal model and if now correlation functions of above operators aren't non-zero they can't be screened. Thus, after twisting when we calculate correlation functions we could forget about dressing by screening operators and do just naive computation. This simplifies the story. Vertex operators are the same, but their dimensions are now different. We have

$$\left[\frac{1}{10}\right] \rightarrow \left[-\frac{2}{5}\right], \quad (5.9)$$

$$\left[\frac{6}{10}\right] \rightarrow \left[-\frac{2}{5}\right], \quad (5.10)$$

$$\left[\frac{3}{2}\right] \rightarrow [0]. \quad (5.11)$$

Note that in particular we learn that the special states we get in the NS sector have total dimension zero in the topological theory:

$$\begin{aligned} \left| \frac{1}{10}, \frac{2}{5} \right\rangle &\rightarrow \left| -\frac{2}{5}, \frac{2}{5} \right\rangle, \\ \left| \frac{6}{10}, \frac{2}{5} \right\rangle &\rightarrow \left| -\frac{2}{5}, \frac{2}{5} \right\rangle, \\ \left| \frac{3}{2}, 0 \right\rangle &\rightarrow |0, 0\rangle, \end{aligned}$$

which is what one would expect of topological observables. Moreover they do seem to form a ring under multiplication. This can be checked explicitly for example for the untwisted moduli of the toroidal compactification discussed in the previous section. Concentrating on left-movers, the states of the first type are written as ψ^i for $i = 1, \dots, 7$. Now under naive product between the ψ^i there would be poles because of contractions, but one can see that they do not contribute to the topological amplitude because they fail to cancel the background charge in the topological theory. In fact the ring they form in this case is

$$\langle \psi^i \psi^j \psi^k \rangle = f_{ijk},$$

where f_{ijk} are defined by $\Phi = f_{ijk} \psi^i \psi^j \psi^k$ (note that the 6/10 states above are nothing but the quadratic fermion terms).

The expressions for the shift in the dimension of the tri-critical piece together with the fact that we have already discussed the tri-critical content of the generators of the chiral algebra means that we can deduce their twisted dimension. We find that they all have shifted to integer dimensions, another hallmark of topological theories: $G - dim.1$, $\Phi - dim.0$, $M - dim.2$, plus we got dimension 1 bosonic operator K . Thus, after twisting, G is a candidate for BRST current of the topological theory and M — for antighost. To prove the last statement we need to show that OPE's of G with itself, as well as M with itself don't have simple poles (or at least do not contribute to the amplitudes) and in addition, G with M have the modified stress-tensor as a residue of simple pole. This would need to be verified. It should also be verified that with this sense of topological BRST invariance the above special states in the NS sector indeed are BRST invariant.

It is not difficult to repeat above procedure for the case of $Spin(7)$. Bosonized Ising stress-tensor has the form:

$$\tilde{X} = (\partial\varphi)^2 + \frac{1}{4\sqrt{3}}\partial^2\varphi \tag{5.12}$$

and chiral primaries are:

$$[0] = I, \tag{5.13}$$

$$\left[\frac{1}{2} \right] = e^{\frac{3i}{2\sqrt{3}}\varphi}, \tag{5.14}$$

$$\left[\frac{1}{16} \right] = e^{\frac{3i}{4\sqrt{3}}\varphi}. \tag{5.15}$$

Background charge is $-2\alpha_0 = -\frac{1}{2\sqrt{3}}$ and screening charges are $\alpha_+ = -\frac{3}{2\sqrt{3}}$, $\alpha_- = \frac{2}{\sqrt{3}}$. Bosonized vertex operators are given by above expressions dressed with n_1 screening charges of type α_+ and n_2 of the type α_- . Insertion of spin fields $\sigma_{\frac{1}{2}}$ according (5.1) in the picture with $n_1 = 6, n_2 = 2$ is equivalent to a change in background charge $-2\alpha_0 \rightarrow -2\tilde{\alpha}_0 = -\frac{5}{2\sqrt{3}}$.⁹ Thus, the new stress-tensor is given by $\tilde{X}_{tw} = (\partial\varphi)^2 + \frac{5}{4\sqrt{3}}\partial^2\varphi$ with central charge $\tilde{c}_{tw} = 1 - 24\tilde{\alpha}_0^2 = -\frac{23}{2}$. If we remember that the central charge of T_r was $12 - \frac{1}{2} = \frac{23}{2}$ and it has remained unchanged under our twist we will find another remarkable coincidence: total central charge after twist is 0! Now we can check other properties discovered above for the case of G_2 . Vertex operators remain the same (5.13), (5.14), (5.15), but now they have different dimensions:

$$\left[\frac{1}{2} \right] \longrightarrow \left[-\frac{1}{2} \right], \quad (5.16)$$

$$\left[\frac{1}{16} \right] \longrightarrow \left[-\frac{7}{16} \right]. \quad (5.17)$$

So, special states we had in NS sector have total dimension zero in the topological theory:

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle \rightarrow \left| -\frac{1}{2}, \frac{1}{2} \right\rangle, \quad (5.18)$$

$$\left| \frac{1}{16}, \frac{7}{16} \right\rangle \rightarrow \left| -\frac{7}{16}, \frac{7}{16} \right\rangle. \quad (5.19)$$

Also, one can use relations (3.47) and show that the dimensions of fermionic operators G and \tilde{M} are shifted properly: $\dim G = 1, \dim \tilde{M} = 2$. This means that once again G is a candidate for the BRST current and \tilde{M} — for antighost.

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9) Note that in G_2 case we had used $n_1 = 0, n_2 = 0$ picture, which was the minimal solution in that case.

Appendix 1

Below we give the result of computations mentioned in the Section 3 and corresponding mode expansion.

For OPE we have:

$$G(z)K(w) = \frac{3}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi \tag{1.1}$$

$$G(z)M(w) = -\frac{7}{2}\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}(T+4X)(w) + \frac{1}{z-w}\partial X(w) \tag{1.2}$$

$$\Phi(z)K(w) = -\frac{3}{(z-w)^2}G(w) - \frac{3}{z-w}\left(M + \frac{1}{2}\partial G\right)(w) \tag{1.3}$$

$$\Phi(z)M(w) = \frac{9}{2}\frac{1}{(z-w)^2}K(w) - \frac{1}{z-w}\left(3 : G(w)\Phi(w) : - \frac{5}{2}\partial K(w)\right) \tag{1.4}$$

$$X(z)K(w) = -\frac{3}{(z-w)^2}K(w) + \frac{3}{z-w}(: G(w)\Phi(w) : - \partial K(w)) \tag{1.5}$$

$$X(z)M(w) = -\frac{9}{2}\frac{1}{(z-w)^3}G(w) - \frac{1}{(z-w)^2}\left(5M + \frac{9}{4}\partial G\right)(w) + \frac{1}{z-w}\left(4 : G(w)X(w) : - \frac{7}{2}\partial M(w) - \frac{3}{4}\partial^2 G(w)\right) \tag{1.6}$$

$$K(z)K(w) = -\frac{21}{(z-w)^4} + \frac{6}{(z-w)^2}(X-T)(w) + \frac{3}{z-w}\partial(X-T)(w) \tag{1.7}$$

$$K(z)M(w) = -\frac{15}{(z-w)^3}\Phi(w) - \frac{11}{2}\frac{1}{(z-w)^2}\partial\Phi(w) + \frac{3}{z-w}(: G(w)K(w) : + 2 : T(w)\Phi(w) :) \tag{1.8}$$

$$M(z)M(w) = -\frac{35}{(z-w)^5} + \frac{1}{(z-w)^3}(20X-9T)(w) + \frac{1}{(z-w)^2}\left(10\partial X - \frac{9}{2}\partial T\right)(w) - \frac{1}{z-w}\left(\frac{3}{2}\partial^2 X(w) - \frac{3}{2}\partial^2 T(w) - 4 : G(w)M(w) : + 8 : T(w)X(w) : \right) \tag{1.9}$$

$$T(z)M(w) = -\frac{1}{2}\frac{1}{(z-w)^3}G(w) + \frac{5}{2}\frac{1}{(z-w)^2}M(w) + \frac{1}{z-w}\partial M(w). \tag{1.10}$$

In the case of *Spin*7 algebra looks simpler:

$$\tilde{X} = \psi^8\Phi - X + 1/2\partial\psi^8\psi^8 \tag{1.11}$$

$$\tilde{M} = J^8\Phi - \psi^8K - M + 1/2\partial J^8\psi^8 - 1/2J^0\partial\psi^0 \quad (1.12)$$

$$\tilde{X}(z)\tilde{X}(w) = 16/(z-w)^4 + 16/(z-w)^2\tilde{X}(w) + 8/(z-w)\partial\tilde{X}(w) \quad (1.13)$$

$$T(z)\tilde{X}(w)2/(z-w)^4 + 1/(z-w)^2(\tilde{X}(w) + \tilde{X}(z)) \quad (1.14)$$

$$G(z)\tilde{X}(w) = 1/2(z-w)^2G(w) + 1/(z-w)\tilde{M}(w) \quad (1.15)$$

$$G(z)\tilde{M}(w) = \frac{4}{(z-w)^4} - \frac{1}{(z-w)^2}(T(w) - 4\tilde{X}(w)) + \frac{1}{z-w}\partial\tilde{X}(w) \quad (1.16)$$

$$\begin{aligned} \tilde{X}(z)\tilde{M}(w) = & -15/2(z-w)^3G(w) - 1/(z-w)^2(15/4\partial G(w) - \\ & 8\tilde{M}(w)) + 1/(z-w)(11/2\partial\tilde{M}(w) - 5/4\partial^2G(w) - \\ & 6 : G(w)\tilde{X}(w) :) \end{aligned} \quad (1.17)$$

$$\begin{aligned} \tilde{M}(z)\tilde{M}(w) = & -64/(z-w)^5 - 1/(z-w)^3(15T(w) + 32\tilde{X}(w)) - \\ & 1/(z-w)^2(15/2\partial T(w) + 16\partial\tilde{X}(w)) - \\ & 1/(z-w)(5/2\partial^2\tilde{X}(w) + \\ & 5/2\partial^2T(w) + 12 : T(w)\tilde{X}(w) : - 6 : G(w)\tilde{M}(w) :). \end{aligned} \quad (1.18)$$

Now, if we use mode expansion for our generators $B(z) = B_n z^{-n-\Delta}$, where Δ is a dimension of operator B , we have (we use the normal ordering prescription $:AB :_n = \sum_{p < -\Delta_A - 1} A_p B_{n-p} + (-1)^{AB} \sum_{p > -\Delta_A} B_{n-p} A_p$):

$$\Phi_n^+ = -\Phi_{-n}, \quad K_n^+ = -K_{-n}, \quad M_n^+ = \frac{1}{2}G_{-n} - M_{-n} \quad (1.19)$$

$$\{G_n, G_m\} = \frac{7}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} + 2L_{n+m} \quad (1.20)$$

$$[L_n, L_m] = \frac{21}{24}(n^3 - n)\delta_{n+m,0} + (n-m)L_{n+m} \quad (1.21)$$

$$[L_n, G_m] = \left(\frac{1}{2}n - m\right)G_{n+m} \quad (1.22)$$

$$\{\Phi_n, \Phi_m\} = -\frac{7}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} + 6X_{n+m} \quad (1.23)$$

$$[X_n, \Phi_m] = -5\left(\frac{1}{2}n - m\right)\Phi_{n+m} \quad (1.24)$$

$$[X_n, X_m] = \frac{35}{24}(n^3 - n)\delta_{n+m,0} - 5(n-m)X_{n+m} \quad (1.25)$$

$$[L_n, X_m] = -\frac{7}{24}(n^3 - n)\delta_{n+m,0} + (n-m)X_{n+m} \quad (1.26)$$

$$\{G_n, \Phi_m\} = K_{n+m} \quad (1.27)$$

$$[G_n, K_m] = (2n-m)\Phi_{n+m} \quad (1.28)$$

$$[G_n, X_m] = -\frac{1}{2}\left(n + \frac{1}{2}\right)G_{n+m} + M_{n+m} \quad (1.29)$$

$$\{G_n, M_m\} = -\frac{7}{12}\left(n^2 - \frac{1}{4}\right)\left(n - \frac{3}{2}\right)\delta_{n+m,0} + \left(n + \frac{1}{2}\right)L_{n+m} + (3n - m)X_{n+m} \quad (1.30)$$

$$[\Phi_n, K_m] = \frac{3}{2}\left(m - n + \frac{1}{2}\right)G_{n+m} - 3M_{n+m} \quad (1.31)$$

$$\{\Phi_n, M_m\} = \left(2n - \frac{5}{2}m - \frac{11}{4}\right)K_{n+m} - 3 : G\Phi :_{n+m} \quad (1.32)$$

$$[X_n, K_m] = 3(m + 1)K_{n+m} + 3 : G\Phi :_{n+m} \quad (1.33)$$

$$[X_n, M_m] = \left[\frac{9}{4}(n + 1)\left(m + \frac{3}{2}\right) - \frac{3}{4}\left(n + m + \frac{3}{2}\right)\left(n + m + \frac{5}{2}\right)\right]G_{n+m} - \left[5(n + 1) - \frac{7}{2}\left(n + m + \frac{5}{2}\right)\right]M_{n+m} + 4 : GX :_{n+m} \quad (1.34)$$

$$[K_n, K_m] = -\frac{21}{6}(n^3 - n)\delta_{n+m,0} + 3(n - m)(X_{n+m} - L_{n+m}) \quad (1.35)$$

$$[K_n, M_m] = \left[\frac{11}{2}(n + 1)\left(n + m + \frac{3}{2}\right) - \frac{15}{2}(n + 1)n\right]\Phi_{n+m} + 3 : GK :_{n+m} - 6 : L\Phi :_{n+m} \quad (1.36)$$

$$\begin{aligned} \{M_n, M_m\} = & -\frac{35}{24}\left(n^2 - \frac{1}{4}\right)\left(n^2 - \frac{9}{4}\right)\delta_{n+m,0} + \left[\frac{3}{2}(n + m + 2)(n + m + 3) - \right. \\ & \left. 10\left(n + \frac{3}{2}\right)\left(m + \frac{3}{2}\right)\right]X_{n+m} + \left[\frac{9}{2}\left(n + \frac{3}{2}\right)\left(m + \frac{3}{2}\right) - \right. \\ & \left. - \frac{3}{2}(n + m + 2)(n + m + 3)\right]L_{n+m} - 4 : GM :_{n+m} + 8 : LX :_{n+m} . \end{aligned} \quad (1.37)$$

We have similar equations for the case of $Spin(7)$. Operator \tilde{M} again has nonstandard conjugation property

$$\tilde{M}_n^+ = -\frac{1}{2}G_{-n} - \tilde{M}_{-n}, \quad (1.38)$$

and commutation relations are given by:

$$[\tilde{X}_n, \tilde{X}_m] = \frac{16}{6}(n^3 - n)\delta_{n+m,0} + 8(n - m)\tilde{X}_{n+m}, \quad (1.39)$$

$$[L_n, \tilde{X}_m] = \frac{1}{3}(n^3 - n)\delta_{n+m,0} + (n - m)\tilde{X}_{n+m}, \quad (1.40)$$

$$[G_n, \tilde{X}_m] = \frac{1}{2}\left(n + \frac{1}{2}\right)G_{n+m} + \tilde{M}_{n+m}, \quad (1.41)$$

$$\{G_n, \tilde{M}_m\} = \frac{2}{3}\left(n^2 - \frac{1}{4}\right)\left(n - \frac{3}{2}\right)\delta_{n+m,0} - \left(n + \frac{1}{2}\right)L_{n+m} + (3n - m)\tilde{X}_{n+m}, \quad (1.42)$$

$$\begin{aligned}
[\tilde{X}_n, \tilde{M}_m] = & \left[\frac{15}{4}(n+1)\left(m + \frac{3}{2}\right) - \frac{5}{4}\left(n+m + \frac{3}{2}\right)\left(n+m + \frac{5}{2}\right) \right] G_{n+m} - \\
& \left[8(n+1) + \frac{11}{2}\left(n+m + \frac{5}{2}\right) \right] \tilde{M}_{n+m} - 6 : G\tilde{X} :_{n+m}, \quad (1.43)
\end{aligned}$$

$$\begin{aligned}
\{\tilde{M}_n, \tilde{M}_m\} = & -\frac{8}{3}\left(n^2 - \frac{9}{4}\right)\left(n^2 - \frac{1}{4}\right) + \left[\frac{15}{2}\left(n + \frac{3}{2}\right)\left(m + \frac{3}{2}\right) - \right. \\
& \left. \frac{5}{2}(n+m+2)(n+m+3) \right] L_{n+m} + \\
& \left[16\left(n + \frac{3}{2}\right)\left(m + \frac{3}{2}\right) - \right. \\
& \left. \frac{5}{2}(n+m+2)(n+m+3) \right] \tilde{X}_{n+m} + 12 : L\tilde{X} :_{n+m} - \\
& 6 : G\tilde{M} :_{n+m}. \quad (1.44)
\end{aligned}$$

Appendix 2

First, let us note that from commutation relations, given in Appendix 1, and (1.19), (1.38) we could derive following identities:

$$|M_{-1/2}(0,0)|^2 = (0,0)^* M_{1/2}^+ M_{-1/2}(0,0) = 0, \quad (2.1)$$

$$|M_{-3/2}(0,0)|^2 = (0,0)^* M_{3/2}^+ M_{-3/2}(0,0) = 0,$$

$$|\tilde{M}_{-1/2}(0,0)|^2 = (0,0)^* \tilde{M}_{1/2}^+ \tilde{M}_{-1/2}(0,0) = 0, \quad (2.2)$$

$$|\tilde{M}_{-3/2}(0,0)|^2 = (0,0)^* \tilde{M}_{3/2}^+ \tilde{M}_{-3/2}(0,0) = 0.$$

These identities are necessary because as we had already seen operators M and \tilde{M} have nonstandard conjugation properties and in principle $M_{-1/2}$, $M_{-3/2}$, $\tilde{M}_{-1/2}$, $\tilde{M}_{-3/2}$ might not annihilate the vacuum. But we see that they do.

Finally, we will show the validity of (3.18), (3.19), (3.46) and (3.47). First two identities in (3.18) and (3.46) are obvious and to derive the last one in (3.18) and (3.19) ((3.47)) we have to apply zero mode $T_0^I = -\frac{1}{5}X_0$ ($\tilde{T}_0^I = \frac{1}{8}\tilde{X}_0$) to the left hand side:

$$\begin{aligned}
-\frac{1}{5}X_0 G_{-3/2}|0,0\rangle &= \frac{1}{10}G_{-3/2}|0,0\rangle - \frac{1}{5}M_{-3/2}|0,0\rangle = \\
&= \frac{1}{10}G_{-3/2}|0,0\rangle, \quad (2.3)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{8}\tilde{X}_0 \tilde{G}_{-3/2}|0,0\rangle &= \frac{1}{16}G_{-3/2}|0,0\rangle + \frac{1}{8}M_{-3/2}|0,0\rangle = \\
&= \frac{1}{16}G_{-3/2}|0,0\rangle. \quad (2.4)
\end{aligned}$$

Here we used (2.1) and (2.2). Another useful relation is (1.29) (for $Spin(7)$ — (1.41)) which leads to:

$$\begin{aligned} M_{-5/2}|0,0\rangle &= -X_{-1}G_{-3/2}|0,0\rangle - \frac{1}{2}L_{-1}G_{-3/2}|0,0\rangle = \\ &= \left(-\frac{7}{10}X_{-1} - \frac{1}{2}L_{-1}^r\right)G_{-3/2}|0,0\rangle, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \tilde{M}_{-5/2}|0,0\rangle &= -\tilde{X}_{-1}G_{-3/2}|0,0\rangle + \frac{1}{2}L_{-1}G_{-3/2}|0,0\rangle = \\ &= \left(-\frac{7}{16}\tilde{X}_{-1} + \frac{1}{2}L_{-1}^r\right)G_{-3/2}|0,0\rangle. \end{aligned} \quad (2.6)$$

For $K_{-2}|0,0\rangle$ we simply use:

$$K_{-2}|0,0\rangle = \Phi_{-1/2}G_{-3/2}|0,0\rangle. \quad (2.7)$$

From (2.3) and (2.4) it follows that $G_{-3/2}|0,0\rangle$ is a linear combination of $|\frac{1}{10}, \frac{14}{10}\rangle$ ($|\frac{1}{16}, \frac{23}{16}\rangle$) and $|0, \frac{3}{2}\rangle$, but latter can be excluded because there is no half-integer chiral spin $\frac{3}{2}$ operator in the T^r sector of our theory. Thus, $G_{-3/2}|0,0\rangle = |\frac{1}{10}, \frac{14}{10}\rangle$ and relations (3.18), (3.19), (3.46) and (3.47) are consequences of above computations.

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