DERIVATIONS OF MONOCOMPOSITION ALGEBRAS

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UDC 519.48

In this paper we establish a connection between the derivations of an arbitrary finite-dimensional nondegenerate monocomposition algebra $\mathfrak{A}=\Phi 1\oplus A$ with unity 1 and the derivations of its associated KM-algebra $A=\langle A, x\times y, f(x,y)\rangle$. Namely, we prove

<u>THEOREM 1.</u> An endomorphism $\mathcal{D}$ of the vector space $\mathfrak{A}$ is a derivation of the algebra $\mathfrak{A}$ if and only if $1\mathcal{D}=0$, $A\mathcal{D}\subseteq A$, and

$$(x\times y)\mathcal{D} = x\mathcal{D}\times y + x\times y\mathcal{D},$$
$$f(x\mathcal{D},y)+f(x,y\mathcal{D})=0$$

for all $x,y\in A$.

This theorem is then used to obtain a description of the Lie derivation algebra $\mathcal{D}er\,\mathfrak{A}$ of the algebra $\mathfrak{A}$ when the finite-dimensional nondegenerate monocomposition algebra $\mathfrak{A}=\Phi 1\oplus A$ decomposes into an orthogonal sum of algebras $\mathfrak{A}=\mathfrak{A}_1\perp\ldots\perp\mathfrak{A}_n$ and, in addition, $A\times A=A$ (Theorem 3).

Suppose $\mathfrak{A}=\langle\mathfrak{A}, x\cdot y\rangle$ is an arbitrary algebra with unity 1 over a field Φ of characteristic $\neq 2$. Then it can be represented in the form

$$\mathfrak{A}=\Phi 1\oplus A, \tag{1}$$

where A - is some subspace complementary to $\Phi 1$. The decomposition (1) induces on the space* A the structure of a linear algebra $A=\langle A, x\times y, f(x,y)\rangle$ with bilinear form $f(x,y)$:

$$x\cdot y=f(x,y)1+x\times y,\quad x\times y\in A, \tag{2}$$

for all $x,y\in A$.

Now let $\mathcal{D}$ be a derivation of the algebra $\mathfrak{A}$. Then $1\mathcal{D}=0$ and, for all $x\in A$,

$$x\mathcal{D}=\ell(x)1+xP, \tag{3}$$

where $\ell(x)$ is a linear form on the space A and P is an endomorphism of the space A.

<u>Proposition 1.</u> An endomorphism $\mathcal{D}$ of the space $\mathfrak{A}$ is a derivation of the algebra $\mathfrak{A}$ if and only if $1\mathcal{D}=0$ and the $\ell(x)$ and P in (3) satisfy the relations

$$\ell(x\times y)=f(xP,y)+f(x,yP), \tag{4}$$
$$(x\times y)P=xP\times y+x\times yP+\ell(x)y+\ell(y)x \tag{5}$$

*By "space" we always mean a vector space over the field Φ.

Translated from Algebra i Logika, Vol. 16, No. 6, pp. 629-636, November-December, 1977. Original article submitted April 20, 1977.

0002-5232/77/1606-0417$07.50

for all $x, y \in A$.

Proof. Suppose $\mathcal{D}$ is a derivation of the algebra $\mathfrak{A}$ and x, y are arbitrary elements of A. Then

$$(xy)\mathcal{D} = x\mathcal{D} \cdot y + x \cdot y\mathcal{D}. \tag{6}$$

In view of (2) and (3), we have

$$\begin{aligned}&(xy)\mathcal{D} = [f(x,y)1 + x \times y]\mathcal{D} = \ell(x \times y)1 + (x \times y)P;\\ &x\mathcal{D} \cdot y + x \cdot y\mathcal{D} = [\ell(x)1 + xP] \cdot y + x \cdot [\ell(x)1 + yP] =\\ &= [f(xP,y) + f(x,yP)]1 + [xP \times y + x \times yP + \ell(x)y + \ell(y)x].\end{aligned}$$

From these two equalities and (6) we obtain the desired equalities (4) and (5).

The proof of the converse is left to the reader.

A derivation $\mathcal{D}$ of the algebra $\mathfrak{A}$ assumes a particularly simple and convenient form when the linear form $\ell(x)$ is zero. Then $P = \mathcal{D}$ on the space A, and the equalities (4) and (5) become

$$f(x\mathcal{D}, y) + f(x, y\mathcal{D}) = 0, \tag{7}$$

$$(x \times y)\mathcal{D} = x\mathcal{D} \times y + x \times y\mathcal{D}. \tag{8}$$

Therefore, the mapping $\mathcal{D}: A \to A$ is a derivation of the algebra A and, in addition, is a skew-symmetric linear transformation of the space A relative to the bilinear form $f(x,y)$. Such derivations of the algebra $\mathfrak{A}$ are called skew-symmetric. It follows immediately from (5) that if $\mathfrak{A}$ is a quadratic algebra with unity 1 and A is its associated anticommutative algebra, then all derivations $\mathcal{D}$ of the algebra $\mathfrak{A}$ are skew-symmetric.

Problem. Are all derivations of a nondegenerate monocomposition algebra with unity skew-symmetric?

It is easy to see that if $\mathcal{D}$ is a derivation of a monocomposition algebra $\mathfrak{A}$ with 1, then $\mathcal{D}$ is also a derivation of the associated algebra $\mathfrak{A}^+$, which is again a monocomposition algebra. If $\mathcal{D}$ is a skew-symmetric derivation of the algebra $\mathfrak{A}^+$, then it is also a skew-symmetric derivation of the algebra $\mathfrak{A}$. Therefore, an affirmative answer to the above question for all commutative nondegenerate monocomposition algebras with unity will imply an affirmative answer to the question in general.

Proposition 2. Suppose $\mathfrak{A} = \Phi 1 \oplus A$ is a commutative monocomposition algebra with unity 1 and $A = \langle A, x \cdot y, f(x,y) \rangle$ is its associated KM-algebra. If $\mathcal{D}$ is a derivation of the algebra $\mathfrak{A}$, then the linear form $\ell(x)$ defined by means of (3) satisfies the following scalar identities:

$$\ell(x^3) - 2\ell(x)f(x,x) = 0, \tag{9}$$

$$\ell(x^2 \cdot xy) = 0 \tag{10}$$

for all $x, y \in A$.

Proof. By definition of commutative KM-algebra (see [1]), the algebra A satisfies the following scalar identities:

$$f(x^2, x) - f(xy, z) + f(yz, x) + f(zx, y) = 0, \tag{11}$$

$$f(x^2, x^2) - f(xy, zt) + f(yz, xt) + f(zx, yt) = 0. \tag{12}$$

Here and from now on, x, y, z, t stand for arbitrary elements of A.

Using these scalar identities and equalities (4) and (5), we obtain

$$\ell(x^3) = f((xx)P, x) + f(xx, xP) = 2f(xP\cdot x, x) + 2\ell(x)f(x,x) + f(xx, xP) - 2\ell(x)f(x,x).$$

Equality (9) is proved. Also,

$$\ell(x^2\cdot xy) = f((xx)P, xy) + f(xx, (xy)P) =$$

$$= 2f(xP\cdot x, xy) + 2\ell(x)f(x, xy) + f(xx, xP\cdot y) + f(xx, x\cdot yP) + \ell(x)f(xx, y) + \ell(y)f(xx, x) = 0.$$

Thus, equality (10) and Proposition 2 are proved.

Let us turn to the proof of Theorem 1, which was stated in the introduction. In view of what has been said above, it is a consequence of the following theorem, which we will prove

THEOREM 2. Suppose $A = \langle A, x\cdot y, f(x,y)\rangle$ is a finite-dimensional commutative nondegenerate KM-algebra. If the linear form $\ell(x)$ on the algebra A satisfies the scalar identities (9) and (10), then it is zero.

Proof. Suppose the linear form $\ell(x)$ satisfies (9) and (10). Since the symmetric bilinear form $f(x,y)$ is nondegenerate and $\dim A < \infty$, there exists an element $a \in A$ such that $\ell(x) = f(x, a)$ for all $x \in A$.

Using this equality, we rewrite the scalar identities (9) and (10) as follows:

$$T(x,x,x) \equiv f(x^3, a) - 2f(x,a)f(x,x) = 0, \tag{13}$$

$$S(x,x,x,y) \equiv f(x^2\cdot xy, a) = 0. \tag{14}$$

Linearization of these identities yields scalar identities (15) and (16), which are also satisfied in A:

$$T(x,y,z) \equiv f(xyz + yzx + zxy, a) - 2f(x,a)f(y,z) - 2f(y,a)f(z,x) - 2f(z,a)f(x,y) = 0, \tag{15}$$

$$S(x,y,z,t) \equiv f(xy\cdot zt + yz\cdot xt + zx\cdot yt, a) = 0. \tag{16}$$

LEMMA 1.

$$a^3 = 0. \tag{17}$$

Proof. From (13) we obtain

$$T(a,a,a)=f(a^3,a)-2[f(a,a)]^2=-2[f(a,a)]^2=0,$$

hence

$$f(a,a)=0. \tag{18}$$

Now $T(a,a,x)=0$ implies $f(a^3,x)=0$. Since $f(x,y)$ is a nondegenerate form, we obtain the desired equality (17).

LEMMA 2.

$$a^2=0. \tag{19}$$

Proof. The equality $T(x,y,a)=0$ and identities (11) and (12) yield

$$f(xaa,y)+f(yaa,x)+f(xa,ya)=-4f(x,a)f(y,a). \tag{20}$$

The equality $S(x,y,a,a)=0$ yields

$$f(xaa,ya)=f(yaa,xa)=0. \tag{21}$$

If in (20) we replace x by xa, then, in view of (21), we obtain

$$f(xaaa,y)=2f(x,a^2)f(y,a).$$

Since $f(x,y)$ is a nondegenerate form, it follows that

$$xaaa=2f(x,a^2)a. \tag{22}$$

By the product $x_1x_2\dots x_n$ we will always mean the product with right-normed arrangement of parentheses, i.e.,

$$(\dots(((x_1x_2)x_3)x_4)\dots)x_n.$$

If in (22) we replace x by xa, we obtain

$$xaaaa=0.$$

Multiplying (22) on the right by a , we have, in view of the last equality, $f(x,a^2)a^2=0$, from which the desired equality (19) follows.

The lemma is proved.

LEMMA 3.

$$f(x,a)=0. \tag{23}$$

Proof. From (19) and (22) we obtain

$$xaaa=0. \tag{24}$$

Equality (20) with $y=x$ yields

$$f(xaa,x)=-2[f(x,a)]^2,$$

from which, after the substitution $x\longrightarrow xax$, we obtain

$$f(xaxaa,xax)=-8[f(x,a)]^4. \tag{25}$$

The equality $T(xa, xa, y) = 0$ yields $f((xa \cdot xa)a; y) = 0$, hence

$$(xa \cdot xa)a = 0. \tag{26}$$

In view of (26), $S(xa, xa, xa, x) = 0$ implies

$$f(xaxa \cdot xa, xa) = 0. \tag{27}$$

From $S(xaxa, x, x, a) = 0$ we obtain

$$f(xaxax, xaa) = 0. \tag{28}$$

If in (12) we make the substitution $x \to xaxa$, $y \to xa$, $z \to x$, $t \to a$, we obtain

$$f(xaxa \cdot xa, xa) + f(xaxaa, xax) + f(xaxax, xaa) = 0.$$

In view of (25), (27), and (28), it follows that $-8[f(x,a)]^4 = 0$, hence $f(x,a) = 0$.

The lemma is proved.

Since $\ell(x) = f(x, a)$ for any $x \in A$, it follows that $\ell(x)$ is the zero form.

Theorem 2 is proved.

By analogy with the concept of orthogonal sum of quadratic algebras, which was introduced by Becker [2], we define the orthogonal sum $\mathfrak{A}_1 \perp \mathfrak{A}_2 \perp \ldots \perp \mathfrak{A}_n$ of monocomposition algebras $\mathfrak{A}_i = [A_i, f_i, e_i]$ with unities e_i $(i = 1, 2, \ldots, n)$, where $[A_i, f_i]$ is the KM-algebra associated with the monocomposition algebra $\mathfrak{A}_i$. Suppose $A = A_1 \oplus \ldots \oplus A_n$ is the direct sum of the algebra $A_1, \ldots, A_n$; $f = f_1 \perp f_2 \perp \ldots \perp f_n$ is the orthogonal sum of the bilinear forms $f_i(x_i, y_i)$, i.e.,

$$f(x_1 + \ldots + x_n, y_1 + \ldots + y_n) = f_1(x_1, y_1) + \ldots + f_n(x_n, y_n)$$

for any $x_i, y_i \in A_i$ $(i = 1, \ldots, n)$. Then $[A, f]$ is also a KM-algebra, and we call its associated monocomposition algebra $\mathfrak{A} = [A, f, e]$ with unity e the orthogonal sum of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and denote it by $\mathfrak{A} = \mathfrak{A}_1 \perp \ldots \perp \mathfrak{A}_n$.

<u>THEOREM 3.</u> Suppose $\mathfrak{A} = [A, f, e]$ is a finite-dimensional nondegenerate monocomposition algebra with unity e, $\mathcal{D}er\,\mathfrak{A}$ is the Lie derivation algebra of $\mathfrak{A}$, and $A \times A = A$. If

$$\mathfrak{A} = \mathfrak{A}_1 \perp \ldots \perp \mathfrak{A}_n, \tag{29}$$

then

$$\mathcal{D}er\,\mathfrak{A} = \Delta_1 \oplus \ldots \oplus \Delta_n \tag{30}$$

is a direct sum of ideals Δ_i, where each ideal Δ_i is isomorphic to the Lie algebra Л and $\mathcal{D}er\,\mathfrak{A}_i$ $(i = 1, 2, \ldots, n)$.

<u>Proof.</u> It follows from (29) that $A = A_1 \oplus \ldots \oplus A_n$, $A_i \lhd A$ $(i = 1, \ldots, n)$. Since $A \times A = A$, we have $A_i \times A_i = A_i$ $(i = 1, \ldots, n)$. This, as is well known [3, Exercise 19], implies the algebra isomorphism

$$\mathcal{D}er\,A \cong \mathcal{D}er\,A_1 \oplus \ldots \oplus \mathcal{D}er\,A_n.$$

From this relation and Theorem 1 it is easy to obtain the desired decomposition (30).

The theorem is proved.

LITERATURE CITED

1. A. T. Gainov, "Subalgebras of nondegenerate commutative KM-algebras," Algebra Logika, 15, No. 4, 371-383 (1976).
2. E. Becker, "Halbeinfache quadratische Algebren und antikommutative Algebren mit assoziativen Bilinearformen," Abh. Math. Sem. Univ. Hamb., 36, 229-256 (1971).
3. N. Jacobson, Lie Algebras, Interscience, New York–London (1962).

ENUMERATION OF THE CLASS $\mathcal{C}_{20}^*$

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UDC 517.01

The main problem of the theory of enumerations (see [1] for all concepts not defined here) is that of finding a "regular" enumeration for one or another class of objects. The class $\mathcal{C}_{20}^*$ of enumerated sets is very useful for defining the concept of a computable functional; the enumerated sets in $\mathcal{C}_{20}^*$ are also an effective version of the concept of a complete f_0-space [2]. However, the whole class (category) $\mathcal{C}_{20}^*$ is too large (it is not even a set) to be able to look for a suitable enumeration of it. Therefore, the correct approach is to look for some countable (concrete) subcategory equivalent to the whole category $\mathcal{C}_{20}^*$ and then an enumeration of this subcategory. In the present note this will be done. We will define a family K of enumerated sets in $\mathcal{C}_{20}^*$ such that any enumerated set in $\mathcal{C}_{20}^*$ is equivalent (even effectively in some sense) to some enumerated set in K. We will define an enumeration τ of this family such that the category operation $\times$ and $\mathfrak{Mor}$ will be morphisms of the enumerated set $\tilde{\mathcal{K}} \times \tilde{\mathcal{K}}$ into $\tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}} \rightleftharpoons (K, \tau)$; moreover, we will show that the enumerated set $\tilde{\mathcal{K}}$ itself belongs to the class $\mathcal{C}_{20}^*$.

Any enumerated set γ in $\mathcal{C}_{20}^*$ is uniquely determined [1] by its approximation γ_0 and the order $\leqslant$ induced on S_0 by the order $\leqslant_\gamma$. The pair $\langle \gamma_0, \leqslant \rangle$ in this case is a constructive sail. This means that: a) the partially ordered set $\langle S_0, \leqslant \rangle$ is a sail, i.e., for any two compatible elements a, b (i.e., elements for which there exists c such that $a \leqslant c$ and $b \leqslant c$) there exists their least upper bound $a \cup^* b$; b) the order $\leqslant$ is partial recursive on γ_0, i.e., the set $\{\langle x, y \rangle \mid \nu_0 x \leqslant \nu_0 y\}$ is recursively enumerable; c) the predicate of compatibility $R \rightleftharpoons \{\langle x, y \rangle \mid \nu_0 x$ and $\nu_0 y$ are compatible$\}$ is recursive; d) there exists a 2-place partial recursive function G such that $\delta G = R$ and, for $\langle x, y \rangle \in R$, $\nu_0 x \cup^* \nu_0 y = \nu_0 G(x, y)$. A constructive sail will be denoted as follows: $\mathcal{P} = \langle \gamma_0, \leqslant, G, R \rangle$. Membership of γ in $\mathcal{C}_{20}^*$ also means that $\langle S_0, \leqslant \rangle$ has a smallest element. We will assume without loss of generality that this element is $\nu_0 0$.

Two constructive sails $\mathcal{P}_0 = \langle \gamma_0, \leqslant_0, G_0, R_0 \rangle$ and $\mathcal{P}_1 = \langle \gamma_1, \leqslant_1, G_1, R_1 \rangle$ are called equivalent if there exists a morphism $\mu : \gamma_0 \to \gamma_1$ such that μ is an isomorphism of the partially ordered sets $\langle S_0, \leqslant_0 \rangle$ and $\langle S_1, \leqslant_1 \rangle$. If $f \in \mathcal{O}$ is such that $\mu \nu_0 = \nu_1 f$, then for any $x, y \in N$:

Translated from Algebra i Logika, Vol. 16, No. 6, pp. 637-642, November-December, 1977. Original article submitted October 27, 1977.

 0002-5232/77/1606-0422$07.50