CRAIG'S THEOREM IN SUPERINTUITIONISTIC LOGICS AND AMALGAMABLE VARIETIES OF PSEUDO-BOOLEAN ALGEBRAS

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In recent years there have appeared many studies of an interesting and important property of logical theories, the so-called Craig interpolation theorem. Craig proved the interpolation theorem for classical predicate logic in 1957 [9]. Schütte [23] proved the interpolation theorem for intuitionistic predicate logic, and Gabbay [12] for certain extensions of this logic. In [10, 13] the interpolation theorem was proved for a number of modal logics, and in [21] for many-valued predicate calculi.

We will study the Craig property in connection with extensions of intuitionistic propositional logic, the so-called superintuitionistic logics. The results of [9, 12, 23] imply the validity of the interpolation theorem for three superintuitionistic logics. It turns out that in the whole continuum of consistent superintuitionistic logics the Craig interpolation theorem is true for only seven (Theorem 3). Consequently, the following problem is solvable: given a finite system of axioms of a superintuitionistic logic, determine whether the Craig interpolation theorem (CIT) is true in this logic.

We obtain a complete description of the superintuitionistic logics with CIT from a description of all amalgamable varieties of pseudo-Boolean algebras (of which there are eight, including the trival one). It is known [1] that there is a one-to-one correspondence between the family of all superintuitionistic logics and the family of varieties of pseudo-Boolean algebras. It turns out that the CIT in superintuitionistic logics is equivalent to the so-called interpolation principle for equalities (IPE) in the corresponding varieties of pseudo-Boolean algebras. Jónsson [17] showed that, under certain conditions, the IPE in varieties of algebras follows from the amalgamation property. It was noted in [8] that the reverse implication for varieties is, in general, false. For varieties of pseudo-Boolean algebras, the amalgamation property and the IPE turn out to be equivalent to each other and equivalent to the strong amalgamation property and to the superamalgamation property (Theorem 1).

In [8] there is given a classification of the different versions of the interpolation theorem. In particular, it is noted that in any variety K, the IPE is equivalent to the property $\mathfrak{Int}(E^+, U^+, 0^+)$:

If α is a positive $\exists$-formula, β a positive $\forall$-formula, and $\models_K \alpha \Rightarrow \beta$, then there exists an unquantified positive formula γ such that $\models_K \alpha \Rightarrow \gamma$ and $\models_K \gamma \Rightarrow \beta$.

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It was also shown there that the amalgamation property of a universally axiomatizable class K is equivalent to the property $\mathfrak{Int}(E, U, O)$, the formulation of which can be obtained from the definition of $\mathfrak{Int}(E^+, U^+, O^+)$ by omitting the word "positive." Therefore, in view of Theorem 1, we obtain for varieties of pseudo-Boolean algebras the equivalence of the amalgamation property, strong amalgamation property, IPE, $\mathfrak{Int}(E, U, O)$, and $\mathfrak{Int}(E^+, U^+, O^+)$.

We should mention that the amalgamation property is being intensively studied at the present time. In particular, all amalgamable varieties of modular lattices [16] and pseudo-complemented lattices [15] have been described.

In [19] there was observed the equivalence of Craig's theorem in the systems of classical predicate logic and the amalgamation property of classes of cylindric algebras, and the amalgamation property for various classes of cylindric algebras was studied.

We mention also that from the results of this present paper it can be shown that there are exactly four amalgamable varieties of implicative lattice (relatively pseudocomplemented lattices) and three consistent extensions of the positive fragment of intuitionistic logic, namely the positive fragments of intuitionistic logic, classical logic, and Dummett's logic LC. By our methods we can also obtain Craig's theorem in certain enrichments of intuitionistic logic (by additional connectives). For example, the CIT is true in the propositional logic $H-B$ [22], and the variety of corresponding algebras is strongly amalgamable.

I. Craig's Theorem and the Amalgamation Property

The formulas of propositional logic will be constructed in the usual way from propositional variables and the propositional constant 1 ("true") by means of the connectives $\&, \vee, \supset, \neg$; we also use the notation $x \equiv y \leftrightharpoons (x \supset y) \& (y \supset x)$. By a superintuitionistic logic we mean any set of formulas containing the axioms of intuitionistic logic and closed under the rules of substitution and modus ponens.

By the Craig interpolation theorem (CIT) in a logic L we mean the following proposition:

For any formulas A and B, if $(A \supset B) \in L$, then there exists a formula C such that $(A \supset C) \in L$ and $(C \supset B) \in L$ and C contains only those variables which occur simultaneously in both A and B.

By a pseudo-Boolean algebra (PBA), or a Heyting algebra, we mean [6] a system $\mathfrak{A} = \langle A, \&, \vee, \supset, \neg, 1 \rangle$, satisfying the following conditions:

1) $\langle A, \&, \vee, 1 \rangle$ is a lattice with largest element 1 and smallest element 0, and $\&$ (respectively, $\vee$) denotes the greatest lower (least upper) bound (as usual, we write $x \leq y$ instead of $x \& y = x$);

2) $z \leq x \supset y \iff z \& x \leq y$;

3) $z \leq \neg x \iff z \& x = 0$.

It follows directly from the definition that in any PBA:

4) $x \leqslant y \Longleftrightarrow x \supset y = 1,$

5) $x = y \Longleftrightarrow x \equiv y = 1,$

6) $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \Longleftrightarrow (x_1 \equiv y_1) \& \ldots \& (x_n \equiv y_n) = 1.$

In the sequel we will denote the carrier $\mathcal{A}$ of the algebra $\mathfrak{A}$ by the same letter $\mathfrak{A}$.

A PBA $\mathfrak{A}$ is called nondegenerate if it contains at least two elements. A PBA $\mathfrak{A}$ is called completely connected if for all $x, y \in \mathfrak{A}$

$$x \vee y = 1 \Longrightarrow \quad x = 1 \text{ or } y = 1.$$

The properties of a PBA used in this paper can be found in [6].

The Class H_1 of all pseudo-Boolean algebras forms a variety. To each superintuitionistic logic L there corresponds in a one-to-one fashion [1] the variety $\mathfrak{M}_L$ of pseudo-Boolean algebras defined within H_1 by the set of identities $\{A = 1 \mid A \in L\}$. If $\mathfrak{M}$ is a variety of PBA, then the logic corresponding to it is

$$L(\mathfrak{M}) = \{A \mid \models_{\mathfrak{M}} A = 1\}.$$

Suppose P is the equality of terms $u = v$. We denote by $\bar{P}$ the term $((u \supset v) \& (v \supset u))$.

LEMMA 1. Suppose $\mathfrak{M}$ is any variety of pseudo-Boolean algebras. Then the quasiidentity $(p_1 \wedge \ldots \wedge p_n) \Rightarrow q$ is true on $\mathfrak{M}$ if and only if the identity $(\bar{p}_1 \& \ldots \& \bar{p}_n) \supset \bar{q} = 1$ is true on $\mathfrak{M}$.

Proof. Suppose the identity $(\bar{p}_1 \& \ldots \& \bar{p}_n) \supset \bar{q} = 1$; does not hold on $\mathfrak{M}$; let $x_1, \ldots, x_m$ be all of the variables occurring in this identity. Take an algebra $\mathfrak{A}$ in $\mathfrak{M}$ with generators $a_1, \ldots, a_m$ such that $((\bar{p}_1 \& \ldots \& \bar{p}_n) \supset \bar{q})(a_1, \ldots, a_m) \neq 1$. Let Φ be the filter in $\mathfrak{A}$ generated by the element $(\bar{p}_1 \& \ldots \& \bar{p}_n)(a_1, \ldots, a_m)$. Then (see [6]) in the algebra $\mathfrak{A}_1 = \mathfrak{A}/\Phi$ we have

$$\bar{p}_i(a_1/\Phi, \ldots, a_m/\Phi) = 1 \text{ for } i = 1, \ldots, n,$$
$$\bar{q}(a_1/\Phi, \ldots, a_m/\Phi) \neq 1.$$

Since $\mathfrak{A}_1 \in \mathfrak{M}$, we obtain, using properties 4-6 of pseudo-Boolean algebras, $\not\models_{\mathfrak{M}} (p_1 \wedge \ldots \wedge p_n) \Rightarrow q$. The converse is obvious, since $\models_{\mathfrak{M}} (x = 1 \wedge x \supset y = 1) \Longrightarrow y = 1$.

Suppose K is an arbitrary class of algebras. By the interpolation principle for equalities (IPE) in the class K we mean the following proposition:

For any pairwise disjoint sets of variables x, y, z and equalities $p_1(x, y), \ldots, p_n(x, y)$, $q(x, z)$, if

$$\models_K \bigwedge_{i=1}^{n} p_i(x, y) \Longrightarrow q(x, z),$$

then there exist m and equalities $r_1(x), \ldots, r_m(x)$ such that

$$\models_K \bigwedge_{i=1}^{n} p_i(x, y) \Rightarrow \bigwedge_{j=1}^{m} r_j(x) \quad \text{and} \quad \models_K \bigwedge_{j=1}^{m} r_j(x) \Longrightarrow q(x, z).$$

If all algebras in K are partially ordered, then we also define the interpolation principle for inequalities (IPI):

For any terms $t(x,y)$ and $u(x,z)$, if $\vDash_K t(x,y) \leq u(x,z)$, then there exists a term $v(x)$, such that $\vDash_K t(x,y) \leq v(x) \leq u(x,z)$.

We say that a class of algebras K is amalgamable [17] if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in K$ the following condition is satisfied:

(A) For any monomorphisms $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$, $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist an algebra $\mathfrak{A} \in K$ and monomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{A}$, such that $\varepsilon_1 i_1 = \varepsilon_2 i_2$.

The triple $(\mathfrak{A}, \varepsilon_1, \varepsilon_2)$ will be called a common extension of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$. A class K is called strongly amalgamable [14] if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in K$ condition (A) is satisfied and $\varepsilon_1(\mathfrak{A}_1) \cap \varepsilon_2(\mathfrak{A}_2) = \varepsilon_1 i_1(\mathfrak{A}_0)$. A class K of partially ordered algebras is called superamalgamable if condition (A) is satisfied for $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in K$, and if $\varepsilon_j(x) \leq \varepsilon_k(y) \Rightarrow (\exists z \in \mathfrak{A}_0)(x \leq_j i_j(z) \wedge i_k(z) \leq_k y)$, where $\{j,k\} = \{1,2\}$, $\leq_j$ is the order in $\mathfrak{A}_j$, $x \in \mathfrak{A}_j$, $y \in \mathfrak{A}_k$. A class K is called weakly amalgamable if condition (A) is satisfied for finite, subdirectly irreducible $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in K$.

Condition (A) is obviously equivalent to the following:

(A') If $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then there exist $\mathfrak{A} \in K$ and monomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that $\varepsilon_1 \restriction \mathfrak{A}_0 = \varepsilon_2 \restriction \mathfrak{A}_0$.

THEOREM 1. For any superintuitionistic logic L the following conditions are equivalent:

1) Craig's theorem is true in L;

2) the variety $\mathfrak{M}_L$ satisfies the interpolation principle for inequalities;

3) $\mathfrak{M}_L$ satisfies the interpolation principle for equalities;

4) $\mathfrak{M}_L$ is superamalgamable;

5) $\mathfrak{M}_L$ is strongly amalgamable;

6) $\mathfrak{M}_L$ is amalgamable;

7) condition (A) is satisfied for any completely connected $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}_L$.

Proof. The equivalence $1 \Leftrightarrow 2$ is obvious, since $(A \supset B) \in L$ is equivalent to $\vDash_{\mathfrak{M}_L} A \leq B$. From Lemma 1 it is easy to obtain $2 \Leftrightarrow 3$. Obviously, $5 \Rightarrow 6 \Rightarrow 7$.

It is easy to see that $4 \Rightarrow 5$ for any class K. Indeed, if $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in K$, $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and $\varepsilon_1(x) = \varepsilon_2(y)$ in the common extension $(\mathfrak{A}, \varepsilon_1, \varepsilon_2)$, then $x \leq z_1$, $z_1 \leq y$, $y \leq z_2$, $z_2 \leq x$ for certain $z_1, z_2 \in \mathfrak{A}_0$. Therefore, in $\mathfrak{A}$ we have $x \leq z_1 \leq y \leq z_2 \leq x$, i.e., $x = y = z_1 \in \mathfrak{A}_0$. Proof that $1 \Rightarrow 4$ and $7 \Rightarrow 2$ is contained in Lemmas 2 and 4 below.

LEMMA 2. Suppose Craig's theorem is true in L. Then $\mathfrak{M}_L$ is superamalgamable.

Proof. Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}_L$, $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ are monomorphisms. We may assume that $\mathfrak{A}_0 \subset \mathfrak{A}_1 \cap \mathfrak{A}_2$. We associate to any element $a \in \mathfrak{A}_i$ $(i = 0,1,2)$ a variable x_a^i, where $x_a^0 = x_a^1 = x_a^2$ for $a \in \mathfrak{A}_0$ and the remaining variables are distinct. Let $\mathcal{F}_i$ denote the set of propositional formulas in the variables x_a^i, and $\mathcal{F}$ the set of formulas in the consolidated set of variables. We fix an interpretation of the variables, assigning to x_a^i the value a, and we write $\mathfrak{A}_i \models A = B$, if the equality $A=B$ is true in $\mathfrak{A}_i$ under the given interpretation.

For $i=1,2$ let

$$T_i = \{A \mid A \in \mathcal{F}_i \wedge \mathfrak{A}_i \models A = 1\}.$$

Note that the T_i are closed under modus ponens and $T_i \supseteq \mathcal{F}_i \cap L$. Let

$$T = \{A \mid A \in \mathcal{F} \wedge T_1 \cup T_2 \vdash_L A\},$$

where $\Gamma \vdash_L A$ denotes deducibility of A from $\Gamma \cup L$ by means of modus ponens. We will show that if $\{i,j\} = \{1,2\}$, $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$, then

$$T \vdash_L (A \supset B) \Leftrightarrow (\exists C \in \mathcal{F}_0)(\mathfrak{A}_i \models A \leqslant C \wedge \mathfrak{A}_j \models C \leqslant B). \tag{1}$$

Indeed, if $\mathfrak{A}_i \models A \leqslant C$, $\mathfrak{A}_j \models C \leqslant B$ for $C \in \mathcal{F}_0$, then $\mathfrak{A}_i \models (A \supset C) = 1$, $\mathfrak{A}_j \models (C \supset B) = 1$, hence $(A \supset C)$, $(C \supset B) \in T$, and therefore $T \vdash_L (A \supset B)$. On the other hand, suppose $T \vdash_L (A \supset B)$. Then there exist finite subsets $\Gamma_i \subset T_i$, $\Gamma_j \subset T_j$ such that $\Gamma_i, \Gamma_j \vdash_L A \supset B$. If we denote by Γ_k' the conjunction of all formulas of Γ_k and apply the deduction theorem, we obtain

$$\vdash_L \Gamma_i' \supset (\Gamma_j' \supset (A \supset B)),$$

which is equivalent to

$$\vdash_L \Gamma_i' \,\&\, A \supset (\Gamma_j' \supset B).$$

By Craig's theorem, L contains a formula $C \in \mathcal{F}_0$ such that $\vdash_L \Gamma_i' \,\&\, A \supset C$ and $\vdash_L C \supset (\Gamma_j' \supset B)$. Therefore, $\Gamma_i' \vdash_L A \supset C$ and $\Gamma_j' \vdash_L C \supset B$. Since $\Gamma_i' \in T_i$, we have $(A \supset C) \in T_i$ and $\mathfrak{A}_i \models (A \supset C) = 1$, i.e., $\mathfrak{A}_i \models A \leqslant C$. Similarly, $\mathfrak{A}_j \models C \leqslant B$.

Putting $A = 1$, in (1), we obtain for $j=1,2$ and $B \in \mathcal{F}_j$

$$T \vdash_L B \Leftrightarrow (\exists C \in \mathcal{F}_0)(\mathfrak{A}_i \models 1 \leqslant C \wedge \mathfrak{A}_j \models C \leqslant B) \Leftrightarrow \mathfrak{A}_j \models B = 1. \tag{2}$$

We define on the whole set $\mathcal{F}$ the relation

$$A \approx B \Longleftrightarrow T \vdash_L (A \equiv B).$$

In view of the replacement theorem in intuitionistic logic, the relation $\approx$ is a congruence on $\mathcal{F}$.

Suppose $A, B \in \mathcal{F}_i$ $(i=1,2)$. Then, in view of (2),

$$T \vdash_L (A \equiv B) \Leftrightarrow \mathfrak{A}_i \models (A \equiv B) = 1 \Leftrightarrow \mathfrak{A}_i \models A = B,$$

hence

$$A \approx B \Longleftrightarrow \mathfrak{A}_i \models A = B. \tag{3}$$

Let

$$\mathfrak{A} = \mathcal{F}/\approx .$$

We define mappings $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ $(i=1,2)$, by putting for $a \in \mathfrak{A}_i$:

$$\varepsilon_i(a) = x_a^i/\approx.$$

By virtue of (3), ε_i is one-to-one and a homomorphism. If $a \in \mathfrak{A}_0$, then $x_a^1 = x_a^2 = x_a^0$, hence $\varepsilon_1(a) = \varepsilon_2(a)$. Thus, $(\mathfrak{A}, \varepsilon_1, \varepsilon_2)$ is a common extension of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$.

We will show that the superamalgamation property holds. Suppose $\{j,k\} = \{1,2\}$, $a \in \mathfrak{A}_j$, $b \in \mathfrak{A}_k$, and $\varepsilon_j(a) \leq \varepsilon_k(b)$. Then $\varepsilon_j(a) \supset \varepsilon_k(b) = 1$, hence $(x_a^j \supset x_b^k) \approx 1$, i.e., $T \vdash_L ((x_a^j \supset x_b^k) \equiv 1)$ and $T \vdash_L (x_a^j \supset x_b^k)$. In view of (1), $(\exists C \in \mathcal{F}_0)(\mathfrak{A}_j \models x_a \leq C \wedge \mathfrak{A}_k \models C \leq x_b^k)$. Then the value c of formula C under the given interpretation occurs in $\mathfrak{A}_0$ and $a \leq_j c$, $c \leq_k b$.

The lemma is proved.

<u>LEMMA 3.</u> Suppose a PBA $\mathfrak{A}_0$ is a subalgebra of PBA $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Suppose also that $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2$, and there exists no $c \in \mathfrak{A}_0$ such that $a \leq_1 c$ and $c \leq_2 b$. Then there exist prime filters Φ_1 on $\mathfrak{A}_1$ and Φ_2 on $\mathfrak{A}_2$ such that $a \in \Phi_1$, $b \notin \Phi_2$ and $\Phi_1 \cap \mathfrak{A}_0 = \Phi_2 \cap \mathfrak{A}_0$.

<u>Proof.</u> Consider the following two sets:

$$\nabla = \{z \in \mathfrak{A}_0 \mid a \leq_1 z\},$$
$$\Delta = \{z \in \mathfrak{A}_0 \mid z \leq_2 b\}.$$

By hypothesis, $\nabla \cap \Delta = \varnothing$. Now consider

$$\Sigma_1 = \{ J \mid J \text{ is an ideal of } \mathfrak{A}_2,\ \{b\} \cup \Delta \subseteq J,\ J \cap \nabla = \varnothing \}.$$

The family Σ_1 is nonempty, since it contains $J = \{x \in \mathfrak{A}_2 \mid x \leq_2 b\}$. Obviously, Σ_1 is inductive, i.e., the union of any chain of ideals in Σ_1 again belongs to Σ_1. By Zorn's lemma, Σ_1 contains a maximal element J_2. A standard argument shows that J_2 is a prime ideal, i.e., satisfies the condition

$$(x \,\&\, y) \in J_2 \implies [x \in J_2 \text{ or } y \in J_2].$$

Put $\Phi_2 = \mathfrak{A}_2 \setminus J_2$, $\Phi_0 = \Phi_2 \cap \mathfrak{A}_0$, $J_0 = J_2 \cap \mathfrak{A}_0$. We have $\nabla \subseteq \Phi_0$, $\Delta \subseteq J_0$.

Now consider

$$\Sigma_2 = \{\Phi \mid \Phi \text{ is a filter on } \mathfrak{A}_1, \{a\} \cup \Phi_0 \subseteq \Phi, \Phi \cap J_0 = \varnothing\}.$$

We will show that Σ_2 contains

$$\Phi = \{x \in \mathfrak{A}_1 \mid (\exists z \in \Phi_0)(a \,\&\, z \leq_1 x)\}.$$

We will first show that $\Phi \cap J_0 = \varnothing$. Assume that $x \in \Phi \cap J_0$. Then $a \& z \leq_1 x$ for some $z \in \Phi_0$, $a \leq_1 z \supset x$ and $x \in \mathfrak{A}_0$. Therefore, $(z \supset x) \in \nabla \subseteq \Phi_0$, hence $x \in \Phi_0$, since $z \in \Phi_0$. We have obtained $x \in \Phi_0 \cap J_0$, which contradicts $\Phi_0 \cap J_0 = \varnothing$. Obviously, Φ is a filter on $\mathfrak{A}_1$ and $\{a\} \cup \Phi_0 \subseteq \Phi$. Thus, $\Phi \in \Sigma_2$.

Obviously, Σ_2 is inductive. By Zorn's lemma, Σ_2 contains a maximal element Φ_1. A standard argument shows that Φ_1 is a prime filter on $\mathfrak{A}_1$, i.e., $x \vee y \in \Phi_1 \Rightarrow (x \in \Phi_1 \text{ or } y \in \Phi_1)$.

Finally, $\Phi_1 \cap \mathfrak{A}_0 = \Phi_2 \cap \mathfrak{A}_0$. Indeed,

$$x \in \Phi_1 \cap \mathfrak{A}_0 \Rightarrow x \in \mathfrak{A}_0 \wedge x \notin \mathcal{J}_0 \Longrightarrow x \in \Phi_0 = \Phi_2 \cap \mathfrak{A}_0,$$
$$x \in \Phi_2 \cap \mathfrak{A}_0 \Longrightarrow x \in \Phi_0 \subseteq \Phi_1 \cap \mathfrak{A}_0.$$

LEMMA 4. Suppose $\mathfrak{M}$ is an arbitrary variety of PBA and condition (A) is satisfied for any completely connected $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}$. Then the interpolation principle for inequalities holds in $\mathfrak{M}$.

Proof. Suppose x, y, z are pairwise disjoint sets of variables, $t(x,y)$ and $u(x,z)$ are terms, and there exists no term $v(x)$ such that $\models_{\mathfrak{M}} t(x,y) \leqslant v(x) \leqslant u(x,z)$. Let $\mathcal{F}_0$ be a free algebra in $\mathfrak{M}$ with free generators x, and $\mathcal{F}$ a free algebra with free generators x, y, z. Then, by Lemma 3, there exist prime filters Φ_1 and Φ_2 on $\mathcal{F}$ such that $t(x,y) \in \Phi_1$, $u(x,z) \notin \Phi_2$ and $\Phi_1 \cap \mathcal{F}_0 = \Phi_2 \cap \mathcal{F}_0$. Put (see [6]) $\mathfrak{A}_1 = \mathcal{F}/\Phi_1$, $\mathfrak{A}_2 = \mathcal{F}/\Phi_2$. The algebras $\mathfrak{A}_1, \mathfrak{A}_2$ are completely connected, since Φ_1, Φ_2 are prime filters. Moreover, $t(x,y)/\Phi_1 = 1$, $u(x,z)/\Phi_2 \neq 1$. Note that if $v, w \in \mathcal{F}_0$, then

$$v/\Phi_1 = w/\Phi_1 \Longleftrightarrow (v \equiv w) \in \Phi_1 \Longleftrightarrow$$
$$\Longleftrightarrow (v \equiv w) \in \Phi_2 \Longleftrightarrow v/\Phi_2 = w/\Phi_2 .$$

Therefore, there exists a natural isomorphism $i_2(v/\Phi_1) = v/\Phi_2$ of the algebra $\mathfrak{A}_0 = \{v/\Phi_1 \mid v \in \mathcal{F}_0\} \subseteq \mathfrak{A}_1$ into the algebra $\mathfrak{A}_2$. It follows from the hypothesis of the lemma that there exist $\mathfrak{A} \in \mathfrak{M}$ and monomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that $\varepsilon_1 \restriction \mathfrak{A}_0 = \varepsilon_2 i_2$.

Now put

$$\bar{h}(x) \leftrightharpoons \varepsilon_1(x/\Phi_1), \bar{h}(y) \leftrightharpoons \varepsilon_1(y/\Phi_1)$$
$$\bar{h}(z) \leftrightharpoons \varepsilon_2(z/\Phi_2).$$

This mapping can be uniquely extended to a homomorphism $h : \mathcal{F} \to \mathfrak{A}$. We have in $\mathfrak{A}$: $h(t(x,y)) = \varepsilon_1(t(x,y)/\Phi_1) = 1$, $h(u(x,z)) = \varepsilon_2(u(x,z)/\Phi_2) \neq 1$. Therefore, $\not\models_{\mathfrak{M}} t(x,y) \leqslant u(x,z)$.

The lemma is proved.

II. Amalgamable Varieties of PBA

In this section we will establish the amalgamation property for the following eight varieties of pseudo-Boolean algebras. We denote by H_1 the variety of all PBA and define $H_2 - H_8$ within H_1 by the following identities:

$$H_2 : \neg x \vee \neg\neg x = 1;$$
$$H_3 : x \vee (x \supset (y \vee \neg y)) = 1;$$
$$H_4 : x \vee (x \supset (y \vee \neg y)) = 1, \ (x \supset y) \vee (y \supset x) \vee (x \equiv \neg y) = 1;$$
$$H_5 : x \vee (x \supset (y \vee \neg y)) = 1, \ \neg x \vee \neg\neg x = 1;$$
$$H_6 : (x \supset y) \vee (y \supset x) = 1;$$
$$H_7 : x \vee \neg x = 1;$$
$$H_8 : x = 1.$$

Obviously, the variety H_8 containing only the trivial PBA is amalgamable. The amalgamation property for H_7, which is precisely the variety of all Boolean algebras, was proved earlier (see [14]). This also follows immediately from Theorem 1, since H_7 contains only two completely connected PBA: the 2-element Boolean algebra and the trivial PBA. The following method of proof of the amalgamation property of varieties of PBA can be used to obtain an explicit construction of the free product over an amalgamated subalgebra [5] in varieties of Boolean algebras and distributive lattices (see the remark after Proposition 1).

To prove the amalgamation property for $H_1 - H_5$ we use the representation of a pseudo-Boolean algebra as an algebra of subsets of a partially ordered (p.o.) set [3].

Suppose S is an arbitrary p.o. set. A subset $X \subseteq S$ is called an (upper) cone if the following condition is satisfied:

$$x \in X \wedge x \leqslant y \Longrightarrow y \in X.$$

Let $B(S)$ be the family of all cones of S. It is known that the following is a PBA:

$$\mathcal{B}(S) = \langle B(S); \&, \vee, \supset, \neg, 1 \rangle,$$

where $\&$ and $\vee$ are the set-theoretic intersection and union,

$$X \supset Y = \{x \mid (\forall y \geqslant x)(y \in X \Longrightarrow y \in Y)\},$$
$$\neg X = X \supset \varnothing, \quad 1 = S.$$

By the representing set of a PBA $\mathfrak{A}$ we mean the set $S_{\mathfrak{A}}$ of all prime filters on $\mathfrak{A}$, ordered by inclusion. The following theorem is well known.

<u>Representation Theorem for PBA.</u> The mapping $\varphi_{\mathfrak{A}} : \mathfrak{A} \to \mathcal{B}(S_{\mathfrak{A}})$ defined by

$$\varphi_{\mathfrak{A}}(a) = \{\Phi \mid \Phi \in S_{\mathfrak{A}},\ a \in \Phi\}$$

is an isomorphism of the PBA $\mathfrak{A}$ into $\mathcal{B}(S_{\mathfrak{A}})$. If $\mathfrak{A}$ is finite, then $\varphi_{\mathfrak{A}}$ is a monomorphism of $\mathfrak{A}$ onto $\mathcal{B}(S_{\mathfrak{A}})$.

Now suppose we are given PBA $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ such that $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Let

$$S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2) = \{\langle \Phi_1, \Phi_2 \rangle \mid \Phi_1 \in S_{\mathfrak{A}_1} \wedge \Phi_2 \in S_{\mathfrak{A}_2} \wedge \Phi_1 \cap \mathfrak{A}_0 = \Phi_2 \cap \mathfrak{A}_0\}.$$

Also, put $\langle \Phi_1, \Phi_2 \rangle \leqslant \langle \Phi_1', \Phi_2' \rangle \Leftrightarrow \Phi_1 \subseteq \Phi_1' \wedge \Phi_2 \subseteq \Phi_2'$.

<u>LEMMA 6.</u> Suppose $\tilde{S} \subseteq S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ possesses the following properties:

1a) $(\forall \Phi_1 \in S_{\mathfrak{A}_1})(\exists \Phi_2 \in S_{\mathfrak{A}_2})(\langle \Phi_1, \Phi_2 \rangle \in \tilde{S})$,

1b) $(\forall \Phi_2 \in S_{\mathfrak{A}_2})(\exists \Phi_1 \in S_{\mathfrak{A}_1})(\langle \Phi_1, \Phi_2 \rangle \in \tilde{S})$,

2a) $\langle \Phi_1, \Phi_2 \rangle \in \tilde{S} \wedge \Phi_1 \subseteq \Phi_1' \in S_{\mathfrak{A}_1} \Rightarrow \exists \Phi_2' [\langle \Phi_1', \Phi_2' \rangle \in \tilde{S} \wedge \Phi_2 \subseteq \Phi_2']$,

2b) $\langle \Phi_1, \Phi_2 \rangle \in \tilde{S} \wedge \Phi_2 \subseteq \Phi_2' \in S_{\mathfrak{A}_2} \Rightarrow \exists \Phi_1' [\langle \Phi_1', \Phi_2' \rangle \in \tilde{S} \wedge \Phi_1 \subseteq \Phi_1']$.

Then the following mappings ψ_κ $(\kappa=1,2)$ are isomorphisms of $\mathfrak{A}_\kappa$ into $\mathcal{B}(\tilde{S})$:

$$\psi_\kappa(x) = \{<\Phi_1, \Phi_2> \mid <\Phi_1, \Phi_2> \in \tilde{S} \wedge x \in \Phi_\kappa\},$$

where $\psi_1(x) = \psi_2(x)$ for $x \in \mathfrak{A}_0$.

The lemma follows from the representation theorem, since for $<\Phi_1,\Phi_2> \in \tilde{S}$, $\kappa = 1,2$, and $x \in \mathfrak{A}_\kappa$

$$<\Phi_1, \Phi_2> \in \psi_\kappa(x) \Longleftrightarrow \Phi_\kappa \in \varphi_{\mathfrak{A}_\kappa}(x).$$

We will show, for example, that $\psi_1(x \supset y) = \psi_1(x) \supset \psi_1(y)$. Suppose $<\Phi_1,\Phi_2> \in \tilde{S}$.

$$\begin{aligned}
&<\Phi_1,\Phi_2> \in \psi_1(x \supset y) \Longrightarrow \Phi_1 \in \varphi_{\mathfrak{A}_1}(x \supset y) = \varphi_{\mathfrak{A}_1}(x) \supset \varphi_{\mathfrak{A}_1}(y) \Longrightarrow \\
&\Rightarrow (\forall \Phi_1' \supseteq \Phi_1)(\Phi_1' \in \varphi_{\mathfrak{A}_1}(x) \Longrightarrow \Phi_1' \in \varphi_{\mathfrak{A}_1}(y)) \Longrightarrow \\
&\Rightarrow (\forall \Phi_1' \supseteq \Phi_1)(\forall \Phi_2' \supseteq \Phi_2)\big[<\Phi_1',\Phi_2'> \in \tilde{S} \wedge <\Phi_1',\Phi_2'> \in \psi_1(x) \Longrightarrow \\
&\Longrightarrow <\Phi_1',\Phi_2'> \in \psi_1(y)\big] \Longleftrightarrow <\Phi_1',\Phi_2'> \in \psi_1(x) \supset \psi_1(y).
\end{aligned}$$

$$\begin{aligned}
&<\Phi_1,\Phi_2> \notin \psi_1(x \supset y) \Rightarrow \Phi_1 \notin \varphi_{\mathfrak{A}_1}(x \supset y) = \varphi_{\mathfrak{A}_1}(x) \supset \varphi_{\mathfrak{A}_1}(y) \Longrightarrow \\
&\Rightarrow (\exists \Phi_1' \supseteq \Phi_1)(\Phi_1' \in \varphi_{\mathfrak{A}_1}(x) \smallsetminus \varphi_{\mathfrak{A}_1}(y)) \underset{(\text{see } 2a)}{\Longrightarrow} \\
&\Rightarrow (\exists \Phi_1' \supseteq \Phi_1)(\exists \Phi_2' \supseteq \Phi_2)\big[<\Phi_1',\Phi_2'> \in \tilde{S} \wedge <\Phi_1',\Phi_2'> \in \psi_1(x) \smallsetminus \psi_1(y)\big] \\
&\Longleftrightarrow <\Phi_1,\Phi_2> \notin \psi_1(x) \supset \psi_1(y).
\end{aligned}$$

Permutability of ψ_κ with the remaining operations is easily proved. Uniqueness of ψ_κ follows from conditions 1a and 1b; ψ_1 and ψ_2 agree on $\mathfrak{A}_0$ since $\tilde{S} \subseteq S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

In view of Lemma 6, we can prove the amalgamation property or, equivalently, property (A') for a given variety $\mathfrak{M}$ as follows: for any PBA $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}$ with common subalgebra $\mathfrak{A}_0$ it suffices to choose $\tilde{S} \subseteq S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ so that the conditions of Lemma 6 are satisfied and $\mathcal{B}(\tilde{S}) \in \mathfrak{M}$. If one of the algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ is degenerate, then so are the others. Therefore, we need only consider nondegenerate $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$. By Theorem 1, we need only consider completely connected $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$.

LEMMA 7. For any nondegenerate PBA $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with common subalgebra $\mathfrak{A}_0$, the set $\tilde{S} = S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ satisfies the conditions of Lemma 6.

Proof. Properties 1a and 1b follow from the fact that $\Phi_1 \cap \mathfrak{A}_0$ and $\Phi_2 \cap \mathfrak{A}_0$ are prime filters on $\mathfrak{A}_0$ and any prime filter Φ_0 on a sublattice L_0 of a distributive lattice L can be extended to a prime filter Φ on L such that $\Phi \cap L_0 = \Phi_0$ [4, Lemma 5].

Let us prove property 2a. Suppose $<\Phi_1,\Phi_2> \in S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ and $\Phi_1 \subseteq \Phi_1' \in S_{\mathfrak{A}_1}$. Consider the family

$$\Sigma = \{\Phi_2' \mid \Phi_2' \text{ is a filter on } \mathfrak{A}_2,\ \Phi_2 \subseteq \Phi_2',\ \Phi_2' \cap \mathfrak{A}_0 = \Phi_1' \cap \mathfrak{A}_0\}.$$

Note that Σ is nonempty, i.e., it contains the filter Φ on $\mathfrak{A}_2$ generated by the set

$\Phi_2 \cup (\Phi_1' \cap \mathfrak{A}_0)$. Indeed, suppose $x \in \Phi \cap \mathfrak{A}_0$. Then $y \& z \leq x$ for certain $y \in \Phi_2$, $z \in \Phi_1' \cap \mathfrak{A}_0$. Therefore, $y \leq z \supset x$ and $(z \supset x) \in \Phi_2 \cap \mathfrak{A}_0 = \Phi_1 \cap \mathfrak{A}_0 \in \Phi_1'$. It follows that $x \in \Phi_1' \cap \mathfrak{A}_0$. On the other hand, by construction, $\Phi_1' \cap \mathfrak{A}_0 \subseteq \Phi \cap \mathfrak{A}_0$.

Obviously, the family Σ is inductive and therefore, by Zorn's lemma, contains a maximal element Φ_2' . A standard argument shows that Φ_2' is a prime filter on $\mathfrak{A}_2$.

Property 2b is proved analogously.

From Lemma 7 we immediately obtain

Proposition 1. The variety H_1 of all PBA is amalgamable.

Remark. Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are Boolean algebras (distributive lattices with 0 and 1) and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then the subalgebra $\mathfrak{A}$ of $\mathcal{B}(S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2))$ generated by the set $\psi_1(\mathfrak{A}_1) \cup \psi_2(\mathfrak{A}_2)$ is the free product of its subalgebras $\psi_1(\mathfrak{A}_1)$ and $\psi_2(\mathfrak{A}_2)$ over the amalgamated subalgebra $\psi_1(\mathfrak{A}_0)$.

To prove the amalgamation property of the remaining varieties $H_2 - H_6$ it suffices, in view of Theorem 1, to show that condition (A) is satisfied for nondegenerate, completely connected PBA.

We obviously have

LEMMA 8. A nondegenerate PBA $\mathfrak{A}$ is completely connected if and only if $S_{\mathfrak{A}}$ has a smallest element, namely $\Phi = \{1\}$.

Well known (see [6]) is

LEMMA 9. If $\mathfrak{A}$ is a PBA, then $S_{\mathfrak{A}}$ satisfies the following condition: for any $\Phi \in S_{\mathfrak{A}}$ there exist a maximal (with respect to inclusion) filter $\Phi' \in S_{\mathfrak{A}}$ such that $\Phi \subseteq \Phi'$.

LEMMA 10. Suppose $\mathfrak{A}_1, \mathfrak{A}_2$ are PBA with common subalgebra $\mathfrak{A}_0$.

a) If Φ_{10} and Φ_{20} are the smallest elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$, respectively, then $<\Phi_{10}, \Phi_{20}> \in S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

b) For any maximal $\Phi_1 \in S_{\mathfrak{A}_1}$ there exists a maximal $\Phi_2 \in S_{\mathfrak{A}_2}$ such that $<\Phi_1, \Phi_2> \in S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

c) For any maximal $\Phi_2 \in S_{\mathfrak{A}_2}$ there exists a maximal $\Phi_1 \in S_{\mathfrak{A}_1}$ such that $<\Phi_1, \Phi_2> \in S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

Proof. a) By Lemma 7, $<\Phi_{10}, \Phi_2> \in \tilde{S} = S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ and $<\Phi_1, \Phi_{20}> \in \tilde{S}$ for certain $\Phi_1 \in S_{\mathfrak{A}_1}$, $\Phi_2 \in S_{\mathfrak{A}_2}$. Then

$$\Phi_{10} \cap \mathfrak{A}_0 \subseteq \Phi_1 \cap \mathfrak{A}_0 = \Phi_{20} \cap \mathfrak{A}_0 \subseteq \Phi_2 \cap \mathfrak{A}_0 = \Phi_{10} \cap \mathfrak{A}_0,$$

hence $<\Phi_{10}, \Phi_{20}> \in \tilde{S}$.

b) Suppose Φ_1 is maximal in $S_{\mathfrak{A}_1}$. By Lemma 7, $<\Phi_1, \Phi_2'> \in \tilde{S}$ for some $\Phi_2' \in S_{\mathfrak{A}_2}$. By Lemma 9, there exists a maximal $\Phi_2 \in S_{\mathfrak{A}_2}$ such that $\Phi_2' \subseteq \Phi_2$. It follows from property 2b that

$\langle \Phi_1', \Phi_2 \rangle \in \tilde{S}$ for some $\Phi_1' \in S_{\mathfrak{A}_1}$, $\Phi_1' \supseteq \Phi_1$. Since Φ_1 is maximal, we obtain $\Phi_1' = \Phi_1$, i.e., $\langle \Phi_1, \Phi_2 \rangle \in \tilde{S}$. Assertion c) is proved analogously.

We will now show that H_2 has the amalgamation property.

LEMMA 11. a) Suppose a nondegenerate PBA $\mathfrak{A}$ is completely connected and satisfies the identity $\neg x \vee \neg\neg x = 1$. Then $S_{\mathfrak{A}}$ has a largest element.

b) Suppose S is a p.o. set having a largest element. Then $\mathcal{B}(S) \in H_2$.

Proof. a) We will show that $\Phi = \{x \mid x \neq 0\} \in S_{\mathfrak{A}}$.

We need only show that Φ is closed under $\&$. Suppose $x \neq 0$, $y \neq 0$. Then $\neg x \neq 1$, $\neg y \neq 1$ hence, by the hypothesis of the lemma, $\neg\neg x = 1$ and $\neg\neg y = 1$. Therefore $\neg(x \& y) = \neg(\neg\neg x \& \neg\neg y) = 0 \neq 1$, and $x \& y \neq 0$.

Obviously, $\Phi' \subseteq \Phi$ for any filter Φ' on $\mathfrak{A}$.

b) Suppose Φ is the largest element of S and X is any cone in S. If $\Phi \in X$, then $\neg\neg X = S$; if $\Phi \notin X$, then $X = \emptyset$ and $\neg X = S$.

The lemma is proved.

Proposition 2. The variety H_2 of pseudo-Boolean algebras satisfying the identity $\neg x \vee \neg\neg x = 1$ is amalgamable.

Proof. Suppose $\mathfrak{A}_1, \mathfrak{A}_2$ are completely connected algebras in H_2 and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then, by Lemma 11a, $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$ have largest element Φ_1 and Φ_2.

By Lemma 10b, $\langle \Phi_1, \Phi_2 \rangle \in \tilde{S} = S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ i.e., $\tilde{S}$ has a largest element. By Lemma 11a, $\mathcal{B}(\tilde{S}) \in H_3$, and it follows from Lemmas 6 and 7 that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be isomorphically embedded in $\mathcal{B}(\tilde{S})$, where the embeddings agree on $\mathfrak{A}_0$. By Theorem 1, H_2 is amalgamable.

LEMMA 12. a) Suppose a nondegenerate PBA $\mathfrak{A}$ is completely connected and satisfies the identity $x \vee (x \supset (y \vee \neg y)) = 1$. Then $S_{\mathfrak{A}}$ has a smallest element and all other elements are maximal.

b) Suppose S is a p.o. set in which all $<$-chains have length at most 2. Then $\mathcal{B}(S) \in H_3$.

The lemma follows from [3, Proposition 1].

Proposition 3. The variety H_3 of pseudo-Boolean algebras satisfying the identity $x \vee (x \supset (y \vee \neg y)) = 1$ is amalgamable.

Proof. Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are nondegenerate, completely connected PBA in H_3. Let Φ_{10} and Φ_{20} be the smallest elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$. Let $\tilde{S} = \{\langle \Phi_{10}, \Phi_{20} \rangle\} \cup \{\langle \Phi_1, \Phi_2 \rangle \in S(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2) \mid \Phi_1$ is maximal in $S_{\mathfrak{A}_1}$, Φ_2 is maximal in $S_{\mathfrak{A}_2}\}$. By Lemmas 10 and 12a, $\tilde{S}$ satisfies all conditions of Lemma 6, hence $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be embedded in $\mathcal{B}(\tilde{S})$, where the embeddings agree on $\mathfrak{A}_0$. By Lemma 12b, $\mathcal{B}(\tilde{S}) \in H_3$, and, by Theorem 1, H_3 is amalgamable.

LEMMA 13. a) Suppose a nondegenerate PBA $\mathfrak{A}$ is completely connected and belongs to H_4. Then $S_{\mathfrak{A}}$ contains at most three elements, one of which is the smallest and the rest maximal.

b) Suppose S contains at most three elements, one of which is the smallest and the rest maximal. Then $\mathcal{B}(S)\in H_4$.

Proof. a) By Lemma 12a, $S_{\mathfrak{A}}$ contains a smallest element, the rest being maximal.

Obviously, if $\mathfrak{A}$ is linearly ordered, then $S_{\mathfrak{A}}$ contains only one maximal filter. Suppose $\mathfrak{A}$ contains incomparable elements x and y. Then $(x\equiv \neg y)=1$ and $(y\equiv\neg x)=1$ i.e., $x=\neg y$ and $\neg x=y$, hence $\mathfrak{A}$ cannot contain another pair of incomparable elements. But then $\mathfrak{A}$ has only one pair of incomparable (with respect to inclusion) filters, hence $S_{\mathfrak{A}}$ contains at most two maximal elements.

b) Suppose S contains a smallest element Φ_0 and at most two maximal ones. Let X, Y be cones in S, and assume that $X\not\subseteq Y$ and $Y\not\subseteq X$. Then $\Phi_1\in X\setminus Y, \Phi_2\in Y\setminus X$ for certain $\Phi_1,\Phi_2\in S$. Obviously, $\Phi_1\neq\Phi_0$, since $\Phi_0\in X\Rightarrow X=S$; similarly $\Phi_2\neq\Phi_0$ and also $\Phi_1\neq\Phi_2$. Thus, $X=\{\Phi_1\}, Y=\{\Phi_2\}$, hence $X=\neg Y$ and $\mathcal{B}(S)\in H_4$.

Proposition 4. The variety H_4 is amalgamable.

Proof. Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are nondegenerate, completely connected PBA in H_4 and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Let Φ_{10} and Φ_{20} be the smallest elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$, and let S_1 and S_2 be the sets of maximal elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$. By Lemma 13a, $S_i\leq 2, S_{\mathfrak{A}_i}=\{\Phi_{i0}\}\cup S_i\ (i=1,2)$.

Assume that $S_1=\{\Phi_{11},\Phi_{12}\}$, where Φ_{11} and Φ_{12} are not necessarily distinct. By Lemma 10, there exist $\Phi_{21},\Phi_{22}\in S_2$ such that

$$S'=\{<\Phi_{11},\Phi_{21}>,<\Phi_{12},\Phi_{22}>\}\subseteq S(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2).$$

If $\{\Phi_{21},\Phi_{22}\}=S_2$, then we take $\tilde{S}=\{<\Phi_{10},\Phi_{20}>\}\cup S'$. If $S_2\setminus\{\Phi_{21},\Phi_{22}\}=\{\Phi_2\}$, then $<\Phi_1,\Phi_2>\in S(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2)$ for some $\Phi_1\in S_1$. Put $S''=\{<\Phi_1,\Phi_2>,<\Phi_1',\Phi_{21}>\}$, where $\{\Phi_1'\}=S_1\setminus\{\Phi_1\}$ $\tilde{S}=\{<\Phi_{10},\Phi_{20}>\}\cup S''$. Since in this case $\Phi_{21}=\Phi_{22}$, we again obtain $\tilde{S}\subseteq S(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2)$.

In either case, $\tilde{S}$ satisfies the conditions of Lemma 6, and, by Lemma 13, $\mathcal{B}(\tilde{S})\in H_4$, as required.

Proposition 5. The variety H_5 is amalgamable.

Proof. Suppose $\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2$ are nondegenerate, completely connected PBA in H_5 and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Using Lemmas 11 and 12, we obtain that $S_{\mathfrak{A}_1}=\{\Phi_{10},\Phi_{11}\}$, where $\Phi_{10}\subseteq\Phi_{11}$ and $S_{\mathfrak{A}_2}=\{\Phi_{20},\Phi_{21}\}$, where $\Phi_{20}\subseteq\Phi_{21}$. Take $\tilde{S}=\{<\Phi_{10},\Phi_{20}>,<\Phi_{11},\Phi_{21}>\}$. By Lemma 10, $\tilde{S}\subseteq S(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2)$. Obviously, $\tilde{S}$ satisfies the conditions of Lemma 6, and, by Lemmas 11 and 12, $\mathcal{B}(\tilde{S})\in H_5$, as required.

The amalgamation property can be established for the variety H_6 by the same method. However, we will establish it without using the representation theorem.

Proposition 6. The variety H_6 is amalgamable.

Proof. Note first that the completely connected PBA in H_6 are precisely the chained PBA, i.e., linearly ordered sets with smallest and largest elements. The operations $\supset$ and $\neg$ in a chained PBA are defined as follows: $x \supset y = 1$ if $x \leq y$, $x \supset y = y$ if $x > y$; $\neg x = 0$ if $x > 0$, $\neg 0 = 1$. Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are chained PBA and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. We may assume that $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \mathfrak{A}_0$ (otherwise we can replace $\mathfrak{A}_2$ by a suitable isomorphic PBA). Let $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$. For $x, y \in \mathfrak{A}$ we put

$$x \leq y \iff (x, y \in \mathfrak{A}_1 \wedge x \leq_1 y) \vee (x, y \in \mathfrak{A}_2 \wedge x \leq_2 y) \vee$$
$$\vee\ (x \in \mathfrak{A}_1 \wedge y \in \mathfrak{A}_2 \wedge (\exists z \in \mathfrak{A}_0)(x \leq_1 z \wedge z \leq_2 y)) \vee$$
$$\vee\ (x \in \mathfrak{A}_2 \wedge y \in \mathfrak{A}_1 \wedge (\exists z \in \mathfrak{A}_0)(y \geq_1 z \wedge x \leq_2 z)).$$

It is easy to see that $\leq$ is a partial order on $\mathfrak{A}$, and for $i = 1, 2$ and $x, y \in \mathfrak{A}_i$ we have $x \leq y \iff x \leq_i y$. Extending $\leq$ to a linear order on $\mathfrak{A}$, we obtain the desired PBA in H_6.

III. Characterization of the Varieties $H_1 - H_8$

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two pseudo-Boolean algebras. By the ordered sum $\mathfrak{A} + \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ we mean the algebra $\mathfrak{C} = \langle C, \&, \vee, \supset, \neg, 1 \rangle$ defined as follows: $C = \mathfrak{A} \cup \mathfrak{B}'$, where $\mathfrak{B}'$ is isomorphic to $\mathfrak{B}$ and $\mathfrak{B}' \cap \mathfrak{A} = \{1_{\mathfrak{A}}\} = \{0_{\mathfrak{B}'}\}$, and the order $\leq_{\mathfrak{C}}$ is defined by the condition:

$$x \leq_{\mathfrak{C}} y \iff (x \in \mathfrak{A} \wedge y \in \mathfrak{B}') \vee (x, y \in \mathfrak{A} \wedge x \leq_{\mathfrak{A}} y) \vee (x, y \in \mathfrak{B}' \wedge x \leq_{\mathfrak{B}'} y).$$

Consequently, $0_{\mathfrak{C}} = 0_{\mathfrak{A}}$, $1_{\mathfrak{C}} = 1_{\mathfrak{B}'}$,

$$x \supset_{\mathfrak{C}} y = \begin{cases} 1, & \text{if } x \leq_{\mathfrak{C}} y, \\ x \supset_{\mathfrak{A}} y, & \text{if } x, y \in \mathfrak{A}, x \not\leq_{\mathfrak{C}} y, \\ x \supset_{\mathfrak{B}'} y, & \text{if } x, y \in \mathfrak{B}', \\ y, & \text{if } x \in \mathfrak{B}' \setminus \{0_{\mathfrak{B}'}\}, y \in \mathfrak{A}. \end{cases}$$

$$\neg_{\mathfrak{C}} x = \begin{cases} 0, & \text{if } x \in \mathfrak{B}' \setminus \{0_{\mathfrak{B}'}\}, \\ \neg_{\mathfrak{A}} x, & \text{if } x \in \mathfrak{A} \setminus \{0_{\mathfrak{A}}\}, \\ 1, & \text{if } x = 0_{\mathfrak{A}} = 0_{\mathfrak{C}}. \end{cases}$$

The operation $+$ is obviously associative. We will denote the 2-element Boolean algebra by B_0, and will sometimes denote $\mathfrak{A} + B_0$ by $\mathfrak{A}^+$.

Recall [18] that a nondegenerate PBA $\mathfrak{A}$ is subdirectly irreducible if and only if $\mathfrak{A}$ has a penultimate element, i.e., an element $\omega \neq 1$ such that $x \leq \omega$ for any $x \in \mathfrak{A}, x \neq 1$. Thus, $\mathfrak{A}$ is subdirectly irreducible if and only if $\mathfrak{A} = \mathfrak{A}_1^+$ for some PBA $\mathfrak{A}_1$.

For a given finite PBA $\mathfrak{B}$, we define the family $K(\mathfrak{B})$ of finite, subdirectly irreducible PBA as the smallest class satisfying the following conditions:

1) $(\mathfrak{B} + B_0) \in K(\mathfrak{B})$,
2) $(\mathfrak{B} + \mathfrak{A}_1), \ldots, (\mathfrak{B} + \mathfrak{A}_n) \in K(\mathfrak{B}) \Longrightarrow (\mathfrak{B} + (\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n) + B_0) \in K(\mathfrak{B})$.

LEMMA 14. Any finite PBA of the form $\mathcal{L}+\mathfrak{A}+\mathcal{B}_0$ is isomorphic to a subalgebra of a suitable algebra in $K(\mathcal{L})$.

We will first prove the lemma in the case where $\mathcal{L}$ is the 1-element PBA E, i.e., $\mathcal{L}+\mathfrak{A}+\mathcal{B}_0=\mathfrak{A}+\mathcal{B}_0$. The proof is by induction on the number of elements in $\mathfrak{A}$. If $\mathfrak{A}=E$, then $\mathfrak{A}+\mathcal{B}_0=\mathcal{B}_0\in K(E)$ by definition. Suppose $\mathfrak{A}$ is a nondegenerate PBA. Then $\mathfrak{A}$ is a subdirect product of subdirectly irreducible PBA $\mathfrak{A}_1^+,\dots,\mathfrak{A}_n^+$. Since each of $\mathfrak{A}_1,\dots,\mathfrak{A}_n$ has cardinality less than that of $\mathfrak{A}$, it follows from the induction assumption that the algebra $\mathfrak{A}_i^+=\mathfrak{A}_i+\mathcal{B}_0$ can be isomorphically embedded in a suitable PBA $\mathfrak{A}_i'\in K(E)$. Since $\mathfrak{A}$ is isomorphic to a subalgebra of the PBA $\mathfrak{A}_1^+\times\dots\times\mathfrak{A}_n^+$, we have that $\mathfrak{A}+\mathcal{B}_0$ can be isomorphically embedded in $((\mathfrak{A}_1'\times\dots\times\mathfrak{A}_n')+\mathcal{B}_0)\in K(E)$.

In the case where $\mathcal{L}$ is a nondegenerate PBA, we use the obvious equality $K(\mathcal{L})=\{\mathcal{L}+\mathfrak{A}\mid\mathfrak{A}\in K(E)\}$. Clearly, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}'$, then $\mathcal{L}+\mathfrak{A}$ is a subalgebra of $\mathcal{L}+\mathfrak{A}'$.

The lemma is proved.

Proposition 7. 1) The variety H_1 is generated by the family $K(E)$, where E is the 1-element PBA;

2) H_2 is generated by the family $K(\mathcal{B}_0)$;

3) H_3 is generated by the family $\{C_n\mid n\geq 0\}$, where $C_n=\mathcal{B}_0^n+\mathcal{B}_0$;

4) H_4 is generated by the algebra $C_2=\mathcal{B}_0^2+\mathcal{B}_0$;

5) H_5 is generated by the algebra $C_1=\mathcal{B}_0+\mathcal{B}_0$;

6) H_6 is generated by the family $\{L_n\mid n\geq 2\}$, where L_n is a linearly ordered PBA of cardinality n;

7) H_7 is generated by the algebra $\mathcal{B}_0$;

8) H_8 contains only the 1-element PBA.

Proof. 1) It is known [6] that the variety H_1 is generated by all finite PBA. Therefore, it follows from Lemma 14 that the family $K(E)$ generates H_1.

2) It is shown in [7] that H_2 is generated by the finite, subdirectly irreducible PBA of the form $\mathcal{B}_0+\mathfrak{A}$. It remains to apply Lemma 14.

3) This follows from [3, Lemma 10].

4) This follows from Lemma 13 and the representation theorem.

5) This follows from Lemmas 11 and 12.

6) This follows from [11].

7) H_7 is the variety of Boolean algebras.

LEMMA 15. A PBA $\mathfrak{A}\notin H_2$ if and only if $C_2=\mathcal{B}_0^2+\mathcal{B}_0$ can be isomorphically embedded in $\mathfrak{A}$.

If $\mathfrak{A} \notin H_2$, then $\neg a \vee \neg\neg a < 1$ for some $a \in \mathfrak{A}$. Then $A = \{0, \neg a, \neg\neg a, \neg a \vee \neg\neg a, 1\}$ is a subalgebra of C_2 isomorphic to $\mathfrak{A}$. On the other hand, it is obvious that $C_2 \notin H_2$.

LEMMA 16. A PBA $\mathfrak{A} \notin H_3$ if and only if L_4 can be isomorphically embedded in $\mathfrak{A}$.

If $x \vee (x \supset (y \vee \neg y)) < 1$ for certain $x, y \in \mathfrak{A}$, then the set $\{0, y \vee \neg y, x \vee (x \supset (y \vee \neg y)), 1\}$ is a 4-element chained subalgebra of $\mathfrak{A}$.

LEMMA 17. A completely connected PBA $\mathfrak{A} \notin H_4$ if and only if L_4 or $C_3 = B_0^3 + B_0$ can be isomorphically embedded in $\mathfrak{A}$.

If $\mathfrak{A} \notin H_3$, then, by Lemma 16, L_4 can be embedded in $\mathfrak{A}$. If $\mathfrak{A} \in H_3 \setminus H_4$, then $\mathfrak{A}$ contains a finitely generated subalgebra $\mathfrak{A}_1 \in H_3 \setminus H_4$. By Lemmas 12a and 13b, $S_{\mathfrak{A}_1}$ has a smallest element and the others, of which there are at least three, are maximal. The algebra $\mathfrak{A}_1$ is finite, since H_3 is locally finite [2]. It follows from the proof of Lemma 8 of [3] that C_3 is a subalgebra of $\mathfrak{A}_1$ and hence of $\mathfrak{A}$.

LEMMA 18. A PBA $\mathfrak{A} \notin H_5$ if and only if L_4 or $C_2 = B_0^2 + B_0$ can be isomorphically embedded in $\mathfrak{A}$.

This follows from Lemmas 15 and 16.

LEMMA 19. A completely connected $\mathfrak{A} \notin H_6$ if and only if $C_2 = B_0^2 + B_0$ or $B_3 = B_0 + B_0^2 + B_0$ can be isomorphically embedded in $\mathfrak{A}$.

If $\mathfrak{A} \notin H_2 \supseteq H_6$, then, by Lemma 15, $\mathfrak{A}$ contains a subalgebra isomorphic to C_2. Suppose $\mathfrak{A} \in H_2 \setminus H_6$. Then $(x \supset y) \vee (y \supset x) < 1$ for certain $x, y \in \mathfrak{A}; x \nleq y$, $y \nleq x$. Therefore, $x \neq 0$, $y \neq 0$, hence $y \supset x \neq 0$ and $x \supset y \neq 0$. Since $\neg(y \supset x) \vee \neg\neg(y \supset x) = 1$, we obtain $\neg\neg(y \supset x) = 1$; similarly, $\neg\neg(x \supset y) = 1$, i.e., $\neg(y \supset x) = \neg(x \supset y) = 0$. Consequently, the set

$$A_0 = \{0, (x \supset y) \& (y \supset x), (x \supset y), (y \supset x), (x \supset y) \vee (y \supset x), 1\}$$

is a subalgebra of $\mathfrak{A}$, all elements are distinct, and $(x \supset y)$ and $(y \supset x)$ constitute the only pair of incomparable elements. Therefore, this subalgebra is isomorphic to B_3.

LEMMA 20. A PBA $\mathfrak{A} \notin H_7$ if and only if C_1 can be isomorphically embedded in $\mathfrak{A}$.

If $\mathfrak{A} \notin H_7$, then $x \vee \neg x \neq 1$ for some x, hence the set $\{0, x \vee \neg x, 1\}$ is a subalgebra isomorphic to C_1.

LEMMA 21. A PBA $\mathfrak{A} \notin H_8$ if and only if B_0 is a subalgebra of $\mathfrak{A}$.

Any nondegenerate PBA contains B_0 as a subalgebra (see [6]).

IV. Necessary Conditions for Varieties of PBA To Be Amalgamable

LEMMA 22. Suppose $\mathfrak{M}$ is (weakly) amalgamable, $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are (finite) subdirectly irreducible PBA in $\mathfrak{M}$, $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and ω is a penultimate element of all three algebras. Then there exist a subdirectly irreducible $\mathfrak{A} \in \mathfrak{M}$ with penultimate element ω and monomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{A}$, which are the identity mappings on $\mathfrak{A}_0$.

Proof. Since $\mathfrak{M}$ is (weakly) amalgamable, there exist an $\mathfrak{M}$ in $\overline{\mathfrak{A}}$ and monomorphisms $\delta_1 : \mathfrak{A}_1 \to \overline{\mathfrak{A}}$, $\delta_2 : \mathfrak{A}_2 \to \overline{\mathfrak{A}}$ such that $\delta_1 \upharpoonright \mathfrak{A}_0 = \delta_2 \upharpoonright \mathfrak{A}_0$. Let $\overline{\omega} = \delta_1(\omega)$. Consider the set

$$\nabla = \{x \vee (x \supset \bar{\omega}) \mid x \in \mathfrak{A}\}.$$

Since $(x \vee (x \supset \bar{\omega})) \supset \bar{\omega} = \bar{\omega}$ and $(y \& z) \supset \bar{\omega} = y \supset (z \supset \bar{\omega})$, we see that for all $k \geq 1$ and $y_1, \ldots, y_k \in \nabla$ we have $(y_1 \& \ldots \& y_k) \supset \bar{\omega} = \bar{\omega} < 1$, i.e., $y_1 \& \ldots \& y_k \nleq \bar{\omega}$. Therefore, there exists a prime filter $\Phi \supseteq \nabla$, $\bar{\omega} \notin \Phi$. Let

$$\tilde{\mathfrak{A}} = \bar{\mathfrak{A}}/\Phi,$$

$\tilde{\varepsilon}_i(x) = \delta_i(x)/\Phi$ for $i = 1, 2$, $x \in \mathfrak{A}_i$.

Suppose $x, y \in \mathfrak{A}_i$ $(i = 1, 2)$, $x \neq y$. We have $x \equiv y \leq \omega$ in $\mathfrak{A}_i$, hence $\delta_i(x \equiv y) \leq \bar{\omega} \notin \Phi$, hence $\tilde{\varepsilon}_i(x \equiv y) \leq \bar{\omega}/\Phi < 1$ in $\tilde{\mathfrak{A}}$, i.e., $\tilde{\varepsilon}_i(x) \neq \tilde{\varepsilon}_i(y)$. Therefore, the homomorphisms $\tilde{\varepsilon}_i$ are monomorphisms.

We have $x \in \Phi$ or $(x \supset \bar{\omega}) \in \Phi$ for any $x \in \bar{\mathfrak{A}}$. Therefore, $x/\Phi = 1$ or $x/\Phi \supset \bar{\omega}/\Phi = 1$ for any $x \in \tilde{\mathfrak{A}}$, i.e., $z \leq \bar{\omega}/\Phi$ for any $z \in \tilde{\mathfrak{A}}$, $z \neq 1$. Thus, $\tilde{\varepsilon}_1(\omega) = \bar{\omega}/\Phi$ is a penultimate element of $\tilde{\mathfrak{A}}$, and $\tilde{\mathfrak{A}}$ is subdirectly irreducible. Finally, replacing in $\tilde{\mathfrak{A}}$ the subalgebra $\tilde{\varepsilon}_1(\mathfrak{A}_0) = \tilde{\varepsilon}_2(\mathfrak{A}_0)$ by the isomorphic subalgebra $\mathfrak{A}_0$, we obtain an algebra $\mathfrak{A}$ and monomorphisms $\varepsilon_1, \varepsilon_2$ which are the identity mappings on $\mathfrak{A}_0$.

The lemma is proved.

<u>LEMMA 23.</u> Suppose $\mathfrak{M}$ is (weakly) amalgamable and $L_4 \in \mathfrak{M}$. Then for any finite) PBA $\mathfrak{B}$,

$$(\mathfrak{B} + B_0) \in \mathfrak{M} \Longrightarrow (\mathfrak{B} + B_0 + B_0) \in \mathfrak{M}.$$

<u>Proof.</u> Take $\mathfrak{A}_0 = L_3 = \{0, a, 1\}$, $\mathfrak{A}_1 = L_4 = \{0, a, b, 1\}$, where $a < b$, $\mathfrak{A}_2 = \mathfrak{B} + B_0$, and let ω be a penultimate element of $\mathfrak{A}_2$. For $x \in \mathfrak{A}_0$:

$$i_1(x) = x,$$
$$i_2(x) = x \text{ if } x \neq a, \; i_2(a) = \omega.$$

Then $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$, $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ are monomorphisms. Therefore, $\mathfrak{M}$ contains a common extension $(\mathfrak{A}, \varepsilon_1, \varepsilon_2)$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$. We have in $\mathfrak{A}$: $\varepsilon_2(\omega) = \varepsilon_1(a) < \varepsilon_1(b) < \varepsilon_1(1) = 1$. Therefore, the set $\bar{A} = \varepsilon_2(\mathfrak{A}_2) \cup \{\varepsilon_1(b)\}$ is in one-to-one correspondence with $\mathfrak{A}_2 + B_0$, where this correspondence preserves order. We will show that $\bar{A}$ is a subalgebra of $\mathfrak{A}$. Since $\varepsilon_1(b)$ is comparable with all elements of $\varepsilon_2(\mathfrak{A}_2)$ it follows that $\bar{A}$ is closed under $\&$ and $\vee$. Also, if $x \in \varepsilon_2(\mathfrak{A}_2)$ and $x \neq 1$, then $x \leq \varepsilon_2(\omega) < \varepsilon_1(b)$, $x \supset \varepsilon_1(b) = 1$; $x \leq \varepsilon_1(b) \supset x \leq \varepsilon_2(\omega) \supset x = x$, hence $\varepsilon_1(b) \supset x = x$. Thus, $\bar{A}$ is closed under all operations, hence the subalgebra of $\mathfrak{A}$ with carrier $\bar{A}$ is isomorphic to $\mathfrak{A}_2^+$. Therefore, $\mathfrak{A}_2^+ \in \mathfrak{M}$, as required.

<u>LEMMA 24.</u> Suppose $\mathfrak{M}$ is (weakly) amalgamable, $\mathfrak{B}$ is an arbitrary (finite) PBA, and $(\mathfrak{B} + B_0^n + B_0) \in \mathfrak{M}$ for some $n \geq 3$. Then $(\mathfrak{B} + B_0^m + B_0) \in \mathfrak{M}$ for all $m \geq 1$.

<u>Proof.</u> In the algebras $\mathfrak{A}_1 = \mathfrak{B} + B_0^n + B_0$ and $\mathfrak{A}_0 = \mathfrak{B} + B_0^2 + B_0$ we denote by τ the largest element of $\mathfrak{B}$, and by ω the pneumatic element of $\mathfrak{A}_1$ and $\mathfrak{A}_0$; let $a_1, \ldots, a_n$ be all atoms of B_0^n, i.e., all elements directly following τ in $\mathfrak{A}_1$; let a, b be elements directly following τ in $\mathfrak{A}_0$; $\mathfrak{A}_2 = \mathfrak{A}_1$. For $x \in \mathfrak{A}_0$ put

$$i_1(x) = \begin{cases} x, & \text{if } x \notin \{a, b\}, \\ a_1, & \text{if } x = a, \\ a_2 \vee \ldots \vee a_n, & \text{if } x = b, \end{cases}$$

$$i_2(x) = \begin{cases} x, & \text{if } x \notin \{a, b\}, \\ a_1 \vee \ldots \vee a_{n-1}, & \text{if } x = a \\ a_n, & \text{if } x = b. \end{cases}$$

It is easy to verify that $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ are monomorphisms.

Suppose $(\mathfrak{A}, \varepsilon_1, \varepsilon_2)$ is a common extension of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$ and $\mathfrak{A} \in \mathfrak{M}$. Let $\varepsilon_1(\tau) = \varepsilon_2(\tau) = \tilde{\tau}$, $\varepsilon_1(\omega) = \varepsilon_2(\omega) = \tilde{\omega}$, $\varepsilon_1(a_i) = c_i$, $\varepsilon_2(a_i) = c_{n+i}$ $(i = 1, \ldots, n)$. Consider

$$\tilde{A} = \varepsilon_1(\mathcal{L}) \cup \tilde{B} \cup \{1\},$$

where $\tilde{B} = \{\bigvee_{i \in I} c_i \mid \emptyset \neq I \subseteq \{2, \ldots, 2n-1\}\}$.

Note that $\tilde{\omega} = \varepsilon_1 i_1(b) \vee \varepsilon_1 i_1(a) = \varepsilon_1 i_1(b) \vee \varepsilon_2 i_2(a) = c_2 \vee \ldots \vee c_n \vee c_{n+1} \vee \ldots \vee c_{2n-1} \in \tilde{B}$.

We will show that the set $\tilde{A}$ is a subalgebra of $\mathfrak{A}$.

First of all,

$$2 \le i, j \le n \Longrightarrow c_i \& c_j = \varepsilon_1(a_i \& a_j) = \varepsilon_1(\tau) = \tilde{\tau},$$

$$n < i, j \le 2n-1 \Longrightarrow c_i \& c_j = \varepsilon_2(a_i \& a_j) = \varepsilon_2(\tau) = \tilde{\tau},$$

$$2 \le i \le n < j \le 2n-1 \Longrightarrow \tilde{\tau} \le c_i \& c_j \le \varepsilon_1 i_1(b) \& \varepsilon_2 i_2(a) =$$
$$= \varepsilon_1 i_1(b) \& \varepsilon_1 i_1(a) = \varepsilon_1 i_1(b \& a) = \tilde{\tau}.$$

Thus, $c_i \& c_j = \tilde{\tau}$ for all i, j $(2 \le i, j \le 2n-1)$. Therefore, the set $\{\tau\} \cup \tilde{B}$ is closed under $\&$ and $\vee$, hence all of $\tilde{A}$ is closed under these operations, since $\varepsilon_1(\mathcal{L}) \cup \{1\}$ is a subalgebra of $\mathfrak{A}$ and

$$x \in \varepsilon_1(\mathcal{L}),\ y \in \tilde{B} \Longrightarrow x \le y.$$

Note that $0_{\mathfrak{A}} \in \varepsilon_1(\mathcal{L}) \subseteq \tilde{A}$. It remains to show that $\tilde{A}$ is closed under $\supset$. If $x, y \in \varepsilon_1(\mathcal{L}) \cup \{1\}$, then $x \supset y \in \varepsilon_1(\mathcal{L}) \cup \{1\}$. If $x \in \varepsilon_1(\mathcal{L})$ and $y \in \tilde{B} \cup \{1\}$, then $x \supset y = 1$. If $x \in \tilde{B} \cup \{1\}$ and $y \in \varepsilon_1(\mathcal{L}) \setminus \{\tilde{\tau}\}$, then $x \supset y = y$. We now consider the case where $x \in \tilde{B}$, $y \in \{\tilde{\tau}\} \cup \tilde{B}$.

$$2 \le i \le n \Longrightarrow c_i \supset \tilde{\tau} = \varepsilon_1(a_i \supset \tau) = \varepsilon_1\Big(\bigvee_{\substack{j=1 \\ j \neq i}}^{n} a_j\Big) =$$
$$= \varepsilon_1 i_1(a) \vee \bigvee_{\substack{j=2 \\ j \neq i}}^{n} c_j = \varepsilon_2 i_2(a) \vee \bigvee_{\substack{j=2 \\ j \neq i}}^{n} c_j = \bigvee_{\substack{j=2 \\ j \neq i}}^{2n-1} c_j.$$

Similarly, $n+1 \le i \le 2n-1 \Longrightarrow c_i \supset \tilde{\tau} = \bigvee_{\substack{j=2 \\ j \neq i}}^{2n-1} c_j$. Now if $\emptyset \neq J \subseteq \{2, \ldots, 2n-1\}$, then

$$c_i \supset \bigvee_{j \in J} c_j = c_i \supset \bigvee_{j \in J} (c_i \& c_j) = \begin{cases} 1 & \text{if } i \in J \\ c_i \supset \tilde{\tau} & \text{if } i \notin J. \end{cases}$$

Finally, $(\bigvee_{i\in I} c_i)\supset y = \&_{i\in I}(c_i\supset y)$. Thus, $\widetilde{A}$ is closed under $\supset$.

If $I,J\subseteq\{2,\dots,2n-1\}$; $I,J\neq\emptyset$; $i_0\in I\setminus J$, then

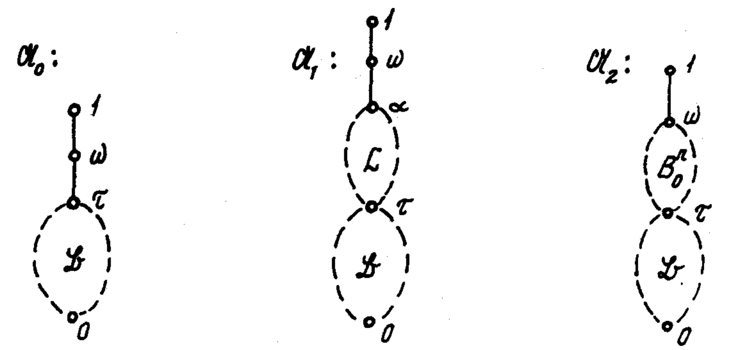
$$\bigvee_{i\in I} c_i \supset \bigvee_{j\in J} c_j \le c_{i_0}\supset \bigvee_{j\in J} c_j = c_{i_0}\supset\widetilde{\tau} < 1 ,$$

i.e., $\bigvee_{i\in I}c_i\not\le\bigvee_{j\in J}c_j$. Therefore, all elements of $\widetilde{B}$ are distinct and different from τ , hence $\{\tau\}\cup\widetilde{B}$ and B_0^{2n-2} are isomorphic as partially ordered sets. Consequently, the pseudo-Boolean algebra $\widetilde{A}$ is isomorphic to $\mathcal{B}_1=\mathcal{B}+B_0^{2n-2}+B_0$, and $\mathcal{B}_1\in\mathfrak{M}$. Note that $2n-2>n$. Repeating the construction, we obtain that the algebras $\mathcal{B}+B_0^m+B_0$ belong to $\mathfrak{M}$ for arbitrarily large m . If $0<k\le m$, then B_0^k can be isomorphically embedded in B_0^m , hence $\mathcal{B}+B_0^k+B$ is isomorphic to a subalgebra of $\mathcal{B}+B_0^m+B_0$. Therefore, the algebras $\mathcal{B}+B_0^m+B_0$ belong to $\mathfrak{M}$ for any $m>0$, as required.

<u>LEMMA 25.</u> Suppose $\mathfrak{M}$ is (weakly) amalgamable, $\mathcal{B}$ and $\mathcal{L}$ are arbitrary (finite) PBA, $(\mathcal{B}+\mathcal{L}^+ +B_0)\in\mathfrak{M}$ and $(\mathcal{B}+B_0^n+B_0)\in\mathfrak{M}$ for some $n\ge 1$. Then $(\mathcal{B}+(B_0^{n-1}\times\mathcal{L}^+)+B_0)\in\mathfrak{M}$.

<u>Proof.</u> The algebras $\mathfrak{A}_1=\mathcal{B}+\mathcal{L}^++B_0$ and $\mathfrak{A}_2=\mathcal{B}+B_0^n+B_0$ contain the common subalgebra $\mathfrak{A}_0=\mathcal{B}+B_0+B_0$.

Diagrams of $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are illustrated below.



By Lemma 22, there exists a common extension $(\mathfrak{A},\varepsilon_1,\varepsilon_2)$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, where $\mathfrak{A}\in\mathfrak{M}$, ω is a penultimate element of $\mathfrak{A}$, and $\varepsilon_1\restriction\mathfrak{A}_0=\varepsilon_2\restriction\mathfrak{A}_0=id$. Note that $\omega=\varepsilon_2(\omega)=\bigvee_{i=1}^n\bar a_i$, where $\bar a_i=\varepsilon_2(a_i)$ and the a_i are atoms of B_0^n , i.e., elements covering τ in $\mathfrak{A}_2$. Since $\varepsilon_1(\alpha)<\varepsilon_1(\omega)=\omega=\bigvee_{i=1}^n\varepsilon_2(a_i)$, there exists i $(1\le i\le n)$ such that $\varepsilon_2(a_i)\not\le\varepsilon_1(\alpha)$. We may assume that $i=n$. Any element of the set $\varepsilon_2(B_0^n)$ can be uniquely represented in the form $\bigvee_{i\in I}\bar a_i$, where $I\subseteq\bar n\rightleftharpoons\{1,\dots,n\}$ (we regard $\bigvee_{i\in\emptyset}\bar a_i=\tau$).

We will show that the algebra $\mathfrak{A}_3=\mathcal{B}+(B_0^{n-1}\times\mathcal{L}^+)+B_0$ can be isomorphically embedded in $\mathfrak{A}$. Any element of the set B_0^{n-1} can be uniquely represented in the form $\bigvee_{j\in J}b_j$, where $J\subseteq\overline{n-1}$, and $b_1,\dots,b_{n-1}$ are all of the atoms in B_0^{n-1} ; an element of the set $B_0^{n-1}\times\mathcal{L}^+$ is a pair $\langle x,y\rangle$, where $x\in B_0^{n-1}$, $y\in\mathcal{L}^+$. For $z\in\mathfrak{A}_3$ we put $h(z)=z$ if $z\in\mathcal{B}\cup\{1\}$; if $z=\langle\bigvee_{j\in J}b_j,y\rangle$, where $J\subseteq\overline{n-1}$, then

$$h(z)=(\varepsilon_1(y)\&\bar a_n)\vee\bigvee_{j\in J}\bar a_j .$$

Note that

$$(\mathcal{L} \cup \{1\}) \cap (B_0^{n-1} \times \mathcal{L}^+) = \{\tau\} = \{< \bigvee_{j\in\emptyset} b_j , 0_{\mathcal{L}^+} >\},$$
$$h(\tau) = (\varepsilon_1 (0_{\mathcal{L}^+}) \& \bar{a}_n) \vee \bar{\tau} = (\tau \& \bar{a}_n) \vee \tau = \tau,$$

i.e., h is correctly defined.

Moreover,

$$h(< \bigvee_{j\in\overline{n-1}} b_j , 1_{\mathcal{L}^+} >) = (\varepsilon_1(\omega) \& \bar{a}_n) \vee \bigvee_{j\in\overline{n-1}} \bar{a}_j = \bigvee_{j\in\bar{n}} \bar{a}_j = \omega.$$

We will show that h is a homomorphism. Clearly, $z \leq t \Rightarrow h(z) \leq h(t)$. Also, for $i \neq j$, $i,j \in \bar{n}$ we have

$$\bar{a}_i \& \bar{a}_j = \varepsilon_2 (a_i \& a_j) = \tau.$$

It follows easily that

$$h(z \& t) = h(z) \& h(t).$$

Obviously, $h(z \vee t) = h(z) \vee h(t)$ and $h(0) = 0$.

It is easy to verify the equality

$$h(z \supset t) = h(z) \supset h(t)$$

in the cases where $z \leq t$ or $z, t \in \mathcal{L} \cup \{1\}$. If $z \notin \mathcal{L}$, $t \in \mathcal{L}$, then $h(z) \geq h(\tau) = \tau \geq t$ $h(z) \supset h(t) \leq \tau \supset t = t = h(z \supset t) = h(t) \leq h(z) \supset h(t)$.

Assume now that $z, t \in B_0^{n-1} \times \mathcal{L}^+$ and $z \nleq t$. Suppose $z = < \bigvee_{j\in J} b_j , y >$, $t = < \bigvee_{\kappa\in K} b_\kappa , \upsilon >$; $J, K \subseteq \overline{n-1}$. Then

$$z \supset t = < \bigvee_{\kappa\in (\overline{n-1} \setminus J) \cup K} b_\kappa , y \supset_{\mathcal{L}^+} \upsilon >, \tag{*}$$

$$h(z) \supset h(t) = [(\varepsilon_1 (y) \& \bar{a}_n) \vee \bigvee_{j\in J} \bar{a}_j] \supset [(\varepsilon_1 (\upsilon) \& \bar{a}_n) \vee \bigvee_{\kappa\in K} \bar{a}_\kappa] =$$
$$= [\varepsilon_1 (y) \& \bar{a}_n \supset \varepsilon_1 (\upsilon) \vee \bigvee_{\kappa\in K} \bar{a}_\kappa] \& [\bigvee_{j\in J} \bar{a}_j \supset \varepsilon_1 (\upsilon) \vee \bigvee_{\kappa\in K} \bar{a}_\kappa] \&$$
$$\& [\varepsilon_1 (y) \& \bar{a}_n \supset \bar{a}_n \vee \bigvee_{\kappa\in K} \bar{a}_\kappa] \& [\bigvee_{j\in J} \bar{a}_j \supset \bar{a}_n \vee \bigvee_{\kappa\in K} \bar{a}_\kappa]. \tag{**}$$

Using known identities for PBA, we obtain

$$\varepsilon_1 (y) \& \bar{a}_n \supset \varepsilon_1 (\upsilon) \vee \bigvee_{\kappa\in K} \bar{a}_\kappa = \varepsilon_1 (y) \& \bar{a}_n \supset \bar{a}_n \& (\varepsilon_1 (\upsilon) \vee \bigvee_{\kappa\in K} \bar{a}_\kappa) =$$
$$= \varepsilon_1 (y) \& \bar{a}_n \supset (\bar{a}_n \& \varepsilon_1 (\upsilon)) \vee \tau = \varepsilon_1 (y) \& \bar{a}_n \supset \varepsilon_1 (\upsilon) = \bar{a}_n \supset \varepsilon_1 (y \supset \upsilon). \tag{***}$$

Case a) $J \nsubseteq K$.

By definition of $\mathcal{A}_2$ and $\mathcal{A}$:

$$\bigvee_{j\in J}\bar a_j \supset \bar a_n \vee \bigvee_{\kappa\in K}\bar a_\kappa = \bigvee_{i\in(\bar n-J)\cup K}\bar a_i \leq \bigvee_{j\in J}\bar a_j \supset \varepsilon_1(v)\vee\bigvee_{\kappa\in K}\bar a_\kappa .$$

Therefore

$$h(z)\supset h(t) = (\bar a_n \supset \varepsilon_1(y\supset v)) \,\&\, \bigvee_{i\in(\bar n-J)\cup K}\bar a_i =$$

$$= \Big[(\bar a_n\supset\varepsilon_1(y\supset v))\,\&\bigvee_{i\in(\overline{n-1}-J)\cup K}\bar a_i\Big]\vee\big[\bar a_n \,\&\, (\bar a_n\supset\varepsilon_1(y\supset v))\big] =$$

$$= \Big[\bigvee_{i\in(\overline{n-1}-J)\cup K}\bar a_i\Big] \vee \big[\bar a_n \,\&\, \varepsilon_1(y\supset v)\big],$$

since

$$\bar a_n\supset\varepsilon_1(y\supset v) \geq \bar a_n\supset\tau = \bigvee_{j\in\overline{n-1}}a_j \geq \bigvee_{j\in(\overline{n-1}-J)\cup K}a_j ,$$

$$a \,\&\, (a\supset b) = a\,\&\, b .$$

If $y\not\leq v$, then $y\supset v = y\supset_{\mathcal{L}^+} v$, and from (*) we obtain $h(z)\supset h(t) = h(z\supset t)$. If $y\leq v$, then $y\supset v = 1$; $y\supset_{\mathcal{L}^+}v = 1_{\mathcal{L}^+}$, $\varepsilon_1(y\supset_{\mathcal{L}^+}v)=\omega$, hence, in view of (*),

$$h(z\supset t) = (\omega\,\&\,\bar a_n)\vee\bigvee_{\kappa\in(\overline{n-1}-J)\cup K}\bar a_\kappa = h(z)\supset h(t).$$

Case b) $J\subseteq K$, and therefore $y\not\leq v$. Then

$$z\supset t = \Big\langle\bigvee_{\kappa\in\overline{n-1}} b_\kappa,\ y\supset_{\mathcal{L}^+}v\Big\rangle = \Big\langle\bigvee_{\kappa\in\overline{n-1}} b_\kappa,\ y\supset v\Big\rangle,$$

$$h(z\supset t) = (\varepsilon_1(y\supset v)\,\&\,\bar a_n)\vee\bigvee_{\kappa\in\overline{n-1}}\bar a_\kappa = \Big(\varepsilon_1(y\supset v)\vee\bigvee_{\kappa\in\overline{n-1}}\bar a_\kappa\Big)\,\&\,\omega =$$

$$= \varepsilon_1(y\supset v)\vee\bar a_n' \qquad \text{, where } \bar a_n' = \bigvee_{\kappa\in\overline{n-1}}\bar a_\kappa .$$

It follows from (**) and (***) that

$$h(z)\supset h(t) = \bar a_n\supset\varepsilon_1(y\supset v).$$

Since $\bar a_n' \,\&\, \bar a_n = \tau \leq \varepsilon_1(y\supset v)$, we have $\varepsilon_1(y\supset v)\vee\bar a_n' \leq \bar a_n\supset\varepsilon_1(y\supset v)$. We will prove the reverse inequality. Since $y\supset v\leq\alpha$ and $\bar a_n\not\leq\varepsilon_1(\alpha)$, we have

$$\bar a_n\supset\varepsilon_1(y\supset v)\leq\bar a_n\supset\varepsilon_1(\alpha)\leq\omega,$$

$$\bar a_n\supset\varepsilon_1(y\supset v)\leq \bar a_n\vee\bar a_n'\supset\varepsilon_1(y\supset v)\vee\bar a_n' = \omega\supset\varepsilon_1(y\supset v)\vee\bar a_n',$$

hence

$$\bar a_n\supset\varepsilon_1(y\supset v)\leq\omega\,\&\,(\omega\supset\varepsilon_1(y\supset v)\vee\bar a_n')\leq\varepsilon_1(y\supset v)\vee\bar a_n'.$$

Thus, in case b) also we have

$$h(z\supset t) = h(z)\supset h(t).$$

In addition,

$$h(\neg z) = h(z\supset 0) = h(z)\supset h(0) = h(z)\supset 0 = \neg h(z).$$

Thus, h is a homomorphism.

If $z \nleq t$, then $z \supset t \leqslant \langle \bigvee_{j \in \overline{n-1}} b_j, 1_{\mathcal{L}^+} \rangle$,

$$h(z) \supset h(t) = h(z \supset t) \leqslant (\omega \,\&\, \bar{a}_n) \vee \bar{a}_n' = \omega < 1,$$

i.e., $h(z) \nleq h(t)$.

The lemma is proved.

LEMMA 26. Suppose $\mathfrak{M}$ is (weakly) amalgamable, $n \geqslant 1$, $\mathcal{B}, \mathcal{L}_1, \ldots, \mathcal{L}_n$ are arbitrary (finite) PBA, $(\mathcal{B} + B_0^n + B_0) \in \mathfrak{M}$, $(\mathcal{B} + \mathcal{L}_1^+), \ldots, (\mathcal{B} + \mathcal{L}_n^+) \in \mathfrak{M}$, and $L_4 \in \mathfrak{M}$. Then

$$(\mathcal{B} + (\mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+) + B_0) \in \mathfrak{M}.$$

Proof. By Lemma 23, the algebras $\mathcal{B} + \mathcal{L}_1^+ + B_0, \ldots, \mathcal{B} + \mathcal{L}_n^+ + B_0$ belong to $\mathfrak{M}$. By Lemma 25, $\mathfrak{M}$ contains the algebras $\mathfrak{A}_i = \mathcal{B} + (B_0^{i-1} \times \mathcal{L}_i^+ \times B_0^{n-i}) + B_0$ for $i = 1, \ldots, n$. The algebra $\mathfrak{A}_0 = \mathcal{B} + B_0^n + B_0$ is a common subalgebra of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$; a penultimate element in $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is $\omega = \langle 1 \ldots 1 \rangle = \langle 1^n \rangle$. The elements of B_0^n can be represented by processions $\langle x_1, \ldots x_n \rangle$, where $x_i \in \{0, 1\}$; $\mathcal{B} \cap B_0^n = \{\tau\}$, where $\tau = \langle 0^n \rangle$. Applying Lemma 22 $(n-1)$ times, we obtain a subdirectly irreducible PBA $\mathfrak{A}$ with penultimate element ω and monomorphisms $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ $(i = 1, \ldots, n)$ such that $\varepsilon_i(x) = x$ for $x \in \mathfrak{A}_0$.

Let $\bar{\mathfrak{A}} = \mathcal{B} + (\mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+) + B_0$. We will show that the following mapping $g : \bar{\mathfrak{A}} \to \mathfrak{A}$ is a monomorphism:

$$g(x) = x \quad , \text{ if } x \in \mathcal{B} \cup \{1\},$$

$$g(\langle x_1 \ldots x_n \rangle) = \mathop{\&}_{i=1}^{n} \varepsilon_i (\langle 1^{i-1} x_i 1^{n-i} \rangle) \quad , \text{ if}$$

$$x = \langle x_1, \ldots x_n \rangle \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+.$$

Note that $\mathfrak{A}_0$ is a subalgebra of $\bar{\mathfrak{A}}$ and $g(x) = x$ for any $x \in \mathfrak{A}_0$. Therefore, the mapping g on $\mathfrak{A}_0$ commutes with the operations $\&, \vee, \supset, \neg$, and to verify the equality

$$g(x \,\&\, y) = g(x) \,\&\, g(y)$$

it suffices to consider cases where $\{x, y\} \not\subseteq \mathfrak{A}_0$.

If $x \in \mathcal{B}$, $y \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+$, then

$$g(x) \,\&\, g(y) = (g(x) \,\&\, \tau) \,\&\, g(y) = g(x) \,\&\, \tau = g(x) = x = g(x \,\&\, y).$$

If $x = 1$, $y \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+$, then also $g(x) \,\&\, g(y) = g(x \,\&\, y)$.

If $x, y \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+$, then

$$g(x) \,\&\, g(y) = \mathop{\&}_{i=1}^{n} \varepsilon_i (\langle 1^{i-1} x_i 1^{n-i} \rangle) \,\&\, \mathop{\&}_{i=1}^{n} \varepsilon_i (\langle 1^{i-1} y_i 1^{n-i} \rangle) =$$

$$= \mathop{\&}_{i} \varepsilon_i \left[\langle 1^{i-1} x_i 1^{n-i} \rangle \,\&\, \langle 1^{i-1} y_i 1^{n-i} \rangle \right] = g(x \,\&\, y).$$

Thus,

$$g(x \& y) = g(x) \& g(y), \tag{\&}$$

hence

$$x \leq y \implies g(x) \leq g(y). \tag{$\leq$}$$

We will show that

$$g(x \vee y) = g(x) \vee g(y). \tag{$\vee$}$$

Note first that for $i \neq j$

$$\begin{aligned} \omega = <1^n> = \varepsilon_i\left(<1^{i-1} x_i\, 1^{n-i}>\right) \vee \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) \geq \\ \geq \varepsilon_i\left(<1^{i-1} 0\, 1^{n-i}>\right) \vee \varepsilon_j\left(<1^{j-1} 0\, 1^{n-j}>\right) = <1^n> = \omega. \end{aligned} \tag{+}$$

Therefore, for $x, y \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+$

$$\begin{aligned} g(x) \vee g(y) = \mathop{\&}_i \varepsilon_i\left(<1^{i-1} x_i\, 1^{n-i}>\right) \vee \mathop{\&}_j \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) = \\ = \omega \& \mathop{\&}_i \varepsilon_i\left(<1^{i-1} (x_i \vee y_i)\, 1^{n-i}>\right) = g(x \vee y). \end{aligned}$$

In the remaining cases the equality follows from $(\leq)$ and from $g \restriction \mathfrak{A}_0 = id$.

Finally, we will prove that

$$g(x \supset y) = g(x) \supset g(y). \tag{$\supset$}$$

a) Suppose $x \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+,\ y \in \mathcal{L} \setminus \{\tau\}$. Then

$$y = g(y) \leq g(x) \supset g(y) \leq g(\tau) \supset g(y) = \tau \supset y = y = g(x \supset y).$$

b) Suppose $x, y \in \mathcal{L}_1^+ \times \ldots \times \mathcal{L}_n^+$ and $x \nleq y$. Then

$$g(x) \supset g(y) = \mathop{\&}_{j=1}^{n} \left[\mathop{\&}_{i=1}^{n} \varepsilon_i\left(<1^{i-1} x_i\, 1^{n-i}>\right) \supset \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right)\right].$$

Using $(+)$, we obtain for any j $(1 \leq j \leq n)$

$$\begin{aligned} z_j \leftrightharpoons \left[\mathop{\&}_i \varepsilon_i\left(<1^{i-1} x_i\, 1^{n-i}>\right) \supset \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right)\right] = \\ = \left[\varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) \vee \left(\mathop{\&}_i \varepsilon_i\left(<1^{i-1} x_i 1^{n-i}>\right)\right)\right] \supset \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) = \\ = \left[\varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) \vee \varepsilon_j\left(<1^{j-1} x_j\, 1^{n-j}>\right)\right] \supset \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) = \\ = \varepsilon_j\left(<1^{j-1} x_j\, 1^{n-j}>\right) \supset \varepsilon_j\left(<1^{j-1} y_j\, 1^{n-j}>\right) = \varepsilon_j\left(<1^{j-1} x_j\, 1^{n-j}> \supset <1^{j-1} y_j\, 1^{n-j}>\right). \end{aligned}$$

Therefore, $z_j = 1$ if $x_j \leq y_j$, and $z_j = \varepsilon_j(<1^{j-1}(x_j \supset y_j)1^{n-j}>) \leq \omega$ if $x_j \not\leq y_j$; in either case, $z_j \,\&\, \omega = \varepsilon_j(<1^{j-1}(x_j \supset y_j)1^{n-j}>)$.

Since $x \not\leq y$, we have $\&_j z_j \leq \omega$, hence

$$g(x) \supset g(y) = \&_j z_j = \&_j (z_j \,\&\, \omega) = \&_j \varepsilon \;(<1^{j-i}(x_j \supset y_j)1^{n-j}>).$$

On the other hand,

$$x \supset y = <(x_1 \supset y_1)\dots(x_n \supset y_n)>$$

and $g(x) \supset g(y) = g(x \supset y)$.

In the remaining cases, equality ($\supset$) follows from ($\leq$), since $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$. Thus, ($\supset$) is proved. Moreover, $g(0) = 0$, hence $g(\neg x) = \neg g(x)$.

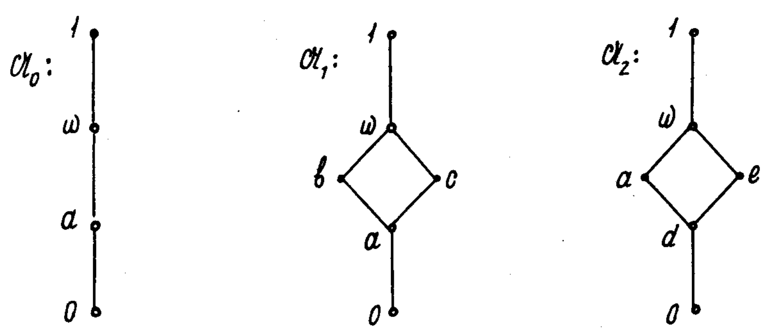
Finally,

$$x \not\leq y \Rightarrow x \supset y \leq \omega \Rightarrow g(x) \supset g(y) = g(x \supset y) \leq g(\omega) < 1 \Rightarrow g(x) \not\leq g(y).$$

Therefore, g is a monomorphism and Lemma 26 is proved.

LEMMA 27. Suppose $\mathfrak{M}$ is weakly amalgamable and $B_3 = (B_0 + B_0^2 + B_0) \in \mathfrak{M}$. Then $B_4 = (B_0 + B_0^3 + B_0) \in \mathfrak{M}$.

Proof. Take in $\mathfrak{M}$ algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$, and $\mathfrak{A}_2$ with the following diagrams:



It is easy to verify that $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. By Lemma 22, there exists a PBA $\mathfrak{A}$ with penultimate element ω and monomorphisms $\varepsilon_i : \mathfrak{A}_i \rightarrow \mathfrak{A}$ $(i = 1, 2)$, which are the identity on $\mathfrak{A}_0$. Let $\bar{b} = \varepsilon_1(b)$, $\bar{c} = \varepsilon_1(c)$, $\tilde{e} = \varepsilon_2(e)$, $\tilde{d} = \varepsilon_2(d)$. Consider the set

$$\bar{A} = \{0, a, \omega, 1, \bar{b}, \bar{c}, \tilde{d}, \tilde{e}, \tilde{e} \,\&\, \bar{b}, \tilde{e} \,\&\, \bar{c}\}.$$

We will show that it is a subalgebra of $\mathfrak{A}$. From the definition we at once obtain the following relations among the elements of $\bar{A}$ ($\longrightarrow$ stands for $<$, and $\Longrightarrow$ for $\leq$):

$$\begin{tikzcd}
 & & \tilde{e}\,\&\,\bar{b} \arrow[r] \arrow[dr, Rightarrow] & \bar{b} \arrow[dr] & & \\
0 \arrow[r] & \tilde{d} \arrow[r] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & a \arrow[ur] \arrow[dr] & \tilde{e} \arrow[r] & \omega \arrow[r] & 1 \\
 & & \tilde{e}\,\&\,\bar{c} \arrow[r] \arrow[ur, Rightarrow] & \bar{c} \arrow[ur] & &
\end{tikzcd} \tag{D}$$

Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are monomorphisms, we have in $\mathfrak{A}$:

$$(\tilde{e}\&\bar{b})\&a = \tilde{e}\&(\bar{b}\&a) = \tilde{e}\&a = \tilde{d},$$
$$(\tilde{e}\&\bar{c})\&a = \tilde{d},\quad (\tilde{e}\&\bar{b})\&(\tilde{e}\&\bar{c}) = \tilde{e}\&a = \tilde{d},$$
$$(\tilde{e}\&\bar{b})\&\bar{c} = \tilde{e}\&a = \tilde{d},\ (\tilde{e}\&\bar{c})\&\bar{b} = \tilde{d},\ \bar{b}\&\bar{c} = a,$$

hence $\bar{A}$ is closed under $\&$.

Also

$$(\tilde{e}\&\bar{b})\vee a = (\tilde{e}\vee a)\&(\bar{b}\vee a) = \omega\&\bar{b} = \bar{b},$$

$$(\tilde{e}\&\bar{c})\vee a = \bar{c},\quad (\tilde{e}\&\bar{b})\vee(\tilde{e}\&\bar{c}) = \tilde{e}\&\omega = \tilde{e};$$

$\tilde{e}\vee\bar{c} \geq \tilde{e}\vee a = \omega$, i.e., $\tilde{e}\vee\bar{c} = \omega$, and similarly $\tilde{e}\vee\bar{b} = \omega$; $\bar{b}\vee\bar{c} = \omega$,

$$(\tilde{e}\&\bar{b})\vee\bar{c} = (\tilde{e}\vee\bar{c})\&(\bar{b}\vee\bar{c}) = \omega,$$
$$(\tilde{e}\&\bar{c})\vee\bar{b} = \omega,$$

hence $\bar{A}$ is closed under $\vee$.

Finally, if $x\in\bar{A}$, $x\neq 0$, then $\neg x \leq \neg d = d\supset 0 = 0$. If $x,y\in\mathcal{E}_i(\mathfrak{A}_i)$ $(i=1,2)$ or $x\leq y$, then $(x\supset y)\in\bar{A}$. Also, $\tilde{e}\leq\bar{b}\supset\tilde{e}\leq a\supset\tilde{e} = \tilde{e}$, i.e., $\bar{b}\supset\tilde{e} = \tilde{e}$, and similarly $\bar{c}\supset\tilde{e} = \tilde{e}$. We have

$$(\tilde{e}\supset\bar{b})\&\bar{c} \leq (\tilde{e}\supset\bar{b})\&(\tilde{e}\supset\bar{c}) = \tilde{e}\supset(\bar{b}\&\bar{c}) = \tilde{e}\supset a = a,$$

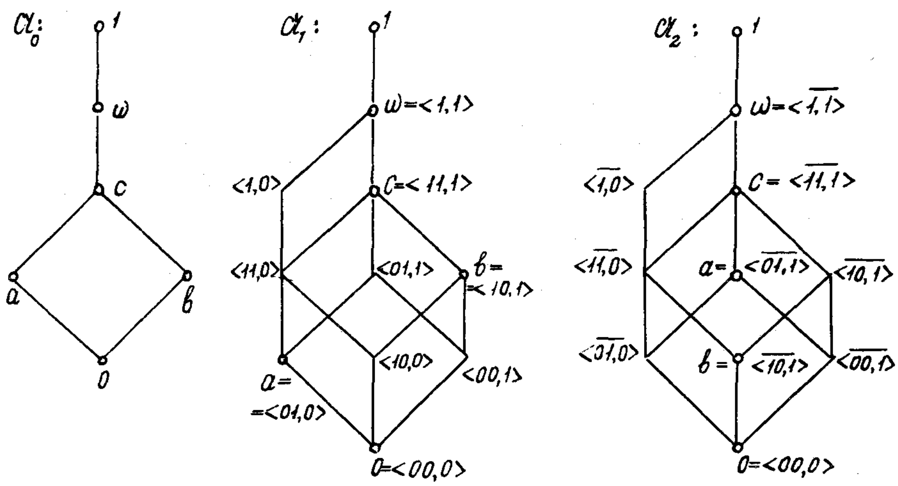
hence $\tilde{e}\supset\bar{b} \leq \bar{c}\supset a = \bar{b}$ and $\tilde{e}\supset\bar{b} = \bar{b}$, and similarly $\tilde{e}\supset\bar{c} = \bar{c}$. Thus, $x\supset y\in\bar{A}$ for $x,y\in\{0,1,a,\omega,\bar{b},\bar{c},\tilde{e}\}$. Using the identities $(x\&y)\supset z = (x\supset(y\supset z)) = y\supset(x\supset z)$, $z\supset(x\&y) = (z\supset x)\&(z\supset y)$ and the equality $\tilde{d} = a\&\tilde{e}$, we obtain the following table for $x\supset y$:

$x \backslash y$	0	$\tilde{d}$	$\tilde{e}\&\bar{b}$	$\tilde{e}\&\bar{c}$	a	$\bar{b}$	$\bar{c}$	$\tilde{e}$	ω	1
0	1	1	1	1	1	1	1	1	1	1
$\tilde{d}$	0	1	1	1	1	1	1	1	1	1
$\tilde{e}\&\bar{b}$	0	$\bar{c}$	1	$\bar{c}$	$\bar{c}$	1	$\bar{c}$	1	1	1
$\tilde{e}\&\bar{c}$	0	$\bar{b}$	$\bar{b}$	1	$\bar{b}$	$\bar{b}$	1	1	1	1
a	0	$\tilde{e}$	$\tilde{e}$	$\tilde{e}$	1	1	1	$\tilde{e}$	1	1
$\bar{b}$	0	$\tilde{e}\&\bar{c}$	$\tilde{e}$	$\tilde{e}\&\bar{c}$	$\bar{c}$	1	$\bar{c}$	$\tilde{e}$	1	1
$\bar{c}$	0	$\tilde{e}\&\bar{b}$	$\tilde{e}\&\bar{b}$	$\tilde{e}$	$\bar{b}$	$\bar{b}$	1	$\tilde{e}$	1	1
$\tilde{e}$	0	a	$\bar{b}$	$\bar{c}$	a	$\bar{b}$	$\bar{c}$	1	1	1
ω	0	$\tilde{d}$	$\tilde{e}\&\bar{b}$	$\tilde{e}\&\bar{c}$	a	$\bar{b}$	$\bar{c}$	$\tilde{e}$	1	1
1	0	$\tilde{d}$	$\tilde{e}\&\bar{b}$	$\tilde{e}\&\bar{c}$	a	$\bar{b}$	$\bar{c}$	$\tilde{e}$	ω	1

It can be seen from this table that in diagram (D) we can replace every $\Rightarrow$ by $\rightarrow$ and that those elements between which there are no arrows are incomparable. Thus, $\bar{A}$ is a subalgebra of $\mathfrak{A}$ and has the same diagram as the PBA B_4, hence B_4 and $\mathfrak{A}$ are isomorphic. Therefore, $B_4 \in \mathfrak{M}$, as required.

<u>LEMMA 28.</u> Suppose $\mathfrak{M}$ is weakly amalgamable, $L_4 \in \mathfrak{M}$, and $C_2 = B_0^2 + B_0 \in \mathfrak{M}$. Then $C_3 = B_0^3 + B_0 \in \mathfrak{M}$.

<u>Proof.</u> By Lemma 26, $\mathfrak{M}$ contains the PBA $\mathfrak{A}_1 = ((B_0^2 + B_0) \times B_0) + B_0$. Let $\mathfrak{A}_0$ be isomorphic to $B_0^2 + B_0 + B_0$, $\mathfrak{A}_2$ isomorphic to $\mathfrak{A}_1$. The diagrams of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are illustrated below.



It is easy to verify that with this notation $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. By Lemma 22, there exist a PBA $\mathfrak{A}$ with penultimate element ω and monomorphisms $\varepsilon_i : \mathfrak{A}_i \rightarrow \mathfrak{A}\ (i = 1, 2)$ which are the identity mappings on $\mathfrak{A}_0$. Now let

$$x = \varepsilon_1(\langle 1,0 \rangle) \,\&\, \varepsilon_2(\langle \overline{1,0} \rangle),$$
$$y = \varepsilon_1(\langle 00,1 \rangle),$$
$$z = \varepsilon_2(\langle \overline{00,1} \rangle)$$

and consider the set

$$\tilde{A} = \{0, 1, \omega, x, y, z, x \vee y, x \vee z, y \vee z\}.$$

We observe at once that in $\mathfrak{A}$:

$$x = \neg y \,\&\, \neg z, \quad \text{hence} \quad x \,\&\, y = 0 \text{ and } x \,\&\, z = 0;$$
$$y \,\&\, z \leqslant b \,\&\, a = 0, \quad \text{i.e.,} \quad y \,\&\, z = 0.$$

Therefore, $\tilde{A}$ is closed under $\&$.

Also

$$x \vee y = \varepsilon_1(\langle 1,0 \rangle \vee \langle 00,1 \rangle) \,\&\, (\varepsilon_2(\langle \overline{1,0} \rangle) \vee \varepsilon_1(\langle 00,1 \rangle)) = \omega \,\&\, \varepsilon_2(\langle \overline{1,0} \rangle),$$

since $(\langle 00,1 \rangle) < b < \varepsilon_2(\langle \overline{1,0} \rangle)$, hence

$$x\vee y=\varepsilon_2(\langle\overline{1,0}\rangle)=\neg\varepsilon_2(\langle\overline{00,1}\rangle)=\neg z.$$

Similarly,

$$x\vee z=(\varepsilon_1(\langle 1,0\rangle)\vee\varepsilon_2(\langle\overline{00,1}\rangle))\&\omega=\varepsilon_1(\langle 1,0\rangle)=\neg y.$$

We will show that $y\vee z=\neg x$. We have

$$\neg x=\varepsilon_1(\langle 1,0\rangle)\supset\neg\varepsilon_2(\langle\overline{1,0}\rangle)=\varepsilon_1(\langle 1,0\rangle)\supset z\le$$
$$\le a\supset z=\varepsilon_2(\langle\overline{10,1}\rangle)=b\vee z,$$
$$\neg x=\varepsilon_2(\langle\overline{1,0}\rangle)\supset\neg\varepsilon_1(\langle 1,0\rangle)=\varepsilon_2(\langle\overline{1,0}\rangle)\supset y\le$$
$$\le b\supset y=\varepsilon_1(\langle 01,1\rangle)=a\vee y.$$

Therefore,

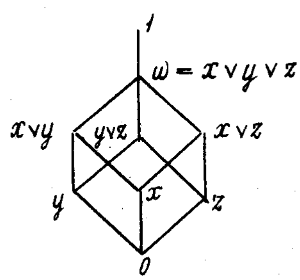
$$\neg x\le(b\vee z)\&(a\vee y)=(b\&a)\vee(z\&a)\vee(b\&y)\vee(z\&y)=$$
$$=z\vee y\vee(z\&y)=z\vee y=\neg\neg z\vee\neg\neg y\le\neg(\neg y\&\neg z)=\neg x.$$

Thus, $\neg x=y\vee z$. It follows from the identity $\neg(u\vee v)=\neg u\&\neg v$ that $\tilde{A}$ is closed under $\neg$. Finally, $x\vee y\vee z=\neg z\vee z=\omega$, hence $\tilde{A}$ is closed under $\vee$. Moreover, $\neg u\vee u\ge x\vee y\vee z=\omega$ for all $u\in\tilde{A}$. Therefore, for $u,v\in\tilde{A}$ it follows from $u\not\le v$ that

$$u\supset v=\omega\&(u\supset v)\le(\neg u\vee u)\&(u\supset v)=(\neg u\&(u\supset v))\vee$$
$$\vee(u\&(u\supset v))\le\neg u\vee v\le u\supset v,\ \text{i.e.,}\ u\supset v=\neg u\vee v.$$

Consequently, $\tilde{A}$ is closed under $\supset$.

It follows from $\neg x=y\vee z\le\omega<1$ that $x>0$, hence $x\not\le y\vee z$ and $y\vee z<\omega$. Similarly, $y\not\le x\vee z$, $z\not\le x\vee y$. Therefore, the diagram of $\tilde{A}$ has the form:



Therefore, the algebra $\tilde{A}$ is isomorphic to the PBA $C_3=B_0^3+B_0$, hence $C_3\in\mathfrak{M}$, as required.

Proposition 8. Any weakly amalgamable variety of PBA coincides with one of the varieties H_1-H_8.

Proof. We use Proposition 7. Suppose $\mathfrak{M}$ is weakly amalgamable. If $\mathfrak{M}$ is trivial, then $\mathfrak{M}=H_8$. If $\mathfrak{M}$ contains a nondegenerate PBA, then $B_0\in\mathfrak{M}$ by Lemma 21 and $\mathfrak{M}\supseteq H_7$. If $\mathfrak{M}\supseteq H_7$, $\mathfrak{M}\ne H_7$, then, by Lemma 20, $C_1\in\mathfrak{M}$, hence $\mathfrak{M}\supseteq H_5$. If $\mathfrak{M}\supseteq H_5$, $\mathfrak{M}\ne H_5$, then, by Lemma 18,

$L_4 \in \mathfrak{M}$ or $C_2 \in \mathfrak{M}$.

Case a) $L_4 \notin \mathfrak{M}$. Then $\mathfrak{M} \supseteq H_4$. If $\mathfrak{M} \neq H_4$, then, by Lemma 17, $C_3 = B_0^3 + B_0 \in \mathfrak{M}$ and, by Lemma 24, all of the algebras $C_m = B_0^m + B_0$ $(m \geq 1)$ belong to $\mathfrak{M}$. Therefore, since $C_0 = B_0 \in \mathfrak{M}$, $\mathfrak{M} \supseteq H_3$. By Lemma 16, since $L_4 \notin \mathfrak{M}$, we obtain $\mathfrak{M} = H_3$.

Case b) $L_4 \in \mathfrak{M}$. By Lemma 23, $L_n \in \mathfrak{M}$ for any $n \geq 2$, since $L_{n+1} = L_n^+$, and therefore $\mathfrak{M} \supseteq H_6$. If $\mathfrak{M} \neq H_6$, then, by Lemma 19, $C_2 \in \mathfrak{M}$ or $B_3 \in \mathfrak{M}$. If $C_2 \in \mathfrak{M}$, then, by Lemma 28, we also have $C_3 \in \mathfrak{M}$; by Lemma 24, all of the algebras $C_m = B_0^m + B_0$, belong to $\mathfrak{M}$. Using Lemma 26 with $\mathcal{L} = E$, we obtain $K(E) \subseteq \mathfrak{M}$, hence $\mathfrak{M} = H_1$. Suppose $C_2 \notin \mathfrak{M}$. Then $B_3 \in \mathfrak{M}$ and, by Lemma 27, $B_4 = B_0 + B_0^3 + B_0 \in \mathfrak{M}$; by Lemma 24, all of the algebras $B_n = B_0 + B_0^{n-1} + B_0 \in \mathfrak{M}$ $(n \geq 1)$. Putting $\mathcal{L} = B_0$ in Lemma 26, we obtain $K(B_0) \subseteq \mathfrak{M}$ and $\mathfrak{M} \supseteq H_2$. In view of Lemma 15, $\mathfrak{M} = H_2$, since $C_2 \notin \mathfrak{M}$.

The proposition is proved.

From Proposition 1-6 and 8 we obtain

THEOREM 2. For any variety $\mathfrak{M}$ of pseudo-Boolean algebras the following conditions are equivalent:

a) $\mathfrak{M}$ is amalgamable;

b) $\mathfrak{M}$ is weakly amalgamable;

c) $\mathfrak{M}$ coincides with one of the varieties $H_1 - H_8$.

From this we obtain a corollary which contrasts with the result [20] on the unsolvability of the amalgamation problem for varieties of groups.

COROLLARY 1. The amalgamation problem for varieties of PBA is solvable, this problem being: for a given finite basis of the identities of a variety, to determine whether the variety is amalgamable.

Proof. In view of Lemmas 15-21, it is easy to verify the relations $\mathfrak{M} \subseteq H_i$ for all $i = 1, \ldots, 8$ and for any finitely based variety $\mathfrak{M}$ (it suffices to verify the identities defining $\mathfrak{M}$ in the algebras $C_2, C_3, L_4, B_3, C_1, B_0$). By Proposition 7, the problems $\mathfrak{M} \supseteq H_i$ are solvable for $i = 1, \ldots, 8$ and for any finitely based variety $\mathfrak{M}$ of pseudo-Boolean algebras. The decision procedure consists, on the one hand, of checking in succession whether the algebras generating H_i belong to $\mathfrak{M}$ and, on the other, of trying to deduce the defining identities of $\mathfrak{M}$ from the identities of H_i.

Remark 1. Note also that Lemmas 23-28 enable us to find, for any nonamalgamable variety $\mathfrak{M}$, finite, subdirectly irreducible $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ for which there is no common extension.

From Corollary 1 and Theorem 1 we at once obtain

COROLLARY 2. The following problem is solvable: for a given formula A, to determine whether Craig's theorem is true in the superintuitionistic logic generated by A.

From Theorems 1 and 2 we obtain

THEOREM 3. There exist precisely seven consistent superintuitionistic logics in which Craig's theorem is true. These logics can be axiomatized by the formulas:

1) $(x \supset x)$;
2) $(\neg x \vee \neg\neg x)$;
3) $(x \vee (x \supset (y \vee \neg y)))$;
4) $(x \vee (x \supset (y \vee \neg y))), ((x \supset y) \vee (y \supset x) \vee (x \equiv \neg y))$;
5) $(x \vee (x \supset (y \vee \neg y)))$; $(\neg x \vee \neg\neg x)$;
6) $((x \supset y) \vee (y \supset x))$;
7) $(x \vee \neg x)$.

Note that the validity of Craig's theorem in the logics 1), 2), and 7) follows from [23, 12, 9].

Remark 2. The proof of Lemma 2 and Remark 1 enable us to construct, for any superintuitionistic logic L in which Craig's theorem is false, an effective counterexample to this theorem. Namely, we can effectively find formulas A and B such that $(A \supset B) \in L$ and show that there exists no intermediate formula C.

V. Positive Logics

The methods developed in this paper enable us to prove Craig's theorem for certain fragments and extensions (by the addition of new connectives) of superintuitionistic logics and to prove the amalgamation property of the corresponding classes of algebras. Consider the positive logics intermediate between J^+ and K^+, the positive fragments of intuitionistic logic J and classical logic K, respectively. Formulas are constructed by means of the connectives $\&, \vee, \supset, 1$. To these logics there correspond varieties of implicative lattices $\mathfrak{A} = \langle A; \&, \vee, \supset, 1 \rangle$, the definition of which can be obtained from that of a PBA by eliminating any mention of the zero 0 and negation $\neg$.

It can be shown that there exist precisely four amalgamable varieties of implicative and, correspondingly, three consistent positive logics with CIT containing J^+, namely J^+, K^+, and the positive fragment LC^+ of Dummett's logic $LC = [(x \supset y) \vee (y \supset x)]$ [11].

Consequently, for positive formulas, from $(A \supset B) \in L$, where L is $J, KC = [\neg x \vee \neg\neg x]$, LC, or K, there follows the existence of a positive interpolated formula C (recall that $KC^+ = J^+$ [7]). In the other three superintuitionistic logics in which Craig's theorem is true, the formula C need not be positive. Consider, for example, the formula

$$D(x, y, z) \leftrightharpoons (A(x, z) \supset B(x, y)),$$

$$\text{where } A(x, z) = (z \supset x) \& ((x \supset z) \supset z),$$
$$B(x, y) = ((x \supset y) \& ((y \supset x) \supset x)) \supset y.$$

Let L be any of the logics 3), 4), or 5) of Theorem 3. It can be shown that $D(x, y, z) \in [x \vee (x \supset (y \vee \neg y))] \subseteq L$. Assume there exists a positive formula $C(x)$ such that $(A(x, z) \supset C(x)) \in L$ and $(C(x) \supset B(x, y)) \in L$. Then $L_3 \vDash (A(x, z) \supset C(x)) = 1$ and $L_3 \vDash (C(x) \supset B(x, y)) = 1$, where $L_3 = \{0, a, 1\}$.

Putting $z=0$, $x=a$ in L_3, we obtain $A(a,0)\supset C(a)=1\supset C(a)=1$ and $C(a)=1$. Putting $x=0$, $y=a$, we obtain $C(0)\supset B(0,a)=C(0)\supset a=1$ and $C(0)\leq a$. Since $C(x)$ does not contain $\neg$, we have $C(a)=1$ and $C(0)\leq a$ in L_3^+, where L_3^+ is obtained from L_3 by eliminating the operation $\neg$. However, the sets $\{a,1\}$ and $\{0,1\}$ are isomorphic subalgebras of L_3^+ under the isomorphism α, where $\alpha(a)=0$, $\alpha(1)=1$. Therefore $1=\alpha C(a)=C\alpha(a)=C(0)\leq a$. Contradiction. Thus, for given A and B an interpolated formula C, which exists by Craig's theorem, must contain $\neg$. Here we can take $C(x)=\neg\neg x$.

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