COMPUTABLE SINGLE-VALUED NUMERATIONS

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The problems of the existence and number of inequivalent univalent computable numerations of families of recursively enumerable sets have attracted the interest of many workers [3, 5, 6-10, etc.].

One of the interesting results along these lines was obtained by Marchenkov [3]: Every computable family of recursive functions has up to equivalence either one or a countable number of univalent computable numerations. So far, all known samples of families of recursively enumerable sets have also had this property. However, in this paper we give an example of a family of recursively enumerable sets which has exactly two inequivalent univalent computable numerations. In this connection the possible number of minimal numerations of families of recursively enumerable sets is of interest.

We now turn to the main results of this paper. We follow the notation and definitions in [1, 2, 4]. First we recall some of the definitions we will need. A numeration $\nu: N \to S$, where S is a family of recursively enumerable sets, is said to be computable if the set $\{\langle n, m\rangle \mid n \in \nu(m)\}$ is recursively enumerable; the numeration is called univalent if $\nu(n) \neq \nu(m)$ for all $n \neq m$. Here and below, N is the set of positive integers $\{0, 1, 2, \ldots\}$. We recall that if ν is a computable numeration of a family S of recursively enumerable sets, then there exists a partial recursive function $f(n, x)$ such that $\nu(n) = \{f(n, x) \mid x \in N\}$. Let K^2 and K^3 [2] be the Kleene universal functions for the families of one- and two-place partial-recursive functions, respectively. For brevity we write simply K in place of K^2. We denote by c, ℓ, r [2] the Cantor functions which numerate pairs of numbers. If $f(x_0, \ldots, x_n)$ is a partial-recursive function then we write $f_t(x_0, \ldots, x_n)$ for the value $f(x_0, \ldots, x_n)$, if it is computed in less than t steps, and $f_t(x_0, \ldots, x_n)$ is not defined otherwise.

We define

$$\gamma_j(n) \rightleftharpoons \{K^3(j, n, x) \mid x \in N\} \text{ and } \gamma_j^t(n) = \{K_t^3(j, n, x) \mid x \leqslant t\}.$$

It is easy to see that for every computable numeration ν of some family S of recursively enumerable sets, there exists a j such that $\nu = \gamma_j$. The numeration ν reduces to a numeration μ $(\nu \leqslant \mu)$ if there exists a recursive function f such that $\nu(n) = \mu f(n)$. Two numerations ν and μ are called equivalent if $\nu \leqslant \mu$ and $\mu \leqslant \nu$.

MAIN THEOREM. There exists a computable family of recursively enumerable sets having precisely two inequivalent univalent computable numerations.

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Proof. The desired set S will be constructed by the priority method. Simultaneous with the construction of S we will construct two univalent inequivalent numerations ν and μ.

At the t-th stage of the construction, we will define finite pieces $\nu^t(n)$ and $\mu^t(n)$ of sets in S so that

$$\{\bigcup_{t\geq 0}\nu^t(n) \mid n\in N\} = \{\bigcup_{t\geq 0}\mu^t(n) \mid n\in N\} = S$$

and $\nu(n) = \bigcup_{t\geq 0}\nu^t(n)$, $\mu(n) = \bigcup_{t\geq 0}\mu^t(n)$. In the construction we will also need some auxiliary constructions. Thus, at step t we will define functions $\varphi^t : N \to N$, $\lambda n \kappa_n^t : N \to N$, partial functions $\boxed{\varkappa}_{ji}^t$ for $\varkappa \in \{\nu, \mu\}$, values Δ_n^i, π_n^i for $i \leq \kappa_n^t$ and $r_\varkappa^t(j,i)$, $s_\varkappa^t(j,i)$, $\ell_\varkappa^t(j,i)$, $d_\varkappa^t(j,i)$ for $\varkappa \in \{\nu,\mu\}$, which take either values in N or an undetermined value; we also construct $p(t,j,i)$, finite sets $\Pi_n^i(t)$ for $i \leq \kappa_n^t$ and $D\boxed{\varkappa}_{j,i}^t$ for $j,i \in N$. In the construction we will use two kinds of pairs: $\langle n,i\rangle$, where $n,i \in N$, and $[j,i]$, where $j,i \in N$, which we will distinguish by the form of brackets used; we consider the lexicographic order $\leq_{lex}$ on pairs. We will also place markers Z_j, $\boxed{+}$, $\boxed{-}$, $\boxed{\nu}$, $\boxed{\mu}$. The markers $\boxed{-}$, $\boxed{\nu}$, and $\boxed{\mu}$ are placed on pairs $[j,i]$, the marker $\boxed{+}$ on pairs $\langle n,i\rangle$, and Z_j is placed on four-tuples $\langle \varkappa, \ell, \ell_1, \ell_2\rangle$, where $\ell, \ell_1, \ell_2, n, i, j \in N$ and $\varkappa \in \{\nu, \mu\}$.

We write

$$M_\varkappa^t(j,i) \rightleftharpoons \{\ell_\varkappa^t(j,i),\ s_\varkappa^t(j,i),\ r_\varkappa^t(j,i),\ d_\varkappa^t(j,i)\}$$

and say that the function $\boxed{\varkappa}_{j,i}^t$ is completely defined on n if $n \in \delta\boxed{\varkappa}_{j,i}^t$ and $\varkappa^t(n) \subseteq \gamma_j^{t+1}(\boxed{\varkappa}_{j,i}^t(n))$, or

$$n \in \bigcup_{t'\leq t} M_\varkappa^{t'}(j',i') \cup \bigcup_{i'\leq \kappa_p^t}\{\Delta_p^{i'}, \pi_p^{i'}\}$$

for $[j',i'] <_{lex} [j,i]$ and $p \leq j$; and the function $\boxed{\varkappa}_{j,i}^t$ is completely defined on a set $D \subseteq N$ if it is completely defined on n for all $n \in D$.

If the marker $\boxed{\varkappa}$ is present on $[j,i]$ at step t then we write

$$X_{j,i}^t = \delta\boxed{\varkappa}_{j,i}^t \cup D\boxed{\varkappa}_{j,i}^t$$

By the $\langle \nu, m, i\rangle$-list at step t for $i \leq \kappa_m^t$, we mean the linearly ordered set $\mathbb{L}_{\nu,m,i}^t = \langle L_{\nu,m,i}^t, \preccurlyeq\rangle$, where

$$L_{\nu,m,i}^t = \{\Delta_m^i, \pi_m^i\} \cup \{\ell_\nu^t(j',i'),\ s_\nu^t(j',i'),\ r_\nu^t(j',i') \mid [j',i'] \in \Pi_m^i(t)\}$$

and we define the order $\preccurlyeq$ on this set by putting $a \prec b$ for $a, b \in L_{\nu,m,i}^t$, provided one of the following cases occurs:

1) $a = \ell_\nu^t(j',i')$ & $b = s_\nu^t(j',i')$;
2) $a = \ell_\nu^t(j',i')$ & $b = r_\nu^t(j',i')$;
3) $a = s_\nu^t(j',i')$ & $b = r_\nu^t(j',i')$;

4) $a = \Delta^i_m$ & $b = \mathcal{T}^i_m$;

5) ($a = \Delta^i_m \vee a = \mathcal{T}^i_m \vee (a \in M^t_i(j'', i'')$ & the marker $\boxed{\nu}$)) appears on $[j'', i'']$ in $\Pi^i_m(t)$) & ($b \in M^t_\nu(j', i')$ & the marker $\boxed{\mu}$) appears on $[j', i']$ in $\Pi^i_m(t)$);

6) ($a \in M^t_\nu(j'', i'')$ & the marker $\boxed{\nu}$ appears on $[j'', i'']$ & $[j'', i''] \in \Pi^i_m(t)$) & $b \in \{\Delta^i_m, \mathcal{T}^i_m\}$;

7) $a \in M^t_\nu(j'', i'')$ & $b \in M^t_\nu(j', i')$ & (the marker $\boxed{\nu}$ appears on $[j', i']$ and $[j'', i'']$ in $\Pi^i_m(t)$) & $j'' < j'$;

8) $a \in M^t_\nu(j'', i'')$ & $b \in M^t_\nu(j', i')$ & (the marker $\boxed{\mu}$ appears on $[j', i']$ and $[j'', i'']$ in $\Pi^i_m(t)$ & $j' < j''$

By the $\langle \mu, m, i \rangle$-list at step t for $i \leqslant \kappa^t_m$ we mean the linearly ordered set $\mathbb{L}^t_{\mu,m,i} = \langle L^t_{\mu,m,i}, \prec \rangle$, where $L^t_{\mu,m,i} = \varphi^t(L^t_{\nu,m,i})$, and $a \prec b$ if there exist a' and b' such that $\varphi^t(a') = a$, $\varphi^t(b') = b$ and $a' \prec b'$ in $\mathbb{L}^t_{\nu,m,i}$.

By the $\langle \varkappa, m, i, j', i' \rangle$-list at step t we mean the submodel $\mathbb{L}^t_{\langle \varkappa,m,i,j',i' \rangle}$ of the model $\mathbb{L}^t_{\langle \varkappa,m,i \rangle}$ with base set consisting of the elements x in $L^t_{\langle \varkappa,m,i \rangle}$ such that $x = \Delta^i_m$ or $x = \mathcal{T}^i_m$ or $x \in M^t_\varkappa(j'', i'')$, where $[j'', i''] \in \Pi^i_m(t)$ and $[j', i'] \leqslant_{lex} [j'', i'']$. We denote μ and ν, respectively, by $\hat{\nu}$ and $\hat{\mu}$.

At step $t+1$ we say that we leave the following unchanged:

1) at the point n, the function $\lambda n \kappa^{t+1}_n$ is unchanged if we put $\kappa^{t+1}_n = \kappa^t_n$;

2) at the point n, the numeration $\varkappa$ is unchanged if we put $\varkappa^{t+1}(n) = \varkappa^t(n)$;

3) the set $\Pi^i_n(t+1)$, if we put $\Pi^i_n(t+1) = \Pi^i_n(t)$;

4) the value $p(t+1, j, i)$, if we put $p(t+1, j, i) = p(t, j, i)$;

5) the value $f^{t+1}_\varkappa(j, i)$, if we put $f^{t+1}_\varkappa(j, i) = f^t_\varkappa(j, i)$, where $\varkappa \in \{\nu, \mu\}$;

6) the set $D\boxed{\varkappa}^{t+1}_{j,i}$, if we put $D\boxed{\varkappa}^{t+1}_{j,i} = D\boxed{\varkappa}^t_{j,i}$;

7) at the point n, the function φ^{t+1} is unchanged if $\varphi^{t+1}(n) = \varphi^t(n)$;

8) the marker is unchanged if it is neither inserted nor removed at step $t+1$.

We write δf for the domain of the function f, and ρf is the range of f.

In the construction, six types of steps will be used: the zero step, and types $5t+1$, $5t+2$, $5t+3$, $5t+4$ and $5t+5$.

We say that the $\varkappa$-number n is used in the construction at step t if

$$\varkappa^t(n) \neq \varkappa^{t-1}(n) \vee n \in \delta\boxed{\varkappa}^t_{j,i} \cup D\boxed{\varkappa}^t_{j,i} \cup \{\Delta^i_m \mid m \in N$$

$$\text{and } i \leqslant \kappa^t_m\} \cup \{\mathcal{T}^i_m \mid m \in N \text{ and } i \leqslant \kappa^t_m\} \cup M^t_\varkappa(j,i)$$

for $j, i \in N$.

n is used at step t as an index if it is used as a ν- or μ-index.

A pair $\langle m, i \rangle$ is used at step t if we perform a construction of type $5t+2$ (Case 2 or 3) or $5t+1$ for $\langle m, i \rangle$.

The pair $[j, i]$ is used at step t if it is added to or removed from some set $\Pi_m^{i_*}$, or else we carry out a construction of type $5t+2$, $5t+3$, $5t+4$, or $5t+5$, for the pair $[j, i]$, or the marker $\boxminus$ is attached to $[j, i]$.

The pair $[j, i]$ is said to be defined at step t if the value $S_x^t(j, i)$ is defined.

Before giving the construction, we informally give some idea of the construction of the desired set.

In order to construct our family S , we must constantly bear in mind the following three properties during the construction of S and the two numerations ν and μ :

1) ν and μ numerate in a univalent way the same family S ;

2) the numerations ν and μ are inequivalent;

3) for every univalent computable numeration ξ of the family S , either $\nu \leq \xi$, or $\mu \leq \xi$.

Simultaneous fulfillment of these three properties is made difficult because they are not all that compatible. The main difficulty is to get them to be satisfied together.

If we wanted to limit ourselves so that only property 1) holds, one step would suffice. By defining $\nu(n) = \mu(n) = \{2n\}$, we get fulfillment of condition 1). If we also want to satisfy property 2), we must define ν and μ in steps. At the zeroth step we put $\nu^0(n) = \mu^0(n) = \{2n\}$, and then during the construction we spoil the reducibility of ν to μ by means of the function $\lambda x K(n, x)$ for every n . To this end it suffices to find a sequence $\Delta_0 < \mathfrak{T}_0 < \Delta_1 < \ldots < \Delta_n < \mathfrak{T}_n < \ldots$ of ν -indices and arrange that on the pair of ν -indices Δ_n and $\mathfrak{T}_n$, the function $\lambda x K(n, x)$ does not reduce ν to μ . Therefore, it is enough to wait until $K(n, \Delta_n)$ and $K(n, \mathfrak{T}_n)$ have been defined. Then if $K(n, \Delta_n) \neq \Delta_n$ or $K(n, \mathfrak{T}_n) \neq \mathfrak{T}_n$ the function $\lambda x K(n, x)$ is easily seen to be nonreducing. If $K(n, \Delta_n) = \Delta_n$ and $K(n, \mathfrak{T}_n) = \mathfrak{T}_n$, however, we can make the following correction:

$$\mu(\mathfrak{T}_n) = \nu(\Delta_n) \Leftrightarrow \nu^0(\Delta_n) \cup \nu^0(\mathfrak{T}_n) \cup \{1\},$$
$$\mu(\Delta_n) = \nu(\mathfrak{T}_n) \Leftrightarrow \nu^0(\Delta_n) \cup \nu^0(\mathfrak{T}_n) \cup \{3\}.$$

It is clear that if we don't do anything more, the numerations ν and μ will be inequivalent univalent numerations numerating the same family.

No additional constructions are necessary to get conditions 1) and 3) to hold simultaneously. It suffices to take $\lambda n \gamma_n$ to be a computable numeration of all computable numerations of families of recursively enumerable sets and, defining the numerations by $\nu(n) = \mu(n) = \{2n\}$, a reducing function $\boxed{\nu}_n$ for reducing ν to γ_n can be constructed as follows. We wait until a step t such that $\nu(m) \subseteq \gamma_n^t(d_m)$ for some $d_m \in N$, and we define $\boxed{\nu}_n(m) = d_m$. It is clear that when the function γ_n numerates the same set as ν , $\boxed{\nu}_n$ will be reducing. Our problem is to combine the constructions for satisfying properties 1)

2), 1) and 3). However, as is seen from the construction, they contradict one another since after $\boxed{\nu}_n(m)$ has been defined for some n, $\nu(m)$ can no longer be changed after this step. In order to overcome this difficulty, we introduce a "stopping" function $\lambda t s_x^t(n)$, where $x \in \{\nu, \mu\}$, which so to speak absorbs all the changes within itself. This should be interpreted as follows: if reducibility by means of $\boxed{\nu}_n^t$ or $\boxed{\mu}_n^t$ (depending on what reduces to γ_n at step t) breaks down at some point, then it breaks down in a corresponding way on $s_\nu^t(n)$ or $s_\mu^t(n)$. In this case, if reducibility breaks down infinitely many times, we can arrange by choosing successive values of the "stopping" function that there exists a set in the numeration γ_n which is not present in our family.

We consider the simplest case showing what kind of effects a change of the set $\nu^t(\ell)$ has for ℓ on which some $\boxed{\nu}_n^t$ on $s_\nu^t(n)$ is defined. Take the situation in which only a single function $\lambda x K(m,x)$ keeps us from preventing reducibility of ν to μ by means of $\boxed{\nu}_n^t$. Then we find a step t' such that $K^{t'}(n,\Delta_n) = \Delta_n$ and

$$K^{t'}(n,\mathcal{F}_n) = \mathcal{F}_n,\ \nu^{t'}(\Delta_n) \subseteq \gamma_n^{t'}(a),\ \nu^{t'}(\mathcal{F}_n) \subseteq \gamma_n^{t'}(b),\ \nu^{t'}(s_\nu^t(n)) \subseteq \gamma_n^{t'}(c),$$

and at this step we define $\boxed{\nu}_n^{t'}(\Delta_n) = a$, $\boxed{\nu}_n^{t'}(\mathcal{F}_n) = b$, $\boxed{\nu}_n^{t'}(s_\nu^{t'}(n)) = c$.

After this, we do the following construction:

$$\begin{aligned}
\nu^{t'+1}(\mathcal{F}_n) &= \mu^{t'+1}(\varphi^{t'}(s_\nu^{t'}(n))) \rightleftharpoons \nu^{t'}(s_\nu^{t'}(n)) \cup \nu^{t'}(\mathcal{F}_n) \cup \{x\},\\
\mu^{t'+1}(\varphi^{t'}(\mathcal{F}_n)) &\rightleftharpoons \nu^{t'+1}(\Delta_n) \rightleftharpoons \nu^{t'}(\Delta_n) \cup \nu^{t'}(\mathcal{F}_n) \cup \{y\},\\
\mu^{t'+1}(\varphi^{t'}(\Delta_n)) &= \nu^{t'+1}(s_\nu^{t'}(n)) \rightleftharpoons \nu^{t'}(s_\nu^{t}(n)) \cup \nu^{t'}(\Delta_n) \cup \{z\},
\end{aligned}$$

where the numbers x, y, z are pairwise distinct and prior to this step are not contained in any set. If γ_n is a univalent computable numeration of the family which we construct, then

$$\begin{aligned}
&\gamma_n(a) \supseteq \nu^{t'+1}(\Delta_n) &&\text{or} && \gamma_n(a) \supseteq \nu^{t'+1}(s_\nu^t(n)),\\
&\gamma_n(c) \supseteq \nu^{t'+1}(s_\nu^{t'}(n)) &&\text{or} && \gamma_n(c) \supseteq \nu^{t'+1}(\mathcal{F}_n),\\
&\gamma_n(b) \supseteq \nu^{t'+1}(\mathcal{F}_n) &&\text{or} && \gamma_n(b) \supseteq \nu^{t'+1}(\Delta_n).
\end{aligned}$$

Thus, the univalence implies that breakdown of reducibility for a single point automatically implies breakdown at other points, and therefore on $s_\nu^t(n)$. In this case we replace the marker $\boxed{\nu}$ by $\boxed{\mu}$, and the value $s_\mu^t(n)$ is already "stopping."

We now say a few words about the objects which we define in our construction. The strongly computable numerations $\nu^t(n)$ and $\mu^t(n)$ in the limit give numerations $\nu(n) \rightleftharpoons \bigcup_{t\geq 0} \nu^t(n)$ and $\mu(n) \rightleftharpoons \bigcup_{t\geq 0} \mu^t(n)$ which are univalent numerations of the same family, but are inequivalent. The functions φ^t have the property of establishing an equivalence between ν^t and μ^t, and $\varphi = \lim_{t\to\infty} \varphi^t$ defines a reduction of ν and μ. Informally, the pair $[j,i]$ will correspond to the i-th attempt to reduce ν or μ to γ_j; correspondingly, a marker $\boxed{\nu}$ or $\boxed{\mu}$ on $[j,i]$ indicates that at a given moment we are reducing

ν (respectively, μ) to γ_j . The pair $\langle n,i\rangle$ will correspond to the i-th attempt to spoil reducibility of ν to μ via the function $\lambda x K(n,x)$. The value κ_n^t gives the number of attempts at step t to spoil reducibility of ν to μ by means of the function $\lambda x K(n,x)$. The pair Δ_n^i, π_n^i will indicate the ν -indices at which we want to spoil reducibility of ν to μ by $\lambda x K(n,x)$ in the i-th attempt. The functions $\boxed{\varkappa}_{j,i}^t$ will reduce the numeration $\varkappa$ to γ_j at step t in the i-th attempt. The function $\lambda t s_{\varkappa}^t(j,i)$ will be a "stopping function" for reducing $\varkappa$ to γ_j in the i-th attempt, and $d_{\varkappa}^t(j,i)$ will define the value in the numeration $\varkappa$ which corresponds to the set with index $s_{\hat{\varkappa}}^t(j,i)$ in the numeration $\hat{\varkappa}$. $\hat{\varkappa}$ is defined throughout as follows: $\hat{\nu} \leftrightharpoons \mu$ and $\hat{\mu} \leftrightharpoons \nu$. The functions $r_{\varkappa}^t(j,i)$ and $\ell_{\varkappa}^t(j,i)$ will define "adjacent" values. The set $\Pi_n^i(t)$ will "count" the attempts to reduce ν or μ to numerations $\gamma_j, j<n$ which are obstructions to preventing reducibility of ν to μ by means of $\lambda x K(n,x)$ at the i-th attempt. The function $p(t,j,i)$ will determine the degree of cycling in the definition of the "stopping function," in order that ν and μ should numerate the same family. The function $\lambda t \mathcal{D}\boxed{\varkappa}_{j,i}^t$ will define a counter for defining the function $\boxed{\varkappa}_{j,i}^t$. The marker $\boxed{+}$ is associated to the pair $\langle n,i\rangle$ and n whenever we have prevented reducibility of ν to μ by means of $\lambda x K(n,x)$ in some i-th attempt. The marker $\boxed{-}$ will be associated to a pair $[j,i]$ if the i-th attempt to reduce ν or μ to γ_j is unsuccessful, and we will not thereafter return to the marker. The marker χ_j is added in cases when we have learned either that γ_j is not a univalent numeration, or that it does not numerate the set we wish to construct. At steps of type $5t+1$ we will attempt to spoil reducibility of ν to μ by means of the function $\lambda x K(n,x)$. At steps of type $5t+2$ we will define a reduction of ν or μ to certain γ_j at points where reducibility is spoiled. At steps of type $5t+3$ we define a counter $\mathcal{D}\boxed{\varkappa}_{j,i}^t$; at steps of type $5t+4$ we extend the definition of $\boxed{\varkappa}_{j,i}^t$ to elements in $\mathcal{D}\boxed{\varkappa}_{j,i}^t$. At step $5t+5$ we introduce the 0 -th attempt to reduce ν to γ_t .

We now turn to the formal constructions.

<u>Step 0.</u> Define $\mu^0(n) = \nu^0(n) = \{2n\}$, $\varphi^0(n) = n$, $\Delta_n^0 = 4n$, $\pi_n^0 = 4n+1$, $\kappa_n^0 = 0$, $\Pi_n^0 = \emptyset$, $p(0,j,i) = 0$, $\boxed{\varkappa}_{j,i}^0$ is nowhere defined, $\mathcal{D}\boxed{\varkappa}_{j,i}^0 = \emptyset$, $r_{\varkappa}^0(0,0) = 2$, $s_{\varkappa}^0(0,0) = 6$, $\ell_{\varkappa}^0(0,0) = 10$ for all $n,j,i \in N$ and $\varkappa \in \{\nu, \mu\}$. We place a marker $\boxed{\nu}$ on all pairs $[j,i]$ where $j,i \in N$.

<u>Step $5t+1$.</u> We consider $T=5t$ and verify whether there exists an $n<T$ such that no marker $\boxed{+}$ is present at n and one of the following cases holds.

<u>Case 1.</u> The function $\lambda x K_T(n,x)$ is defined on Δ_n^κ and π_n^κ and $K(n,\Delta_n^\kappa) = \Delta_n^\kappa$, $K(n,\pi_n^\kappa) = \pi_n^\kappa$, $\Pi_n^\kappa(T) = \emptyset$, and neither Δ_n^κ, nor π_n^κ belongs to $\mathcal{D}\boxed{\varkappa}_{j,i}^T$ for $j<n$, if there is no $\boxed{-}$ on $[j,i]$, and neither Δ_n^κ nor π_n^κ are the second coordinate in a four-tuple to which a $\chi_{j'}$ is associated, where $j'<n$, and $\kappa = \kappa_n^T$.

<u>Case 2.</u> The function $\lambda x K_T(n,x)$ is defined on Δ_n^κ and π_n^κ , and $K(n,\Delta_n^\kappa) \neq \Delta_n^\kappa$ or $K(n,\pi_n^\kappa) \neq \pi_n^\kappa$, where $\kappa = \kappa_n^T$.

Case 3. Cases 1 and 2 do not hold, but there exists an $i \leq \kappa_n^T$ such that $\lambda x K_T(n,x)$ is defined on $\{\Delta_n^i, \pi_n^i\}$, $\Pi_n^i(T) = \{<j_0,i_0>, <j_1,i_1>, \ldots, <j_\ell, i_\ell>$ where $j_0 < j_1 < \ldots < j_\ell$, and the following three conditions hold:

a) for all $\delta \leq \ell$, if the marker $\boxed{x}$ is present on $[j_\delta, i_\delta]$ but there is no $\boxed{-}$ marker, then the function $\boxed{x}^T_{j_\delta i_\delta}$ is completely defined on the elements of the set $L^T_{<x,n,i,j_\delta,i_\delta>}$;

b) there exists no pair $[j', i']$ such that $j' < n$, with the following property: there exists a number δ', with $0 \leq \delta' \leq \ell$ and

$$(j' = j_{\delta'} \,\&\, i' < i_{\delta'}) \vee (j_{\delta'} < j' < j_{\delta'+1}) \vee (\delta' = \ell \,\&\, j_\delta < j') \vee (\delta' = 0 \,\&\, j' < j_0),$$
$$L^T_{<x,n,i,j',i'>} \cap (\delta \boxed{x}^T_{j',i'} \cup D \boxed{x}^T_{j',i'}) \neq \emptyset$$

and the marker $\boxed{x}$, but not $\boxed{-}$, is present at $[j', i']$;

c) there is no $j'' < n$ such that $\chi_{j'}$ appears on a four-tuple with second coordinate contained in the $<x', n, i>$-list for $x' \in \{\nu, \mu\}$, and whose first coordinate is x'.

Case 4. The conditions of Cases 1-3 are not satisfied, but there exists an $i \leq \kappa_n^T$ such that $\lambda x K_T(n,x)$ is defined on $\{\Delta_n^i, \pi_n^i\}$, and there exists a pair $[j', i']$ to which the marker $\boxed{x}$ (but no marker $\boxed{-}$) is attached and $j' < n$, and in addition there exists a number δ', $0 \leq \delta' \leq \ell + 1$, where

$$\Pi_n^i(T) = \{[j_0, i_0], \ldots, [j_\ell, i_\ell]\}, \; j_0 < j_1 < \ldots < j_\ell; \; j_{\ell+1} = n, \; i_{\ell+1} = 0,$$

and the function $\boxed{x'}^T_{j_{\delta'} i_{\delta'}}$ is completely defined on $L^T_{<x', n, i, j_{\delta'}, i_{\delta'}>}$ (with the marker $\boxed{x'}$ present at $[j_{\delta'}, i_{\delta'}]$ all such that: either

a) $$((j_{\delta'-1} < j' < j_{\delta'}) \vee (j' = j_{\delta'} \,\&\, i' < i_{\delta'})) \,\&\, (D \boxed{x}^T_{j',i'} \cup \delta \boxed{x}^T_{j',i'}) \cap L^T_{<x,n,i,j',i'>} \neq \emptyset$$

and for $x^* \in \{\nu, \mu\}$ none of the numbers in $L^T_{<x^*, n, i>}$ is the second component of a four-tuple with a marker $\chi_{j'}$ for $j' < n$, in which the first coordinate is x^*;

or

b) $(j' < j_0 \,\&\, \delta' = 0)$ or $\Pi_n^i(T) = \emptyset$ and $[j', i'] \notin \Pi_{m_*}^{i_*}(T)$, where $\Pi_{m_*}^{i_*}(T)$ is such that $<m_*, i_*> <_{lex} <n, i>$ or $\boxplus$ appears on $<m_*, i_*>$, and there exists no $[j'', i'']$ such that the marker $\boxed{x'}$ occurs,

$$(D \boxed{x'}^T_{j'',i''} \cup \delta \boxed{x'}^T_{j'',i''}) \cap L^T_{<x', n, i, j'', i''>} \neq \emptyset,$$

and if $j'' \neq j'$, then

$[j', i'] <_{lex} [j'', i''] < [j_{\delta'}, i_{\delta'}]$, while if $j'' = j'$, then $i'' < i'$;

or

c) $i = \kappa_n^T$ and Δ_n^i or π_n^i is the second coordinate of some four-tuple labeled by a $\chi_{j'}$ with $j' < n$, or else conditions a) and b) are not satisfied for any $[j', i']$ and there exists a $[j',i']$ such that $\{\Delta_n^i, \mathcal{T}_n^i\} \cap (D\boxed{\mathfrak{X}}^T_{j',i'} \cup \delta\boxed{\mathfrak{X}}^T_{j',i'}) \neq \emptyset$, where the marker $\boxed{\mathfrak{X}^T}$ but not $\boxed{-}$ appears at $[j',i']$.

If no such n exists then we leave everything unchanged and pass to the next step.

If there exist n with the above properties, we take the smallest and denote it by n_0.

If the conditions in Case 1) hold for n_0 then we take the first two odd numbers $a<b$, larger than all the odd numbers in $\bigcup_{n\in N} \nu^T(n)$ and define $\Pi_n^{i'}(T+1)=\emptyset$ for $i'<\kappa=\kappa_n^T$,

$$\mu^{T+1}(\mathcal{T}_{n_0}^\kappa) = \nu^{T+1}(\Delta_{n_0}^\kappa) = \nu^T(\Delta_{n_0}^\kappa) \cup \nu^T(\mathcal{T}_{n_0}^\kappa) \cup \{a\},$$
$$\mu^{T+1}(\Delta_{n_0}^\kappa) = \nu^{T+1}(\mathcal{T}_{n_0}^\kappa) = \nu^T(\Delta_{n_0}^\kappa) \cup \nu^T(\mathcal{T}_{n_0}^\kappa) \cup \{b\},$$
$$\varphi^{T+1}(\Delta_{n_0}^\kappa) = \mathcal{T}_{n_0}^\kappa, \quad \varphi^{T+1}(\mathcal{T}_{n_0}^\kappa) = \Delta_{n_0}^\kappa,$$

on n_0 and associate to $\langle n_0,\kappa\rangle$ the marker $\boxed{+}$; we make no other changes and go to the next step.

If the conditions in Case 2) hold for n_0, then we associate the marker $\boxed{+}$ to n_0 and $\langle n_0,\kappa\rangle$ and define $\Pi_n^{i'}(T+1)=\emptyset$ for $i'<\kappa=\kappa_n^T$; leaving everything else unchanged, we go to the next step.

If the conditions of Case 3) are satisfied for n_0 then we consider the smallest i_0, such that Case 3) holds, and make the following instruction:

Let $\mathbb{L}^T_{\langle\nu,n_0,i_0\rangle} = \langle L^T_{\langle\nu,n_0,i_0\rangle}, \prec\rangle$ be the $\langle\nu,n_0,i_0\rangle$-list and $m_0 \prec m_1 \prec \ldots \prec m_\ell$ all the elements in $L^T_{\langle\nu,n_0,i_0\rangle}$ in the order indicated. For i, $0\leq i<\ell$ we define $\varphi^{T+1}(m_i)=\varphi^T(m_{i+1})$ and

$$\nu^{T+1}(m_i) = \mu^{T+1}(\varphi^T(m_{i+1})) = \nu^T(m_i) \cup \nu^T(m_{i+1}) \cup \{a_i\},$$

$$\nu^{T+1}(m_\ell) = \mu^{T+1}(\varphi^T(m_0)) = \nu^T(m_\ell) \cup \nu^T(m_0) \cup \{a_\ell\},$$

$\varphi^{T+1}(m_\ell)=\varphi^T(m_0)$, where $T<a_0<a_1<\ldots<a_\ell$ are the first numbers not contained in $\bigcup_{n\in N}\nu^T(n)$. We associate a marker $\boxed{+}$ to n_0 and $\langle n_0,i_0\rangle$. For all pairs $[j,i]\in\Pi_{n_0}^{i_0}(T)$ and $[j,i']$, where $(*)\, i<i'$, there is no marker $\boxed{-}$ on $[j,i']$ and $[j,i']\in\Pi_{m_*}^{i_*}(T)$, where $\Pi_{m_*}^{i_*}(T)$ is such that $\boxed{+}$ appears on $\langle m^*,i^*\rangle$. Then we put the marker $\boxed{-}$ on $[j,i']$ and

$$\Pi_{m^*}^{i^*}(T+1) = \Pi_{m^*}^{i^*}(T) \setminus \{[j'',i''] \mid [j'',i''] \leq_{lex} [j,i']\},$$

where $[j,i']\in\Pi_{m^*}^{i^*}(T)$ and has the largest coordinate among all the pairs satisfying condition (*). To all $[j,i']$, where $i>i_0$ and $[j,i']$ is defined, we associate to $[j,i']$ the marker $\boxed{-}$. We choose for each pair $[j,i]\in\Pi_{n_0}^{i_0}(T)$ a number i' such that $S^T(j,i')$ is not defined, and taking the first three indices $a<b<c$ greater than T and still not used in the construction, we define

$$\ell_x^{T+1}(j,i') = a, \quad s_x^{T+1}(j,i') = d_x^{T+1}(j,i') = b,$$

$$r_x^{T+1}(j,i') = c, \quad \Pi_n^{i''}(T+1) = \emptyset,$$

where $i'' \neq i_0$ and $i'' \leq \kappa_n^T$, nothing else being changed. Then we pass to the next step.

If the conditions of Case 4) hold for n_0, then we take the smallest i_0 such that one of the conditions of Case 4) is satisfied. Choose the largest j' for which there exists an i', such that one of the cases holds for $[j', i']$. We consider for j' the smallest i' such that one of the conditions is satisfied for $[j', i']$ on the pair $\langle n_0, i_0 \rangle$.

If the condition a) is satisfied for $[j', i']$ and $[j', i'] \notin \Pi_{m^*}^{i^*}(T)$, where $\langle m^*, i^* \rangle <_{lex} \langle n_0, i_0 \rangle$ or $\boxplus$ appears on $\langle m^*, i^* \rangle$, or else if condition b) is satisfied, we define

$$\Pi_{n_0}^{i_0}(T+1) = (\Pi_{n_0}^{i_0}(T) \cup \{[j', i']\}) \setminus \{[j_{\delta''}, i_{\delta''}] \mid \delta'' < \delta'\}$$

and for all $\langle m^*, i^* \rangle >_{lex} \langle n_0, i_0 \rangle$, if we have $\langle j', i' \rangle \in \Pi_{m^*}^{i^*}(T)$ then we put

$$\Pi_{m^*}^{i^*}(T+1) = \Pi_{m^*}^{i^*}(T) \setminus \{[j'', i''] \mid [j'', i''] \leq_{lex} [j', i']\}.$$

On all the $[j', i'']$ such that $i'' > i'$ and $s_x^T(j', i'') \in N$ we place a marker $\boxminus$ and consider the first three indices $T < a < b < c$ which are not equal to indices previously appearing in the construction; we define

$$\ell_x^{T+1}(j', i^*) = a,$$
$$s_x^{T+1}(j', i^*) = d_x^{T+1}(j', i^*) = b,$$
$$r_x^{T+1}(j', i^*) = c,$$

where i^* is the first number such that $s_v^T(j', i^*)$ is not defined. If $i_0 = \kappa_{n_0}^T$, then we define $\kappa_{n_0}^{T+1} = \kappa_{n_0}^T + 1$ and taking the first two indices $a < b$ not yet used in the construction, we set $\Delta_{n_0}^{i_0+1} = a$ and $\mathscr{T}_{n_0}^{i_0+1} = b$, $\Pi_{n_0}^{i_0+1}(T+1) = \emptyset$.

If, however, $[j', i']$ satisfies condition a) but there exists an $\langle m^*, i^* \rangle$ such that $[j', i'] \in \Pi_{m^*}^{i^*}(T)$ for which

$$\langle m^*, i^* \rangle <_{lex} \langle n_0, i_0 \rangle$$

or the marker $\boxplus$. appears on $\langle m^*, i^* \rangle$, then we set

$$\Pi_{n_0}^{i_0}(T+1) = \Pi_{n_0}^{i_0}(T) \setminus \{[j_{\delta''}, i_{\delta''}] \mid \delta'' < \delta'\}$$

Leaving everything else unchanged, we pass to the next step.

If $[j', i']$ satisfies condition c), then we take the first two numbers $a < b$, not yet used in the construction and define

$$\Pi_{n_0}^{i_0}(T+1) = \emptyset, \quad \kappa_{n_0}^{T+1} = \kappa_{n_0}^T + 1$$

and $\Delta_n^{i_0+1} = a$, $\mathscr{T}_n^{i_0+1} = b$; we leave everything else unchanged and go to the next step.

Step $5t+2$. We define $T = 5t+1$, $j = \ell(t)$ and seek i^*, n^* and i such that $i^*, n^*, i \leq T$, the marker χ_j, $j < n^*$, does not appear, and one of the following conditions is satisfied.

Case 1. There exist $\varkappa \in \{\nu, \mu\}$ and ℓ_1, ℓ_2, ℓ_3, such that

$$\ell_2 \neq \ell_3,\ \varkappa^T(\ell_1) \subseteq \gamma_j^{T+1}(\ell_2),$$
$$\varkappa^T(\ell_1) \subseteq \gamma_j^{T+1}(\ell_3),\quad \ell \notin \bigcup_{t \leq T} M_\varkappa^{t'}(j', i'),$$

where $j' < j$, and $\ell \notin \{\Delta_m^i, \pi_m^i \mid i \leq \kappa_m^T$ and $m < j\}$.

Case 2. The pair $[j, i]$ is the element with smallest left coordinate in $\Pi_{n^*}^{i^*}(T)$, the marker $\boxplus$, is present on $\langle n^*, i^* \rangle$ and n^*, and the marker $\boxed{\varkappa}$ appears on $[j, i]$. Case 1 does not hold, but one of the following subcases does.

Subcase 2.1. For all elements $\kappa_0 \prec \kappa_1 \prec \ldots \prec \kappa_q$ in the $\langle \varkappa, n^*, i^*, j, i \rangle$-list, the following conditions hold:

If $0 < i \leq q$ then $\varkappa^T(\kappa_i) \subseteq \gamma_j^{T+1}(\boxed{\varkappa}_{j,i}^T(\kappa_i))$, and there exists a d_0 such that
$$\varkappa^T(\kappa_0) \subseteq \gamma_j^{T+1}(d_0).$$

Subcase 2.2. $\varkappa = \nu$ and $\nu^T(\kappa_0) \subseteq \gamma_j^T(\boxed{\nu}_{j,i}^T(\kappa_1))$.

Subcase 2.3. $\varkappa = \mu$ and $\mu^T(\kappa_0) \subseteq \gamma_j^T(\boxed{\mu}_{j,i}^T(\kappa_1))$.

Subcase 2.4. The marker $\boxminus$ appears on $[j, i]$.

Case 3. The pair $[j, i] \notin \Pi_{m^{**}}^{i^{**}}(T)$, where $\langle m^{**}, i^{**} \rangle <_{lex} \langle n^*, i^* \rangle$; for every $i'' < i$, if the pair $[j, i''] \in \Pi_{n'}^{i'}(T)$ has smallest left coordinate in $\Pi_{n'}^{i'}(T)$, then $\boxed{\varkappa'}_{j,i}^T$ is completely defined on the $\langle \varkappa', n', i', j, i'' \rangle$-list, where $\boxed{\varkappa'}$ is a marker appearing at $[j, i'']$, and there is no $i'' < i$ such that $[j, i''] \notin \Pi_{n'}^{i'}(T)$ for all $\langle n', i' \rangle$. In Case 3 we assume the previous cases are not satisfied; there is no marker $\boxplus$ on $\langle n^*, i^* \rangle$, or $[j, i] \notin \Pi_{n^*}^{i^*}(T)$ and for all ℓ elements in $L_{\langle \varkappa, n^*, i^*, j, i \rangle}^T$ there exists a d_ℓ such that $\varkappa^T(\ell) \subseteq \gamma^{T+1}(d_\ell)$; if $\boxed{\varkappa}_{j,i}^T(\ell)$ is defined then $\boxed{\varkappa}_{j,i}^T(\ell) = d_\ell$, and $\boxed{\varkappa}_{j,i}^T$ is completely defined on all elements in $L_{\langle \varkappa, m^{**}, i^{**} \rangle}^T$ for $\langle m^{**}, i^{**} \rangle <_{lex} \langle n^*, i^* \rangle$ and is not defined on the $\langle \varkappa, n^*, i^* \rangle$-list.

If n^*, i^* and i with the above properties do not exist, then we have everything unchanged and go to the next step. If these numbers do exist then we choose among them the smallest triple (n^*, i^*, i) (under the lexicographic ordering). Let this triple be (n^*, i^*, i).

If the triple satisfies Case 1 then we put a marker χ_j on $\langle \varkappa, \ell_1, \ell_2, \ell_3 \rangle$. If $\ell \in \bigcup_{t \leq T} M_\varkappa^t(j', i')$ where $j' > j$ then we put a marker $\boxminus$ on $[j', i]$; we then take the smallest i'' such that $s_\nu^T(j', i'')$ is not defined, and three still-unused numbers $T < a < b < c$, and define $\ell_{\varkappa'}^{T+1}(j', i'') = a$, $s_{\varkappa'}^{T+1}(j', i'') = d_{\varkappa'}^{T+1}(j', i'') = b$, $r_{\varkappa'}^{T+1}(j', i'') = c$, where $\varkappa' \in \{\nu, \mu\}$.

For $\langle m^{**}, i^{**}\rangle$ such that $[j', i'] \in \Pi^{i^{**}}_{m^{**}}(T)$, we define

$$\Pi^{i^{**}}_{m^{**}}(T+1) = \Pi^{i^{**}}_{m^{**}}(T) \setminus \{[j'', i''] \mid [j'', i''] <_{lex} [j', i']\},$$

if there is no marker $\boxplus$ on $\langle m^{**}, i^{**}\rangle$; we put a marker $\boxminus$ on all the $[j, i']$; and we define $\Pi^{i_{**}}_{m_{**}}(T+1) = \Pi^{i_{**}}_{m_{**}}(T) \setminus \{[j'', i''] \mid [j'', i''] <_{lex} [j, i']\}$, if $[j, i'] \in \Pi^{i_{**}}_{m_{**}}(T)$ and no marker $\boxplus$ appears on $\langle m_{**}, i_{**}\rangle$. Then we go to the next step.

If Case 2 and Subcase 2.1 hold for the triple (n^*, i^*, i) then we put $\boxed{\varkappa}^{T+1}_{j,i}(K_0) = d_0$ and proceed as follows, depending on the value of $\rho(T, j, i)$.

1) For $\rho(T, j, i) = 0$, if a marker $\boxed{\nu}$ appears on $[j, i]$ then $d^{T+1}_{\mu}(j, i) = s^{T+1}_{\mu}(j, i) = r^{T}_{\mu}(j, i)$; on the other hand, if the marker $\boxed{\mu}$ appears, then $d^{T+1}_{\nu}(j, i) = s^{T+1}_{\nu}(j, i) = \ell^{T}_{\nu}(j, i)$;

$$d^{T+1}_{\varkappa}(j, i) = s^{T+1}_{\varkappa}(j, i) = s^{T}_{\varkappa}(j, i),$$
$$\rho(T+1, j, i) = \rho(T, j, i) + 1.$$

Taking the first two indices a and b, not yet used in the construction, we define

$$\ell^{T+1}_{\nu}(j, i) = \ell^{T+1}_{\mu}(j, i) = a, \quad r^{T+1}_{\nu}(j, i) = r^{T+1}_{\mu}(j, i) = b$$

and putting $\Pi^{i^*}_{n^*}(T+1) = \Pi^{i^*}_{n^*}(T) \setminus \{[j, i]\}$ we pass to step A.

2) For $\rho(T, j, i) = 1$, if the marker $\boxed{\nu}$ appears at $[j, i]$ then $s^{T+1}_{\mu}(j, i) = r^{T}_{\mu}(j, i) = d^{T+1}_{\mu}(j, i)$ and $d^{T+1}_{\nu}(j, i) = s^{T}_{\nu}(j, i)$; if the marker $\boxed{\mu}$ appears, then $s^{T+1}_{\nu}(j, i) = \ell^{T}_{\nu}(j, i) = d^{T+1}_{\nu}(j, i)$ and $d^{T+1}_{\mu}(j, i) = s^{T}_{\mu}(j, i)$. Taking the first two still-unused indices with $T < a < b$, we define $\ell^{T+1}_{\mu}(j, i) = \ell^{T+1}_{\nu}(j, i) \leftrightharpoons a$ and $r^{T+1}_{\mu}(j, i) = r^{T+1}_{\nu}(j, i) \leftrightharpoons b$, $\rho(T+1, j, i) \leftrightharpoons 2$ and putting $\Pi^{i^*}_{n^*}(T+1) \leftrightharpoons \Pi^{i^*}_{n^*}(T) \setminus \{[j, i]\}$, we pass to step A.

3) For $\rho(T, j, i) = 2$, if the marker $\boxed{\nu}$ appears at $[j, i]$, then $s^{T+1}_{\mu}(j, i) = r^{T}_{\mu}(j, i)$; if the marker $\boxed{\mu}$ appears at $[j, i]$, then $s^{T+1}_{\nu}(j, i) = \ell^{T}_{\nu}(j, i)$. We take the first two indices $T < a < b$, still unused in the construction and define $\ell^{T+1}_{\mu}(j, i) = \ell^{T+1}_{\nu}(j, i) = a$, $r^{T+1}_{\mu}(j, i) = r^{T+1}_{\nu}(j, i) = b$, $\rho(T+1, j, i) = 3$. We then put

$$\Pi^{i^*}_{n^*}(T+1) \leftrightharpoons \Pi^{i^*}_{n^*}(T) \setminus \{[j, i]\},$$

and go to step A.

4) For $\rho(T, j, i) = 3$, if $\varkappa = \nu$, then

$$s^{T+1}_{\mu}(j, i) = r^{T}_{\mu}(j, i), \quad r^{T+1}_{\mu}(j, i) = d^{T}_{\mu}(j, i),$$
$$r^{T+1}_{\nu}(j, i) = d^{T}_{\nu}(j, i), \quad \ell^{T+1}_{\nu}(j, i) = \ell^{T+1}_{\mu}(j, i) = a;$$

if $\varkappa = \mu$, then $s^{T+1}_{\nu}(j, i) = \ell^{T}_{\nu}(j, i)$,

$$\ell^{T+1}_{\nu}(j, i) = d^{T}_{\nu}(j, i), \quad \ell^{T+1}_{\mu}(j, i) = d^{T}_{\mu}(j, i),$$

$$r^{T+1}_{\nu}(j, i) = r^{T+1}_{\mu}(j, i) = a,$$

where $T < a$ is the first index still unused in the construction, $p(T+1,j,i) = 1$, $\Pi_{n*}^{i*}(T+1) = \Pi_{n*}^{i*}(T) \setminus \{[j,i]\}$, and we pass to step A.

If our triple satisfies Case 2, Subcase 2.2, then we set $p(T+1,j,i) = 0$, put a marker $\boxed{\mu}$ on $[j,i]$, and remove $\boxed{\nu}$, $s_{\nu}^{T+1}(j,i) = d_{\nu}^{T+1}(j,i) = \ell_{\nu}^{T}(j,i)$; we define $\boxed{\mu}_{j,i}^{T+1}$ by putting $\boxed{\mu}_{j,i}^{T+1}(\varphi^{T}(\kappa_1)) = \boxed{\nu}_{j,i}^{T}(\kappa_1)$ with $\boxed{\mu}_{j,i}^{T+1}$ undefined in the remaining cases,

$$D\boxed{\mu}_{j,i}^{T+1} = \emptyset, \quad \Pi_{n*}^{i*}(T+1) = \Pi_{n*}^{i*}(T) \setminus \{[j,i]\}$$

and pass to step A.

If Case 2 and Subcase 2.3 hold, then we put a marker $\boxed{\nu}$ on $[j,i]$, remove $\boxed{\mu}$, and set $p(T+1,j,i) = 0$,

$$s_{\mu}^{T+1}(j,i) = d_{\mu}^{T+1}(j,i) = \tau_{\mu}^{T}(j,i), \; d_{\nu}^{T+1}(j,i) = s_{\nu}^{T}(j,i),$$

then we define $\boxed{\nu}_{j,i}^{T+1}$ by $\boxed{\nu}_{j,i}^{T+1}((\varphi^{T})^{-1}(\kappa_1)) = \boxed{\mu}_{j,i}^{T}(\kappa_1)$, with $\boxed{\nu}_{j,i}^{T+1}$ undefined at other points, $D\boxed{\nu}_{j,i}^{T} = \emptyset$,

$$\Pi_{n*}^{i*}(T+1) = \Pi_{n*}^{i*}(T) \setminus \{[j,i]\}.$$

We then go to step A.

If Case 2, Subcase 2.4 holds, then we set

$$\Pi_{n*}^{i*}(T+1) = \Pi_{n*}^{i*}(T) \setminus \{[j,i]\}$$

and pass to step A.

If Case 3 holds for the numbers n^*, i^*, i, then we extend the definition of $\boxed{\varkappa}_{j,i}^{T}$ by putting (for the ℓ indicated in the condition) $\boxed{\varkappa}_{j,i}^{T+1}(\ell) = d_{\ell}$, the values which have been found in the present case. We then go to step A.

Step A. Leaving all undefined objects unchanged, we go to the following step.

Step $5t+3$. We put $T = 5t+2$ and $j = \ell(t)$ and verify whether there exists an i such that no marker $\boxed{-}$ appears at $[j,i]$ but the marker $\boxed{\varkappa}$ does appear, the function $\boxed{\varkappa}_{j,i}^{T}$ is completely defined on $D\boxed{\varkappa}_{j,i}^{T}$ and $[j,i] \notin \Pi_{m_*}^{i_*}(T)$, where $\Pi_{m_*}^{i_*}(T)$ is such that $L_{\langle \varkappa, m^*, i^* \rangle}^{T} \subseteq D\boxed{\varkappa}_{j,i}^{T}$. If such an i exists then we take the smallest one, say i_0. We define

$$D\boxed{\varkappa}_{j,i_0}^{T+1} = D\boxed{\varkappa}_{j,i_0}^{T} \cup \{\ell\} \cup L_{\langle \varkappa, m, i \rangle}^{T},$$

where ℓ is the smallest number such that $\ell \notin D\boxed{\varkappa}_{j,i_0}^{T}$, and $\langle m,i \rangle$ is the smallest pair in the lexicographic ordering such that $\boxed{\varkappa}_{j,i_0}^{T}$ is not completely defined on $L_{\langle \varkappa, m, i \rangle}^{T}$. For all $i' > i$ such that $s_{\varkappa}^{T}(j,i')$ is defined and all pairs $\langle m^{**}, i^{**} \rangle$ such that $[j,i'] \in \Pi_{m^{**}}^{i^{**}}(T)$, we set

$$\Pi_{m^{**}}^{i^{**}}(T+1) = \Pi_{m^{**}}^{i^{**}}(T) \setminus \{[j'',i''] \mid [j'',i''] \leq_{lex} [j,i']\},$$

if no marker $\boxed{+}$ appears on $\langle m^{**}, i^{**} \rangle$, and we put a marker $\boxed{-}$ on $[j,i']$. We make no other changes and go to the next step.

Step $5t+4$. We put $T=5t+3$ and $j=\ell(t)$ and check whether there exists an i such that there is no marker $\boxminus$ on $[j,i]$ but $\boxed{\varkappa}$ appears; whether there exists no $i'<i$ such that the pair $[j,i']$ has minimal left coordinate in some set $\Pi^{i_{**}}_{m_{**}}(T)$ and $\boxed{\varkappa}^T_{j,i'}$ is not completely defined on $L^T_{\langle\varkappa',m_{**},i_{**}\rangle}$, where the marker $\boxed{\varkappa}$ appears on $[j,i']$; or else whether $[j,i']\notin\Pi^{i_{**}}_{m_{**}}(T)$ for all $i_{**},m_{**}\in N$; whether there exist $\ell\in D\boxed{\varkappa}^T_{j,i}$ and $d_\ell\in N$ such that $\boxed{\varkappa}^T_{j,i}(\ell)$ is not defined,

$$\ell\notin\bigcup_{[j'',i'']<_{lex}[j,i]}\bigcup_{t\le T}M^t_\varkappa(j'',i'')\cup\{\Delta^i_m,\pi^i_m\mid m<j$$

and $i\le\kappa^T_m\}$ and $\varkappa^T(\ell)\subseteq\gamma^{T+1}_j(d_\ell)$, and if $[j,i]\in\Pi^{i^*}_{m^*}(T)$, then $(\boxed{\varkappa}^T_{j,i}$ completely defined on $L^T_{\langle\varkappa,m^*,i^*\rangle})\vee(\ell<m^*)\vee\ell\in\bigcup_{\langle m'',i''\rangle\le_{lex}\langle m^*,i^*\rangle}L^T_{\langle\varkappa,m'',i''\rangle}$. In this case we define $\boxed{\varkappa}^{T+1}_{j,i}(\ell)=d_\ell$, while for all $i'>i$ such that $s^T_\varkappa(j,i')$ is defined we place a marker $\boxminus$ on $[j,i']$, and for a pair $\langle m^{**},i^{**}\rangle$ such that $[j,i']\in\Pi^{i^{**}}_{m^{**}}(T)$ and no marker $\boxplus$ is present at $\langle m^{**},i^{**}\rangle$ we put

$$\Pi^{i^{**}}_{m^{**}}(T+1)=\Pi^{i^{**}}_{m^{**}}(T)\setminus\{[j'',i'']\mid[j'',i'']\le_{lex}[j,i']\}.$$

We leave everything else unchanged and go to the next step.

Step $5t+5$. We take three numbers $T<a<b<c$ not yet used in the construction, where $T=5t+4$, and define $\ell^{T+1}_\varkappa(t,0)=a$, $s^{T+1}_\varkappa(t,0)=d^{T+1}_\varkappa(t,0)=b$, $\tau^{T+1}_\varkappa(t,0)=c$ for $\varkappa\in\{\nu,\mu\}$. We make no other changes, and go to the next step.

We make some simple remarks concerning the above construction.

Remark 1. For all n and t we have the equality

$$\mu^t\varphi^t(n)=\nu^t(n).$$

Remark 2. For all n and t there exists an element $a\in\nu^t(n)$ such that $a\notin\nu^t(m)$ for all $m\ne n$.

Remark 3. For all t and all $n\ne m$, the inclusion $\nu^t(n)\subseteq\nu^t(m)$ is false.

Remark 4. For all i,j and t we have the equalities

$$\nu^t s^t_\nu(j,i)=\mu^t s^t_\mu(j,i),\quad \nu^t(\ell^t_\nu(j,i))=\mu^t(\ell^t_\mu(j,i)),$$
$$\nu^t(\tau^t_\nu(j,i))=\mu^t\tau^t_\mu(j,i),\quad \nu^t d^t_\nu(j,i)=\mu^t d^t_\mu(j,i),$$
$$ts^t_\nu(j,i)=s^t_\mu(j,i),\qquad t\tau^t_\nu(j,i)=\tau^t_\mu(j,i),$$
$$t\ell^t_\nu(j,i)=\ell^t_\mu(j,i),\qquad td^t_\mu(j,i)=d^t_\mu(j,i),$$

if the marker $\boxplus$ does not appear on any pair $\langle m^*,i^*\rangle$ such that $[j,i]\in\Pi^{i^*}_{m^*}(t)$, and $\boxminus$ appears at $[j,i]$.

Remark 5. For all t, the value of κ^t_0 is equal to 0 and $\Pi^0_0(t)=\varnothing$.

Remark 6. If $[j,i]\in\Pi^{i^*}_{m^*}(t)$ for all $t\ge t_0$, then starting at some $t'\ge t_0$, the sets $\delta\boxed{\varkappa}^t_{j,i}$ and $D\boxed{\varkappa}^t_{j,i}$, where $\varkappa\in\{\nu,\mu\}$, do not change.

Remark 7. If $[j,i] \in \Pi_{m^*}^{i^*}(t)$ for all $t \geq t_0$, then the pair $[j,i]$ can be used in the construction only finitely many times.

Remark 8. If $[j,i] \in \Pi_{m^*}^{i^*}(t)$ for $t_1 \leq t \leq t_2$, then the marker at $[j,i]$ does not change for steps with such t.

Remark 9. If starting at some step the marker $\boxed{\varkappa}$ appears constantly at the pair $[j,i]$ then the functions $\lambda t s_{\varkappa}^{t}(j,i)$ and $\lambda t d_{\varkappa}^{t}(j,i)$ stabilize.

Remark 10. For every κ there exists at most one i such that the marker $\boxplus$ is placed on $\langle \kappa, i \rangle$ and thereafter not removed.

Remark 11. If the marker $\boxed{\varkappa}$ appears in step t at $[j,i]$ and $\ell \notin L^{t}_{\langle \varkappa, n^*, i^* \rangle}$, where $[j,i] \in \Pi_{n^*}^{i_*}(t)$, $\boxplus$ appears at $\langle n^*, i^* \rangle$, and $\boxed{\varkappa}^{t}_{ji}(\ell)$ is defined, then $\varkappa^{t}(\ell) \subset \gamma_j^{t}(\boxed{\varkappa}^{t}_{j,i}(\ell))$.

Remark 12. If after step t_0 the set $\Pi_{m^*}^{i^*}(t)$ does not change and the marker $\boxplus$ does not appear at $\langle m^*, i^* \rangle$, then $\varkappa^{t}(\ell) = \varkappa^{t_0}(\ell)$ for all $\ell \in L^{t_0}_{\langle \varkappa, m^*, i^* \rangle}$.

We define $M_{ji}^{\varkappa} = \bigcup_{t \geq 0} M_{\varkappa}^{t}(j,i)$.

LEMMA 1. For all unequal pairs $[j',i']$ and $[j'',i'']$ and $\varkappa \in \{\nu, \mu\}$, the sets $M^{\varkappa}_{j',i'}$ and $M^{\varkappa}_{j'',i''}$ are disjoint.

Proof. We define $\hat{M}^{t}_{\varkappa}(j,i) = \bigcup_{t' \leq t} M^{t'}_{\varkappa}(j,i)$. Since $M^{\varkappa}_{ji} = \bigcup_{t \geq 0} \hat{M}^{t}_{\varkappa}(j,i)$, it suffices to show that for all t and i',j',i'',j'' if $[j',i'] \neq [j'',i'']$, then $\hat{M}^{t}_{\varkappa}(j',i') \cap \hat{M}^{t}_{\varkappa}(j'',i'') = \emptyset$. Assume this is false.

We consider pairs $[j,i]$ and $[j',i']$ such that there exists a t, such that $\hat{M}^{t}_{\varkappa}(j,i) \cap \hat{M}^{t}_{\varkappa}(j',i') \neq \emptyset$, and we take the smallest t with this property (call it t_0). Since $M^{\varkappa}_{ji}(0)$ is defined at Step 0 only for $j = i = 0$, we have $t_0 > 0$. By the choice of t_0, we have the condition

$$\hat{M}^{t_0-1}_{\varkappa}(j,i) \cap \hat{M}^{t_0-1}_{\varkappa}(j',i') = \emptyset.$$

Since during a step $M^{t}_{\varkappa}(j,i)$ can change for one pair $[j,i]$ only, we assume for definiteness that

$$M^{t_0}_{\varkappa}(j',i') = M^{t_0-1}_{\varkappa}(j',i'), \text{ and } (\hat{M}^{t_0}_{\varkappa}(j,i) \supset \hat{M}^{t_0-1}_{\varkappa}(j,i)) \vee (\hat{M}^{t_0-1}_{\varkappa}(j,i)$$

is not defined). Therefore either:

1) $s^{t_0}_{\varkappa}(j,i) \in \hat{M}^{t_0-1}_{\varkappa}(j',i')$,

2) $\ell^{t_0}_{\varkappa}(j,i) \in \hat{M}^{t_0-1}_{\varkappa}(j',i')$,

3) $r^{t_0}_{\varkappa}(j,i) \in \hat{M}^{t_0-1}_{\varkappa}(j',i')$,

or 4) $d^{t_0}_{\varkappa}(j,i) \in \hat{M}^{t_0-1}_{\varkappa}(j',i')$.

Since $\ell^{t_0}_{\varkappa}(j,i)$, $s^{t_0}_{\varkappa}(j,i)$, $r^{t_0}_{\varkappa}(j,i)$, $d^{t_0}_{\varkappa}(j,i)$ either take values in $M^{t_0-1}_{\varkappa}(j,i)$ or else are numbers which have not yet been used in the construction, conditions 1)-4) cannot hold in any of these cases. This contradiction proves the lemma.

LEMMA 2. For all n and i the limits $\lim_{t\to\infty} \kappa_n^t < \infty$, $\lim_{t\to\infty} \Pi_{n+1}^i(t)$ exist and the set of pairs $X_n = \{[j',i'] \mid [j',i'] \in \lim_{t\to\infty} \Pi_{m''}^{i''}(t)$ and $m'' < n+1$ and $i'' \in N\}$ is finite.

Proof. We give the proof by induction on n. Assume the lemma has already been proved for all $n < n_0$; we prove it holds for n_0. Using the induction assumption, we consider a step t_0' such that for all $n < n_0$ we have $\kappa_n^{t_0'} = \lim_{t\to\infty} \kappa_n^t$, and starting at some step t_0' the sets $\Pi_n^i(t)$ for $i \leq \kappa_n^{t_0}$ no longer change.

Assume that $\lim_{t\to\infty} \kappa_{n_0}^t < \infty$ does not exist. By the induction assumption, for every $i \in N$ $\lim_{t\to\infty} \Pi_{n_0}^i(t)$ exists, and therefore for each i we can choose a step t_i, after which $\Pi_{n_0}^i(t)$ does not change. We first show that there exist infinitely many i such that $\Pi_{n_0}^i(t_i) \neq \emptyset$. Where this is not so, there would exist an i_0 such that for all $i > i_0$ we have $\Pi_{n_0}^i(t_i) = \emptyset$. Consider the step $t_1' = \max(\{t_0'\} \cup \{t_i \mid i \leq i_0\})$ and let

$$X = \{[j',i'] \mid [j',i'] \in \Pi_{m_*}^{i_*}(t_1') \,\&\, \langle m_*, i_* \rangle \leq_{lex} \langle n_0, i_0 \rangle\}.$$

It is obvious that X is finite. By Remark 6 we choose a step $t_2' > t_1'$ after which $X_{j',i'}^t = \delta \boxed{\varkappa}_{j',i'}^t \cup D \boxed{\varkappa'}_{j',i'}^t$, where the marker $\boxed{\varkappa}$ appears permanently on $[j',i']$, $\varkappa' \in \{\nu, \mu\}$, and $[j',i'] \in X$ and does not change, and no marker $\chi_{j'}$ with $j' < n_0$ appears. Now choose i such that $i > i_0$ and $\{\Delta_{n_0}^i, \mathcal{T}_{n_0}^i\} \cap X_{j',i'}^{t_2'} = \emptyset$ for all pairs $[j',i'] \in X$ and $\{\Delta_{n_0}^i, \mathcal{T}_{n_0}^i\} \cap \{$the second coordinates of four-tuples on which a $\chi_{j'}$ appears for $j' < n_0\} = \emptyset$.

We now choose a step $t_3' \geq t_2'$ such that for all $i' \leq i$ the set $\Pi_{n_0}^{i'}(t)$ no longer changes after step t_3'.

Consider a step $t_4' > t_3'$ after which the marker $\boxplus$ has already been placed for all $n \leq n_0$ at which $\boxplus$ occurs. Now consider the step $T = 5t_4' + 1$. By the choice of t_4', either Case 1 for i, or Case 4b) must hold at this step. But this is impossible, since in the first case the marker $\boxplus$ is placed on n_0 and $\kappa_{n_0}^t$ does not get any larger, while in the second case $\Pi_{n_0}^i(T)$ changes, but by the choice of t_4' such a change is not possible. Consequently, there exist infinitely many i such that $\Pi_{n_0}^i(t_i) \neq \emptyset$. Since the $\Pi_{n_0}^i(t_i)$ consist of elements in the finite set X_n, there exists an x such that there are at least two i_0 and i_1 such that

$$x \in \Pi_{n_0}^{i_0}(t_{i_0}) \text{ and } x \in \Pi_{n_0}^{i_1}(t_{i_1}),$$

But taking $t = \max\{t_{i_0}, t_{i_1}\}$, we obtain $x \in \Pi_{n_0}^{i_1}(t)$ and $x \in \Pi_{n_0}^{i_0}(t)$. This is impossible by our construction. This contradiction shows that $\lim_{t\to\infty} \kappa_{n_0}^t$ exists and is finite.

We now prove by induction on i that $\lim_{t\to\infty} \Pi_{n_0+1}^i(t)$ exists. Assume that for all $i < i_0$ our assertion has already been proved, and let $t_5' > t_0'$ be a step after which $\Pi_{n_0+1}^i(t)$ no longer changes for $i < i_0$. Consider the sets $\hat{\Pi}_{n_0+1}^{i_0}(t)$ obtained from $\Pi_{n_0+1}^{i_0}(t)$ as follows.

Let $t_5' < t_1 < t_2 < \ldots < t_\kappa < \ldots$ be the steps at which the set $\Pi_{n_0+1}^{i_0}(t)$ changes. For all κ and t such that $t_\kappa \leq t < t_{\kappa+1}$, Case 4 of type $5t+1$ holds on t_κ, and nothing is added to $\Pi_{n_0+1}^{i_0}(t_\kappa+1)$ (elements are only removed); we consider a pair $[j,i]$ for which the

Case 4 step of type 5t + 1 was carried out. In these cases we set

$$\hat{\Pi}^{i_0}_{n_0+1}(t) = \Pi^{i_0}_{n_0+1}(t) \cup \{[j,i]\},$$

otherwise, $\hat{\Pi}^{i_0}_{n_0+1}(t) = \Pi^{i_0}_{n_0+1}(t)$. We define a sequence $\langle a^t_{n_0}, a^t_{n_0-1}, \ldots, a^t_0\rangle$ for $t \geq t'_5$ by setting a^t_κ equal to i' if $\langle \kappa, i'\rangle \in \hat{\Pi}^{i_0}_{n_0+1}(t)$ and $a^t_\kappa = \omega$ otherwise. We remark that for every t we have

$$\langle a^{t+1}_{n_0}, a^{t+1}_{n_0-1}, \ldots, a^{t+1}_0\rangle \leq_{lex} \langle a^t_{n_0}, a^t_{n_0-1}, \ldots, a^t_0\rangle.$$

We thus define a decreasing sequence in the totally ordered set $((\omega+1)\times(\omega+1))\ldots\times(\omega+1)$. Therefore, there exists a t''_0 such that for all $t \geq t''_0$ we have the equality

$$\langle a^{t+1}_{n_0}, a^{t+1}_{n_0-1}, \ldots, a^{t+1}_0\rangle = \langle a^{t''_0}_{n_0}, a^{t''_0}_{n_0-1}, \ldots, a^{t''_0}_0\rangle.$$

Thus the sequence $\hat{\Pi}^{i_0}_{n_0+1}(t)$ stabilizes. However, in this case either $\hat{\Pi}^{i_0}_{n_0+1}(t) = \Pi^{i_0}_{n_0+1}(t)$, starting at some $t'''_0 \geq t''_0$ or $\Pi^{i_0}_{n_0+1}(t) = \hat{\Pi}^{i_0}_{n_0+1} \setminus \{[j,i]\}$ for all $t \geq t''_0$. This occurs because if $[j,i]$ after step t''_0 lies in $\Pi^{i_0}_{n_0+1}(t)$, it cannot be removed, since otherwise $\Pi^{i_0}_{n_0+1}(t)$, would change, and hence so would $\hat{\Pi}^{i_0}_{n_0+1}(t)$, a contradiction. Thus this condition has been proved.

Let us prove the last part of our lemma, i.e., that the set X_{n_0} is finite.

Assume X_{n_0} is infinite. Then there exists a smallest $j_0 \leq n_0+1$ such that there are infinitely many i with $[j_0,i] \in \lim_{t\to\infty} \Pi^{i^*_i}_{n_0+1}(t)$. Consider the set I of all i with this property, and for every $i \in I$ consider the step t_i after which $\Pi^{i}_{n_0+1}(t)$ no longer changes. Since for all $j' < j_0$ there exist only finitely many elements i' such that $[j',i] \in \lim_{t\to\infty} \Pi^{i^*}_{m^*}(t)$ for $m^* \leq n_0+1$ and $i^* \in N$ (by the choice of j_0). Therefore, there exist infinitely many i such that $[j_0,i]$, with $i \in I$, is a pair with minimal left coordinate in $\Pi^{i^*_i}_{n_0+1}(t_{i^*_i})$. There exists an i for which a step t' exists after which the condition

$$[j',i'] \in \lim_{t\to\infty} \Pi^{i^*}_{m^*}(t),\ m^* \leq n_0+1 \text{ and } j' < j_0,$$

implies that $\Pi^{i^*}_{m^*}(t)$ does not change and

$$L^{t'}_{\langle \varkappa, n_0+1, i^*_i\rangle} \cap X^{t'}_{j,i'} = \varnothing,$$

and after step t', $\Pi^{i^{**}}_{m^{**}}(t)$ for $\langle m^{**}, i^{**}\rangle \leq_{lex} \langle n_0+1, i^*_i\rangle$ does not change. Let this i be i_0. Then after step t' the construction cannot be carried out for any pairs $[j_0,i']$ where $i' > i_0$ and $\langle n_0+1, i''\rangle$, where $i'' > i^*_{i_0}$, but there must exist infinitely many $i > i_0$ such that $[j_0,i]$ belongs to $\Pi^{i^*_i}_{n_0+1}(t_{i^*_i})$ and $i^*_i > i^*_{i_0}$. This contradiction proves our assertion.

The lemma is proved.

<u>COROLLARY 1.</u> For every pair $\langle m,i\rangle$ only finitely many pairs $[j',i']$ can lie in $\bigcup_{t\geq 0} \Pi^i_m(t)$.

<u>LEMMA 3.</u> If a marker changes infinitely many times on $[j,i]$ then

$$\lim_{t\to\infty} f^t_{\varkappa}(j,i) = \infty \text{ exists for } f \in \{s, \ell, r, d\} \text{ and } \varkappa \in \{\nu, \mu\}$$

Proof. It is enough to show that for infinitely many t, there exists no $\rho \in M^t_{\varkappa}(j,i)$. We first show that for all ρ, if $\rho \in M^{t_1}_{\varkappa}(j,i)$ and $\rho \notin M^t_{\varkappa}(j,i)$, where $t_1 < t_2$, then $\rho \notin M^t_{\varkappa}(j,i)$ for all $t \geq t_2$. This is true because $M^{t+1}_{\varkappa}(j,i)$ can only contain either elements of $M^t_{\varkappa}(j,i)$ or indices which have not been used previously. Thus, if some $\rho \in M^t_{\varkappa}(j,i)$ for infinitely many t, then starting at some step t_0, we have $\rho \in M^t_{\varkappa}(j,i)$ for $t \geq t_0$. Since the marker on $[j,i]$ changes infinitely often, we consider a step $t_1 > t_0$, at which the marker $\boxed{\varkappa}$ for the pair $[j,i]$ is replaced by another marker. In this case, in order for $\rho \in M^t_{\varkappa}(j,i)$ and $\rho \in M^{t_1-1}_{\varkappa}(j,i)$, the following must hold:

$$\rho(t_1, j, i) = 0, \quad d^{t_1}_{\varkappa}(j,i) = s^{t_1}_{\varkappa}(j,i) = \rho,$$
$$\ell^{t_1}_{\varkappa}(j,i) = a, \quad r^{t_1}_{\varkappa}(j,i) = b,$$

where $a < b$ are previously unused numbers. Consider the steps $t_1 < t_2 < \ldots$ at which the set $M^t_{\varkappa}(j,i)$ changes. If the marker does not change at step t_2, then Subcase 1 of Case 2 holds and $\rho(t_2 - 1, j, i) = 0$. But then $M^{t_2}_{\varkappa}(j,i)$ no longer contains ρ, which contradicts our assumption. Hence the marker changes again at step t_2, and therefore $d^{t_2}_{\varkappa}(j,i) = s^{t_2}_{\varkappa}(j,i) = \rho$, $\ell^{t_2}_{\varkappa}(j,i) = a'$, $r^{t_2}_{\varkappa}(j,i) = b'$, where a', b' are numbers previously unused in the construction and the marker $\boxed{\varkappa}$ appears at $[j,i]$. As long as $\boxed{\varkappa}$ appears on $[j,i]$, $s^t_{\varkappa}(j,i) = \rho$. Consider the first step $t' > t_2$, at which the marker changes. Then

$$d^{t'}_{\varkappa}(j,i) = s^{t'}_{\varkappa}(j,i) = \ell^{t'-1}_{\varkappa}(j,i),$$

if $\varkappa = \nu$, and $d^{t'}_{\varkappa}(j,i) = s^{t'}_{\varkappa}(j,i) = r^{t'-1}_{\varkappa}(j,i)$, if $\varkappa = \mu$, while $\ell^{t'}_{\varkappa}(j,i) = a''$, $r^{t''}_{\varkappa}(j,i) = b''$ where the numbers a'', b'' were not previously used in the construction. But since $s^{t'-1}_{\varkappa}(j,i) = \rho$, we have $\ell^{t'-1}_{\varkappa}(j,i) \neq \rho$ and $r^{t'-1}_{\varkappa}(j,i) \neq \rho$, and consequently, $\rho \notin M^{t'}_{\varkappa}(j,i)$. This contradiction proves the lemma.

LEMMA 4. For every j there exist only finitely many elements i, for which the pair $[j,i] \in \lim_{t\to\infty} \Pi^{i^*}_{m^*}(t)$ for a suitable pair $\langle m^*, i^* \rangle$.

We give the proof by contradiction. Consider the smallest j_0 for which the statement of the lemma is false. For $j' < j_0$ consider the set $I_{j'}$ for all i such that beginning at step t'_i, the pair $[j', i] \in \Pi^{i_*}_{m_*}(t)$ for $t \geq t'_i$, where i_*, m_* are suitable numbers. We now choose a step t_0 such that for all $j' < j_0$ and $i \in I_{j'}$ then the pair $[j', i] \in \Pi^{i_*}_{m_*}(t)$ for $t \geq t_0$, where i^*, m^* are suitable numbers. By the choice of j_0, there exist infinitely many i such that

$$[j_0, i] \in \lim_{t\to\infty} \Pi^{i^*}_{m^*}(t)$$

for suitable i^*, m^*. Let I_0 be the set of all such i. By Remark 7, for every $i \in I_0$

there exists a step t_i'', after which the pair $[j_0, i]$ is no longer used. For each m, the limit $\lim_{t\to\infty} \kappa_m^t < \infty$ exists by Lemma 2, and therefore for every m only a finite number of sets $\Pi_m^i(t)$ are defined. Therefore, there exist infinitely many m such that there exist i^m and i_m with

$$[j_0, i^m] \in \lim_{t\to\infty} \Pi_m^{i_m}(t).$$

Consider the set M of all such numbers m. By Lemma 2, the sequence $\lambda t \Pi_m^i(t)$ stabilizes, and therefore we consider a step t_m such that $\Pi_m^i(t) = \Pi_m^i(t_m)$ for all $t \geq t_m$. Since there are only finitely many pairs $[j', i']$ for $j' < j_0$ such that $[j', i] \in \lim_{t\to\infty} \Pi_{m^*}^{i^*}(t)$ for suitable $i^*, m^* \in N$, there are infinitely many m such that the pair $[j_0, i^m]$ has minimal left coordinate in $\Pi_m^{i_m}(t_m)$.

We consider a step t_0''' such that for every pair $[j', i']$ with $j' < j_0$ and $[j', i'] \in \lim_{t\to\infty} \Pi_{m^*}^{i^*}(t)$ for suitable $i^*, m^* \in N$, the set $X_{j',i'}^t$ does not change; this step exists by Remark 6. Since there are infinitely many m such that $[j_0, i^m] \in \Pi_m^{i_m}(t_m)$, and only finitely many pairs $[j', i']$ with the above property, there exists an m_0 such that the $\langle \varkappa, m_0, i_{m_0} \rangle$-list does not intersect $D\boxed{\varkappa}_{j',i'}^{t_0'''} \cup \delta \boxed{\varkappa}_{j',i'}^{t_0'''}$, where the marker $\boxed{\varkappa}$ appears at $[j', i']$, and $j' < j_0$ and $[j', i'] \in \lim_{t\to\infty} \Pi_{m^*}^{i^*}(t)$ for suitable $i^*, m^* \in N$.

We consider a step t_2''' such that no pair $[j', i']$ such that $[j', i] \notin \lim_{t\to\infty} \Pi_{m^*}^{i^*}(t)$ for suitable $i^*, m^* \in N$ belongs to $\Pi_{m^{**}}^{i^{**}}(t)$ for $t \geq t_2'''$ and $\langle m^{**}, i^{**} \rangle \leq_{lex} \langle m_0, i_{m_0} \rangle$. Consider a step $t_1''' > \max\{t_0''', t_{m_0}, t_2'''\}$ such that $t_1''' = 5\kappa + 1$, for all pairs $\langle m^*, i^* \rangle <_{lex} \langle m_0, i_{m_0} \rangle$ none of the constructions holds any longer, and $\boxed{\varkappa}_{j_0, i^{m_0}}^{t_1'''-1}$ is completely defined on $L_{\langle \varkappa, m_0, i^{m_0} \rangle}^{t_1'''-1}$. If no such step exists, then after step t_1''', the construction of step $5t+1$ cannot be defined for any pair $[j_0, i']$, where $i' > i^{m_0}$. This contradicts our assumption. If such a step t_1''' exists, then Case 3 or 4 holds for the pair $[j_0, i^{m_0}]$, and hence either no pair $[j_0, i'], i' \geq i^{m_0}$ will be used subsequently, or else the set $\Pi_{m_0}^{i^{m_0}}(t)$ will change. But both these cases are impossible, and the lemma is proved.

LEMMA 5. For every j only one of the following possibilities holds:

a) The set I_j of numbers i such that $s_\nu^t(j, i)$ is defined for some t is finite, and all the functions

$$l_\varkappa^t(j,i),\ z_\varkappa^t(j,i),\ s_\varkappa^t(j,i),\ d_\varkappa^t(j,i),$$

where $\varkappa \in \{\nu, \mu\}$, stabilize.

b) There exists an i_0 such that for some t the set $s_\varkappa^t(j, i_0)$ is defined and there is no marker $\boxed{-}$ on $[j, i_0]$. This pair is used in the construction infinitely often, and for all $i' < i_0$ the pair $[j, i]$ is used only finitely many times in the construction; and for all $i' > i_0$ either a marker $\boxed{-}$ occurs on $[j, i]$ or else $s_\varkappa^t(j, i')$ is not defined for all t.

Proof. Consider the smallest number j_0 for which the conditions of the lemma are not satisfied.

Since condition a) is false for j_0, we have either

1) the set I_{j_0} is finite,

or

2) there exists an i_0 such that one of the functions $\lambda t \ell^t_{æ}(j,i)$, $\lambda t s^t_{æ}(j,i)$, $\lambda t r^t_{æ}(j,i)$, $\lambda t d^t_{æ}(j,i)$, where $æ \in \{\nu, \mu\}$, does not stabilize.

But if case 2) holds, then by our construction, both Case 3 and Case 4 for a step of type $5t+1$ are satisfied infinitely often for the pair $[j_0, i_0]$. But then I_{j_0} is infinite. Therefore, it suffices to consider only this case.

By Lemma 4 there exists a finite number of i such that $[j_0, i] \in \lim_{t\to\infty} \Pi^{i^*_i}_{m^*_i}(t)$ for suitable $i^*_i, m^*_i \in N$. We therefore consider the step t_0, after which we have for all pairs $[j_0, i]$ such that

$$[j_0, i] \in \lim_{t\to\infty} \Pi^{i^*_i}_{m^*_i}(t)$$

for suitable $i^*_i, m^*_i \in N$ that $[j_0, i] \in \Pi^{i^*_i}_{m^*_i}(t)$ for $t \geq t_0$ and after t_0 $\chi_{j'}$ does not occur for $j' < j_0$. We also consider pairs at which a marker $\boxminus$ appears at step t_0. There are finitely many such pairs. Pairs of the type $[j_0, i]$ will still be used in the construction subsequently, since I_{j_0} is finite. We consider the smallest i_0, such that after step t_0 the pair $[j_0, i_0]$ is used in the construction and such that there was no marker $\boxminus$ on $[j_0, i_0]$ at step t_0. Then we cannot put the marker $\boxminus$ on $[j_0, i_0]$ any more, since in order to put it there the construction would require that a pair $[j, i']$ with $i' < i_0$ have been used, but such is impossible by the choice of i_0. If at some step $\chi_{j'}$ appears, then I_{j_0} is finite, and by the foregoing, case a) is satisfied. We now show that the marker $[j_0, i_0]$ will be used infinitely many times in the construction. Assume that after step $t_1 > t_0$, $[j_0, i_0]$ is no longer used. Then for all $t \geq t_1$ and $\langle m^*, i^* \rangle$, the pair $[j_0, i_0] \notin \Pi^{i^*}_{m^*}(t)$. In this case, after step t_1 the pair $[j, i']$ with $i' > i_0$ can only be used either in a step of type $5t+1$ (Case 3), in a step of type $5t+2$ (Case 2), or else in a step of type $5t+3$. But in order for a marker to be used in a step of type $5t+1$ (Case 3) or type $5t+2$ (Case 2), it is necessary that Case 3 for a step of type $[j_0, i']$ first hold for $5t+1$. This is only possible for finitely many i', since at steps $t \leq t_1$ only a finite number of pairs can be used. In order that $[j, i']$ satisfy the requirements of a step of type $5t+3$, it is necessary that a step of type $5t+4$ or $5t+1$ first be satisfied; but this is only possible for finitely many i'. Thus we have reached a contradiction. Hence $[j_0, i_0]$ is used infinitely often in the construction; but in this case the marker $\boxminus$ appears infinitely many times on all larger $[j, i']$ for which $s^t_{æ}(j, i')$ is defined. Thus the lemma is proved.

<u>LEMMA 6.</u> If the marker $\boxed{æ}$ is permanently present on $[j, i]$ starting at some step t_0, $[j, i]$ is used in the construction infinitely many times, and the element $a_0 \in æ^{t_0}(s^{t_0}_{æ}(j,i))$ and $a_0 \notin æ^{t_0}(\ell)$ for all $\ell \neq s^{t_0}_{æ}(j,i)$, then for every $t \geq t_0$ and $\ell \neq s^{t_0}_{æ}(j,i)$, if $a_0 \in æ^t(\ell)$,

then there exists a $t_1, t_0 \le t_1 \le t$ such that $\ell = \ell_{\varkappa}^{t_1}(j,i)$ for $\varkappa = \nu$, $\ell = \tau_{\varkappa}^{t_1}(j,i)$ for $\varkappa = \mu$ and at step t_1 Case 3 of a step of type $5t+1$ holds for $[j,i]$.

We give the proof by induction on $t \ge t_0$ for the case $\varkappa = \mu$ (everything is analogous for $\varkappa = \nu$). Everything is satisfied for t_0 by the choice of a_0. Consider the smallest $t_1 > t_0$, such that the lemma is false and let ℓ be such that $a_0 \in \mu^{t_1}(\ell)$ but ℓ does not satisfy the hypothesis of the lemma. Since the lemma is true for all steps less than t_1, we have $a_0 \notin \mu^{t_1-1}(\ell)$. Thus, this number was added to $\mu^{t_1}(\ell)$ at step t_1. New numbers are added to sets only in Case 1 and Case 3 of a step of type $5t+1$; but by Case 1, the number a_0 cannot be added, because of the induction hypothesis and the constructions in a step of type $5t+1$. In Case 3, elements are added to sets from the neighbor to the right in some list together with a number which had no importance anywhere else. Thus in addition to ℓ there exists a neighbor ℓ' containing a_0. But by the induction hypothesis, ℓ' must either be $s_{\mu}^{t_1-1}(j,i)$ or $\tau_{\mu}^{t_1-1}(j,i)$ or $\ell_{\mu}^{t_1-1}(j,i)$. If $\ell' = s_{\mu}^{t_1-1}(j,i)$, then the condition of the lemma holds. If not, then $\ell' = \tau_{\mu}^{t_1-1}(j,i)$ or $\ell' = \ell_{\mu}^{t_1-1}(j,i)$, since otherwise ℓ will not be used in the construction at step t_1. If $\ell' = \tau_{\mu}^{t_1-1}(j,i)$, then we consider the largest step $t'' \le t_1$ such that $\ell' \ne \tau_{\mu}^{t_1-1}(j,i)$, but $a_0 \in \mu^{t''}(\ell')$ or $a_0 \notin \mu^{t''}(\ell')$ & $\ell' = \tau_{\mu}^{t''-1}(j,i)$. Then at step $t''+1$ either $a_0 \in \mu^{t''+1}(\ell')$ & $\ell' = \tau_{\mu}^{t''+1}(j,i)$ or $a_0 \notin \mu^{t''}(\ell')$, $\ell' = \tau_{\mu}^{t''}(j,i)$ must be defined, and $a_0 \in \mu^{t''+1}(\ell')$. In the first case, since the sets do not change if the value of $\tau_{\mu}^{t}(j,i)$ changes, the above situation is only possible for a step of type $5t+2$. But in such a step $\tau_{\mu}^{t''+1}(j,i)$ can take a value ℓ' such that this value was already used in the construction and the marker $\boxed{\nu}$ appears on $[j,i]$; but then after this step the marker $\boxed{\nu}$ must be replaced by $\boxed{\mu}$ and there exists a $t''' > t''$, for which the above-stated condition is also satisfied. This contradicts the maximality of t''. Thus, the second case holds. But then the marker $\boxed{+}$ appears on $\langle m_*^*, i^* \rangle$ during step $t''+1$ and $[j,i] \in \Pi_{m^*}^{i^*}(t''+1)$; after this step $[j,i]$ can again be used in the construction only if Case 2 of a step of type $5t+2$ previously holds for $[j,i]$. But then $\tau_{\mu}^{t}(j,i)$ changes its value to a new value and again our assumption that t'' is maximal is violated.

If $\ell' = \ell_{\mu}^{t_1-1}(j,i)$, then $\ell = s_{\mu}^{t_1-1}(j,i)$. But by assumption, ℓ is not equal to $\lim_{t\to\infty} s_{\mu}^{t}(j,i)$, and hence this case is impossible. This completes the proof of the lemma.

LEMMA 7. For every ℓ the limit $\varphi(\ell) = \lim_{t\to\infty} \varphi^t(\ell)$ exists and $\nu(\ell) = \mu\varphi(\ell)$; the value $\varphi(\ell)$ is the only one which is taken by the function $\lambda t \varphi^t(\ell)$ infinitely many times.

Proof. We consider an arbitrary $\ell \in N$ and some possible cases.

Case 1. $\ell \notin \bigcup_{t>0} \left(\bigcup_{[j,i]} M_{\nu}^{t}(j,i) \cup \bigcup_{n>0} \{\Delta_n^i, \pi_n^i \mid i \le \kappa_n^t\} \right)$.

In this case $\varphi^t(\ell) = \varphi^0(\ell)$, $\nu^t(\ell) = \nu^0(\ell)$ and

$$\varphi(\ell) = \varphi^0(\ell) = \lim_{t\to\infty} \varphi^t(\ell),$$

and since $\nu^t(\ell) = \mu^t \varphi^t(\ell)$ we have $\nu(\ell) = \mu\varphi(\ell)$.

Case 2. $\ell \in \bigcup_{n>0} \{\Delta_n^i, \pi_n^i \mid i \le \lim_{t\to\infty} \kappa_n^t\}$.

Since all the elements Δ_n^i, π_n^i, where $n \in N$ and $i \leq \lim_{t\to\infty} \kappa_n^t$, are pairwise distinct, there exists a unique pair $\langle n, i \rangle$ such that either $\ell = \Delta_n^i$ or $\ell = \pi_n^i$. If the marker $\boxplus$ is never placed on this pair, then $\nu^t(\ell) = \nu^0(\ell)$ and $\varphi^t(\ell) = \varphi^0(\ell)$ for all t, and

$$\varphi(\ell) = \varphi^0(\ell) = \lim_{t\to\infty} \varphi^t(\ell) \text{ and } \nu(\ell) = \mu\varphi(\ell).$$

If, on the other hand, the marker t_1 is affixed at step $\boxplus$, then after this step $\varphi^t(\ell) = \varphi^{t_1}(\ell)$ and $\nu^t(\ell) \doteq \nu^{t_1}(\ell)$ for all $t \geq t_1$ and $\nu^t(\ell) = \mu^t(\varphi^t(\ell))$. Consequently, $\varphi(\ell) = \varphi^{t_1}(\ell)$ and $\nu(\ell) = \mu\varphi(\ell)$.

Case 3. $\ell \in \bigcup_{[j,i]} \bigcup_{t\geq 0} M_\nu^t(j,i)$. By Lemma 1, in this case there exists a unique pair $[j,i]$ such that $\ell \in \bigcup_{t\geq 0} M_\nu^t(j,i)$. If starting at some step t_0, $\ell \notin M_\nu^t(j,i)$ or the pair $[j,i]$ is not used any more in Case 3 of a step of type $5t+1$ in the construction, then as in Case 2, $\varphi^t(\ell) = \varphi^{t_0}(\ell)$ and $\nu^t(\ell) = \nu^{t_0}(\ell)$. Consequently, $\varphi(\ell) = \varphi^{t_0}(\ell)$ and $\nu(\ell) = \mu\varphi(\ell)$. If, on the other hand, the pair $[j,i]$ is used infinitely many times in the construction in steps of type $5t+1$ (Case 3), and if starting at some t_0, $\ell \in M_\nu^t(j,i)$ for all $t \geq t_0$, then by Lemma 3 such a situation is possible only when the marker $\boxed{\varkappa}$ is permanently affixed to $[j,i]$.

If $\varkappa = \nu$, then $\ell = \lim_{t\to\infty} s_\nu^t(j,i)$, since all the other functions take new values infinitely often. Since $[j,i]$ is used infinitely many times in the construction in Case 3 of steps of type $5t+1$, we have $p(t,j,i) = 2$ infinitely often. But one sees easily from the construction that in this case $d_\nu^t(j,i) = s_\nu^t(j,i)$. Therefore, $\nu^t(s_\nu^t(j,i)) = \mu^t d_\mu^t(j,i)$ and $\varphi^t s_\nu^t(j,i) = d_\mu^t(j,i)$ for infinitely many t. But since by Remark 9 the functions $\lambda t s_\nu^t(j,i)$ and $\lambda t d_\mu^t(j,i)$ stabilize, we have $\lim_{t\to\infty} \varphi^t(\ell) = \lim_{t\to\infty} d_\mu^t(j,i)$ and $\nu(\ell) = \mu(\varphi(\ell))$. Since all the values, except for $\lim_{t\to\infty} d_\mu^t(j,i)$, of the functions $\lambda t \ell_\mu^t(j,i)$, $\lambda t s_\mu^t(j,i)$, $\lambda t z_\mu^t(j,i)$ are taken only finitely many times, by our construction, the assertion of the lemma is correct.

If $\varkappa = \mu$, then $\ell = d_\nu^t(j,i)$ and $\lim_{t\to\infty} s_\mu^t(j,i)$ exists, from which as in the previous case

$$\lim \varphi^t(\ell) = \lim_{t\to\infty} s_\mu^t(j,i) \quad \text{and} \quad \nu(\ell) = \mu\varphi(\ell).$$

COROLLARY. For every $\ell \in N$ there exists an $\ell' \in N$ such that $\lim_{t\to\infty} \varphi^t(\ell') = \ell$.

The proof is like that of Lemma 7; it is necessary only to consider $(\varphi^t)^{-1}$ in place of φ^t.

LEMMA 8. For all n and m, if $n \neq m$, then there exists a t_1 for which

$$\nu^{t_1}(n) \not\subseteq \nu(m) \text{ and } \nu^{t_1}(m) \not\subseteq \nu(n),$$

and hence $\nu(n) \not\subseteq \nu(m)$.

Proof. In order to prove the statement of the lemma, we consider four possible cases for n and m.

Case 1. Either n or m fails to belong to the set

$$X = \bigcup_{[j',i']} \bigcup_{t\geq 0} M_\nu^t(j',i') \cup \{\Delta_n^i, \pi_n^i \mid n \in N \text{ and } i \leq \lim_{t\to\infty} \kappa_n^t\}.$$

Let $n \notin X$. Then $\nu^t(n) = \nu^0(n)$ for all t, and since n does not occur in any of the constructions, $\nu^t(n) \cap \nu^t(m) = \emptyset$ for every t and the assertion of the lemma is proved (it suffices to take $t_1 = 0$). The case $m \notin X$ is similar.

Case 2. Either n or m belongs to the set

$$X_1 = \{\Delta_n^i, \mathfrak{N}_n^i \mid i < \lim_{t\to\infty} \kappa_n^t \text{ and } n \in N\}.$$

Assume for definiteness that $n \in X_1$. Then n can only be used once in the construction, at some step t_1, and for all $t \geq t_1$ we have $\nu^t(n) = \nu^{t_1}(n)$. Since n is subsequently not used in the construction, for all $t \geq t_1$ we have $\nu^t(n) \not\supseteq \nu^{t_1}(m)$ and $\nu^t(m) \not\supseteq \nu^{t_1}(n)$. Since at a step $t < t_1$ we have $\nu^t(n) \subseteq \nu^{t_1}(n)$, and so for all t $\nu^t(n) \not\supseteq \nu^{t_1}(m)$ and $\nu^t(m) \not\supseteq \nu^{t_1}(n)$, and consequently, $\nu(n) \not\subseteq \nu(m)$ and $\nu(m) \not\subseteq \nu(n)$.

Case 3. There exist t and a pair $[j,i]$ such that $n \in M_\nu^t(j,i) \setminus M_\nu^{t+1}(j,i)$ or $m \in M_\nu^t(j,i) \setminus M_\nu^{t+1}(j,i)$. Assume for definiteness that $n \in M_\nu^t(j,i) \setminus M_\nu^{t+1}(j,i)$. Then after step t + 1 the ν -index n will not be used again in the construction and $\nu^{t+1}(n) = \nu^{t'}(n)$ for all $t' \geq t+1$. But since the relations $\nu^{t+1}(n) \not\supseteq \nu^{t+1}(m)$ and $\nu^{t+1}(m) \not\supseteq \nu^{t+1}(n)$ hold at step t + 1 for every m and there exists an $x \in \nu^{t+1}(n)$ such that $x \notin \nu^{t+1}(m)$ for all m (and this number cannot be added again to any of the sets), we therefore have $\nu^{t+1}(n) \not\subseteq \nu^{t'}(m)$ for all $m \neq n$ and t' and $\nu^{t+1}(n) \not\supseteq \nu^{t+1}(m)$ for all $m \neq n$. Hence the conclusion of the lemma is valid.

Now for the last possible case.

Case 4. There exist pairs $[j',i'], [j'',i'']$ and a t_0 such that $n \in M_\nu^t(j',i')$ and $m \in M_\nu^t(j'',i'')$ for all $t \geq t_0$. In this case, by Lemma 2 we have starting at some step $t_1 \geq t_0$ that the same markers $\boxed{\varkappa_1}$ and $\boxed{\varkappa_2}$ are permanently present at $[j',i']$ and $[j'',i'']$, respectively.

If for at least one of the pairs $[j',i']$ or $[j'',i'']$, Case 3 of a step of type $5t+1$ is satisfied only finitely many times, then after the step τ_1, (for which Case 3 holds for the last time) an argument similar to that for Case 3 proves the lemma.

Thus, it remains for us to consider the last case when Case 3 for a step of type $5t+1$ holds infinitely often for both pairs. We consider the four possibilities for $\varkappa_1$ and $\varkappa_2$.

1. $\varkappa_1 = \varkappa_2 = \nu$. In this case $\lambda t s_\nu^t(j',i')$ and $\lambda t s_\nu^t(j'',i'')$ stabilize, and the other elements of the sets $M_\nu^t(j',i')$ and $M_\nu^t(j'',i'')$ are constantly renewed. Consequently,

$$n = \lim_{t\to\infty} s_\nu^t(j',i') \text{ and } m = \lim_{t\to\infty} s_\nu^t(j'',i'').$$

Assume that stabilization of both functions starts with step $t_2 \geq t_1$. Since $n \neq m$, $[j',i'] \neq [j'',i'']$, and since both the pairs $[j',i']$ and $[j'',i'']$ are used infinitely many times in the construction, we have $j' \neq j''$ by Lemma 5. Assuming that $j' < j''$, we prove that for every t we have $\nu^t(n) \not\supseteq \nu^{t_2}(m)$ and $\nu^t(m) \not\supseteq \nu^{t_2}(n)$. Consider an $a \in \nu^{t_2}(n)$ and $b \in \nu^{t_2}(m)$ such that $a \notin \nu^{t_2}(\ell)$ for all $\ell \neq n$ and $b \notin \nu^{t_2}(\ell)$ for all $\ell \neq m$; these elements exist by Remark 2. But then by Lemma 6, for every $t \geq t_2$ we have $a \notin \nu^t(m)$ and $b \notin \nu^t(n)$. This gives the statement of the lemma in the obvious way.

2. $\varkappa_1 = \varkappa_2 = \mu$. In this case $\lambda t d_\nu^t(j', i')$ and $\lambda t d_\nu^t(j'', i'')$ stabilize. But then for infinitely many t' and t'', respectively, we have $d_\nu^{t'}(j', i') = s_\nu^{t'}(j', i')$ and

$$d_\nu^{t''}(j'', i'') = s_\nu^{t''}(j'', i'') \qquad \text{and } \varphi^{t'} s_\nu^{t'}(j', i') = s_\mu^{t'}(j', i'),$$
$$\varphi^{t''} s_\nu^{t''}(j'', i'') = s_\mu^{t''}(j'', i'') \qquad \text{and } \nu^{t'}(n) = \mu^{t'} \varphi^{t'}(n),$$

$\varphi^{t''}(m) = \mu^{t''} \varphi^{t''}(m)$. But everything is proved as in the previous case for the elements $\lim\limits_{t\to\infty} s_\mu^t(j', i')$ and $\lim\limits_{t\to\infty} s_\mu^t(j'', i'')$.

This gives the required result.

3. $\varkappa_1 = \nu$ and $\varkappa_2 = \mu$. In this case, $\lambda t s_\nu^t(j', i')$ and $\lambda t d_\nu^t(j'', i'')$ stabilize, and the functions $\lambda t \ell_\nu^t(j'', i'')$, and $\lambda t s_\nu^t(j'', i'')$ take the value $\lim\limits_{t\to\infty} d_\nu^t(j'', i'')$ infinitely often. By Lemma 6 there exists a and b such that $a \in \nu(\lim\limits_{t\to\infty} s_\nu^t(j', i'))$ and if $a \in \nu(\ell')$, then

$$\ell' \in \bigcup_{t\geq 0} M_\nu^t(j', i'), \text{ and } b \in \mu(\lim_{t\to\infty} s_\mu^t(j'', i''))$$

while if $b \in \nu(\ell'')$ then $\ell'' \in \bigcup\limits_{t\geq 0} M_\mu^t(j'', i'')$. Let us show that

$$a \notin \mu(\lim_{t\to\infty} s_\mu^t(j'', i'')) \quad \text{and } b \notin \nu(\lim_{t\to\infty} s_\nu^t(j', i')).$$

We prove that $a \notin \mu(\lim\limits_{t\to\infty} s_\mu^t(j'', i''))$, the second assertion being proved analogously.

Assume that $a \in \mu(\lim\limits_{t\to\infty} s_\mu^t(j'', i''))$ and consider the smallest t' such that

$$\varphi^{t'} s_\nu^{t'}(j'', i'') = s_\mu^{t''}(j'', i''), \ a \in \mu^{t'}(\lim_{t\to\infty} s_\mu^t(j'', i''))$$

and $\lambda t s_\mu^t(j'', i'')$ and $\lambda t s_\nu^t(j', i')$ have already stabilized. Then there exists an ℓ_t' such that

$$\varphi^{t'}(\ell') = \lim_{t\to\infty} s_\mu^t(j'', i'') \quad \text{and } \ell' \in \bigcup_{t\geq 0} M_\nu^t(j', i'),$$

but $\varphi^{t'}(s_\nu^t(j'', i'')) = s_\mu^{t'}(j'', i'')$, and the function $\lambda x \varphi^{t'}(x)$ is not univalent. Therefore, $\ell' \in \bigcup\limits_{t\geq 0} M_\nu^t(j', i')$ and $\ell' \in \bigcup\limits_{t\geq 0} M_\nu^t(j'', i'')$, which contradicts Lemma 1. We have thus proved our assertion in this case also. The last remaining case is analogous to the one just considered, and therefore the lemma is proved.

LEMMA 9. The maps ν and μ numerate the same family and are univalent.

Proof. Let $S_\nu = \{\nu(n) \mid n \in N\}$ and $S_\mu = \{\mu(n) \mid n \in N\}$. We show that $S_\mu \subseteq S_\nu$ (the reverse inclusion is proved analogously). Let $A \in S_\mu$. Then there exists an ℓ such that $\lim\limits_{t\to\infty} \varphi^t(\ell') = \ell$ and $\nu(\ell') = \mu\varphi(\ell') = \mu(\ell)$. Thus, $\nu(\ell') = A$ and $A \in S_\nu$. The fact that the numeration ν is univalent follows directly from Lemma 8, and the numeration μ is univalent by the corollary to Lemma 7 and the univalence of ν.

LEMMA 10. The numerations ν and μ are inequivalent.

Proof. Assuming the contrary, there exists an n_0 such that the function $\lambda x K(n_0, x)$ is general recursive and the equality $\nu(x) = \mu K(n_0, x)$ holds for all x.

Consider the number n_0. If the marker $\boxplus$ is placed on n_0 then there exists an i such that $\boxplus$ is placed on $\langle n_0, i\rangle$. But in this case we have

a) $(K(n_0,\Delta^i_{n_0}) \neq \Delta^i_{n_0} \vee K(n_0,\pi^i_{n_0}) \neq \pi^i_{n_0})$ & $\nu(\Delta^i_{n_0}) = \mu(\Delta^i_{n_0})$ & $\nu(\pi^i_{n_0}) = \mu(\pi^i_{n_0})$

or

b) $K(n_0,\Delta^i_{n_0}) = \Delta^i_{n_0}$ & $K(n_0,\pi^i_{n_0}) = \pi^i_{n_0}$ & $\nu(\pi^i_{n_0}) = \mu(\Delta^i_{n_0})$,

since the marker $\boxplus$ can be affixed only in Case 1, 2, or 3 of a step of type $5t+1$. But in cases a) and b) the function $\lambda x K(n_0,x)$ can no longer be a reducing function, since the numerations ν and μ are univalent. Thus the marker $\boxplus$ is not present on n_0. Consider $i_0 = \lim_{t\to\infty} \kappa^t_{n_0}$, which exists by Lemma 2. We choose the step t_0 such that $\lambda t \Pi^i_m(t)$, $\langle m,i\rangle \leq_{lex} \langle n_0,i_0\rangle$, $\lambda t \kappa^t_{n_0}$ do not change after this step and no markers are placed on $\langle m, i\rangle \leq_{lex} \langle n_0,i_0\rangle$. Such a t_0 exists by the definition of limit, Lemma 2, and properties of the construction.

Consider a step $t_1 > t_0$ such that $K_{t_1}(n_0,\Delta^i_{n_0})$ and $K_{t_1}(n_0,\pi^i_{n_0})$, $i \leq i_0$ are defined. If $K(n_0,\Delta^{i_0}_{n_0}) \neq \Delta^{i_0}_{n_0}$ or $K(n_0,\pi^{i_0}_{n_0}) \neq \pi^{i_0}_{n_0}$, then by Case 2 of a step of type $5t+1$, the marker $\boxplus$ is affixed to n_0. But this is impossible, as we remarked above. Thus, $K(n_0,\Delta^{i_0}_{n_0}) = \Delta^{i_0}_{n_0}$ and $K(n_0,\pi^{i_0}_{n_0}) = \pi^{i_0}_{n_0}$. By the choice of t_1, after this step no other step of type $5t+1$ holds for any pair $\langle m,i\rangle \leq_{lex} \langle n_0,i_0\rangle$.

Consider a step $t_2 > t_1$ of type $5t+1$. At this step either Case 1 or Case 4 holds for the pair $\langle n_0,i_0\rangle$. But in the first case the marker $\boxplus$ must be affixed to n_0, while in the second case the pair $\langle n_0, i_0+1\rangle$ must be defined, which contradicts the choice of i_0. Hence our assumption is false.

The lemma is proved.

<u>LEMMA 11.</u> For every computable univalent numeration of a family $S = \{\nu(n) \mid n \in N\}$, there exists a recursive function g such that $\nu(n) = \xi g(n)$ or $\mu(n) = \xi g(n)$ for every n.

<u>Proof.</u> Consider a computable univalent numeration ξ of the family S. Then there exists a j such that $\xi(n) = \gamma_j(n)$ for every n. We consider the three possible cases (allowed by Lemma 5):

<u>Case A.</u> There exists an i_0 such that $[j,i_0]$ is used infinitely many times in the construction, and starting at some step the marker $\boxtimes$ is added to $[j,i_0]$ and not subsequently removed.

<u>Case B.</u> There exists an i_0 such that the pair $[j,i_0]$ is used infinitely often and the markers on $[j,i_0]$ change infinitely often.

<u>Case C.</u> There exist only finitely many i such that the pair $[j,i]$ is used in the construction, and every such pair is used only finitely many times in the construction.

In Case A, the indicated i_0 is unique by Lemma 5. Choose a step t_0 after which no pair $[j,i]$ with $i < i_0$ is used again in the construction. Such a step exists, since otherwise the marker $\boxminus$ would appear at $[j,i_0]$ and it could not be used infinitely many times in the rest of the construction.

We first show that $[j, i_0]$ satisfies the conditions of type $5t+3$ steps infinitely many times. Assuming the contrary, there exists a step $t_1 \geq t_0$, after which no step of type $5t+3$ holds for $[j, i_0]$. Then starting with this step $D\boxed{\varkappa}^t_{j,i_0}$ no longer changes. Consider the smallest $\ell_0 \in D\boxed{\varkappa}^{t_1}_{j,i_0}$ such that $\boxed{\varkappa}^t_{j,i_0}(\ell_0)$ is not defined for all t and

$$\ell_0 \notin \hat{D}_{j,i_0} = \bigcup_{t\geq 0} \bigcup_{[j',i'] <_{\ell\varkappa} [j,i_0]} M^t_{\varkappa}(j',i') \cup \{\Delta^i_n, \pi^i_n \mid n \leq j \text{ and } i \leq \lim_{t\to\infty} \kappa^t_n\},$$

where the marker $\boxed{\varkappa}$ is permanently affixed to $[j, i_0]$. Choose a step $t_2 \geq t_1$ such that $\boxed{\varkappa}^{t_2}_{j,i_0}$ is defined on all $\ell' < \ell_0$, which do not lie in the set

$$\bigcup_{t<t_2} \bigcup_{[j',i'] <_{\ell\varkappa} [j,i_0]} M^t_{\varkappa}(j',i') \cup \{\Delta^i_n, \pi^i_n \mid n \leq j \text{ and } i \leq \lim_{t\to\infty} \kappa^t_n\}$$

and such that the $\boxed{\varkappa}^{t_2}_{j,i_0}$ at these numbers does not change. After step t_2 the conditions of Case $5t+4$ will no longer hold either. Thus after step t_2 only the conditions of the steps of type $5t+1$ and $5t+2$ can be satisfied for $[j, i_0]$. After step t_2 the set $\varkappa^t(\ell_0)$ no longer changes, since it can only change when $\ell \in L^t_{\langle\varkappa, m^*, i^*\rangle}$, $t \geq t_2$, for suitable m^*, i^* and the marker $\boxed{+}$ is added to $\langle m^*, i^*\rangle$ at step t. But in this case, since $\ell \in L^t_{\langle\varkappa, m^*, i^*\rangle}$ and $\ell \in D\boxed{\varkappa}^t_{j,i_0}$, either $[j, i_0]$ or $[j, i']$, where $i' < i_0$, lie in $\prod^{i^*}_{m^*}(t)$. But since $t \geq t_2 \geq t_0$, no pair $[j, i']$, with $i' < i_0$ is used any more in the construction. Hence $[j, i_0] \in \prod^{i^*}_{m^*}(t)$. But in order for $\boxed{+}$ to be placed on $\langle m^*, i^*\rangle$ at step t, it is necessary that $\boxed{\varkappa}^t_{j,i_0}$ be completely defined on the $\langle\varkappa, m^*, i^*\rangle$-list, and consequently on ℓ_0. This contradicts our assumption. Thus $\varkappa^t(\ell_0)$ does not change any more after step t_2.

Consider $\varkappa(\ell_0) = \varkappa^{t_2}(\ell_0)$. Since γ_j is a numeration of $S_\nu = \{\nu(n) \mid n \in N\}$ and by Lemma 9 $S_\nu = S_\mu = \{\mu(n) \mid n \in N\}$, there exists $t_3 \geq t_2$ and d_0 such that $\gamma^{t_3}_j(d_0) \supseteq \varkappa(\ell_0)$. Since $[j, i_0]$ is used infinitely many times in the construction, after step t_0 for all $i' < i_0$ either the marker $\boxed{-}$ appears on the pair $[j, i']$ or $[j, i'] \in \prod^{i'_*}_{m'_*}(t)$ for all $t \geq t_0$. Then at step $T = 5t'_3 + 4$ the conditions of a step $5t+4$ holds for the pair $[j, i_0]$ and ℓ_0, and $t'_3 = c(j, t_3)$ and $\boxed{\varkappa}^T_{j,i_0}(\ell_0)$ is defined. This contradicts our assumption. Thus $\lim_{t\to\infty} D\boxed{\varkappa}^t_{j,i_0} = N$ and since step $5t+3$ holds infinitely often, $\bigcup_{t\geq 0} \boxed{\varkappa}^t_{j,i_0}$ is defined on all $\ell \in N \setminus \hat{D}_{j,i_0}$. It follows from our construction that $\boxed{\varkappa}^t_{j,i_0}(\ell)$ can only change if $\ell = \ell^t_\nu(j, i_0)$ for $\varkappa = \nu$ or $\ell = \ell^t_\mu(j, i_0)$ for $\varkappa = \mu$, and at step t the conditions of subcase 1 of Case 2 for a step of type $5t+2$ hold for $[j, i_0]$. Therefore, if

a) $\lambda t \ell^t_\varkappa(j, i_0)$ stabilizes, then we consider a step $t_4 \geq t_2$, after which stabilization occurs, and then after step $\boxed{\varkappa}^t_{j,i_0}(\ell)$ will not change. We set $t_\ell = t_4$. If on the other hand

b) $\lambda t \ell^t_\varkappa(j, i_0)$ does not stabilize, then it means that $[j, i_0]$ is used in the construction infinitely often in steps of type $5t+1$ (Case 3). For every ℓ we find a step t_ℓ such that $\ell \notin M^{t_\ell}_\varkappa(j, i_0)$ & ($\boxed{\varkappa}^{t_\ell}_{j,i_0}(\ell)$ is defined) or $\ell = \lim_{t\to\infty} s^t_\varkappa(j, i_0)$ & $\boxed{\varkappa}^{t_\ell}_{j,i_0}(\ell)$ is defined. After this step, ℓ can no longer remain equal to $\ell^t_\varkappa(j, i_0)$ for $t \geq t_0$, since $\ell^t_\varkappa(j, i_0)$ takes values in $M^{t-1}_\varkappa(j, i_0)$ only, or else is not used even once in the construction.

We now describe an algorithm for computing the reducing function g. In order to define the value of g at the point ℓ, we seek a step t'_ℓ such that either

1) $\varkappa^t(\ell)$ is defined and $t'_\ell \geq t_\ell$, or
2) $\ell \in \bigcup_{\substack{n \leq j \\ i \leq \lim_{t\to\infty} \kappa_n^t}} L^{t_5}_{\langle \varkappa, n, i\rangle}$ and $t'_\ell = t_5$, where after step t_5, for $n \leq j$ and $i \leq \lim_{t\to\infty} \kappa_n^t$

the sets $\Pi_n^i(t)$ do not change any more and the marker $\boxplus$ is not affixed to $\langle n, i\rangle$, or

3) $\ell \in M_\varkappa^{t'_\ell}(j', i')$ where $[j', i'] <_{lex} [j, i_0]$.

For each $j' < j$ such that there exists an $i_{j'}$ for which the pair $[j', i_{j'}]$ is used infinitely often in the construction, we first fix such an $i_{j'}$. Let $\mathcal{J}$ be the set of all numbers $j' < j$, for which $i_{j'}$ exists. For each $j' < j$ we fix a step $t^0_{j'}$ after which no pair $[j', i']$ with $j' < j \,\&\, (i' < i_{j'} \vee j' \notin \mathcal{J})$ is subsequently used in the construction.

Consider $\mathcal{J}_0 = \{ j' \in \mathcal{J} \mid$ on $[j', i_{j'}]$, such that the marker $\boxed{\varkappa}$ is permanently affixed starting at some step $\}$.

For each $j' \in \mathcal{J}_0$ we consider a step $t^1_{j'}$ such that some marker $\boxed{\varkappa'}$ is permanently affixed to $[j', i_{j'}]$ thereafter and $p(t^1_{j'}, j', i_{j'}) > 0$.

Now if $\varkappa' = \varkappa$, then after step $t^1_{j'}$ $\lambda t s^t_{\varkappa j}(j', i_{j'})$ stabilizes, while if $\varkappa' \neq \varkappa$, then $\lambda t d^t_{\varkappa j}(j', i_{j'})$ stabilizes after step $t^1_{j'}$. We fix $\ell_{j'} = \lim_{t\to\infty} s^t_{\varkappa j}(j', i_{j'})$ for $\varkappa = \varkappa'$ and put $\ell_{j'} = \lim_{t\to\infty} d_\varkappa(j', i_{j'})$ when $\varkappa \neq \varkappa'$ and find a $d_{j'}$ such that $\varkappa(\ell_{j'}) = \gamma_j(d_{j'})$. For each $\ell \in \bigcup_{\substack{n \leq j \\ i \leq \lim_{t\to\infty} \kappa_n^t}} L^{t_5}_{\langle \varkappa, n, i\rangle}$ we consider a number d^*_ℓ such that $\varkappa(\ell) = \gamma_j(d^*_\ell)$. Since the set

$\bigcup_{\substack{n \leq j \\ i \leq \lim_{t\to\infty} \kappa_n^t}} L^{t_5}_{\langle \varkappa, n, i\rangle}$ is finite, this will not affect the recursiveness of g.

We now define g. If Case 1 holds then we set $g(\ell) = \boxed{\varkappa}^{t'_\ell}_{j, i_0}(\ell)$. If Case 2 holds then we put $g(\ell) = d^*_\ell$. If Case 3 holds, then we consider several subcases.

<u>Subcase 3.1.</u> If $\ell \in M_\varkappa^{t'_\ell}(j', i')$ and

$$(j' = j \,\&\, i' < i) \vee (j' < j \,\&\, j' \notin \mathcal{J}) \vee (j' < j' \,\&\, j' \in \mathcal{J} \,\&\, i' < i_{j'}),$$

then we consider a step $t' = max\{t'_\ell, t^0_{j'}, t_5\}$ and find d_ℓ and t''_ℓ such that $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''_\ell}(d_\ell)$; we set $g(\ell) = d_\ell$.

<u>Subcase 3.2.</u> If $\ell \in M_\varkappa^{t'_\ell}(j', i')$ and $j' < j \,\&\, j' \in \mathcal{J} \,\&\, i_{j'} < i'$, then we find a number d_ℓ and steps t', t'', $t'' > t' \geq max\{t'_\ell, t^0_{j'}\}$ such that the marker $\boxminus$ is affixed to $[j', i']$ at step t', $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''}(d_\ell)$, and we set $g(\ell) = d_\ell$.

<u>Subcase 3.3.</u> If $\ell \in M_\varkappa^{t'_\ell}(j', i_{j'})$ and $j' \notin \mathcal{J}_0$, then we find a number d_ℓ and steps $t'' \geq t' > t'_\ell$ such that $\ell \notin M_\varkappa^{t'}(j', i_{j'})$ and $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''}(d_\ell)$, and set $g(\ell) = d_\ell$.

<u>Subcase 3.4.</u> If $\ell \in M_\varkappa^{t'_\ell}(j', i_{j'})$, $j' \in \mathcal{J}_0$ and $\lambda t M^t_\varkappa(j, i_{j'})$ does not stabilize, then we set $g(\ell) = d_{j'}$ if $\ell = \ell_{j'}$. Otherwise, we find $t'' \geq t' > t_\ell$ and a number d_ℓ such that $\ell \notin M_\varkappa^{t'}(j', i_{j'})$ and $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''}(d_\ell)$, we set $g(\ell) = d_\ell$.

Subcase 3.5. If $\lambda t M^t_{\varkappa}(j', i_{j'})$ stabilizes and conditions 3.1-3.4 do not hold, then we consider a step $t^2_{j'}$, after which the conditions of Case 3 of steps of type $5t+1$ do not hold for $[j', i_{j'}]$. Then we find d_ℓ and $t'' > t^2_{j'}$ such that $t'' \geq t'_\ell$ and $\varkappa^t(\ell) \subseteq \gamma_j^{t''}(d_\ell)$ and set $g(\ell) = d_\ell$.

We now prove that the function g defined in this way is everywhere defined and reducing. The fact that g is recursive follows from the description of the algorithm for computing g. We show that for every ℓ the value $g(\ell)$ is defined and $\varkappa(\ell) = \gamma_j g(\ell)$. Consider the smallest ℓ such that this is false. If Case 1 holds for ℓ, then $g(\ell)$ is defined. Thus $\varkappa(\ell) \neq \gamma_j g(\ell)$. But since $[j, i_0]$ satisfies the conditions of steps of type $5t+3$, infinitely often, we have for infinitely many t that $\varkappa^t(\ell) \subseteq \gamma_j^{t+1}(\boxed{\varkappa}^t_{j,i_0}(\ell))$, and as we have observed, after step t'_ℓ the value $\boxed{\varkappa}^t_{j,i_0}(\ell)$ does not change and is equal to $g(\ell)$. Therefore, $\varkappa^t(\ell) \subseteq \gamma^{t+1}(g(\ell))$ for infinitely many t, and consequently $\varkappa(\ell) \subseteq \gamma_j g(\ell)$.

Since by Lemma 8 there are no proper inclusions among the elements of S, we have $\varkappa(\ell) = \gamma_j(g(\ell))$. Thus Case 1 cannot be satisfied. Case 2 obviously cannot hold, and therefore, only the last case remains.

In Subcase 3.3, since $j' \notin J_0$ there exists by Lemma 3 a t' such that $\ell \notin M^{t'}_{\varkappa}(j', i_{j'})$ and $t' \geq t'_\ell$. Then as in Subcases 3.1 and 3.2, after step t' the set $\varkappa^t(\ell)$ no longer changes and is finite, and there exist t'' and d_ℓ (since γ_j is a numeration of S) such that $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''}(d_\ell)$. But therefore $g(\ell)$ is defined and $\varkappa(\ell) \subseteq \gamma_j(g(\ell))$. But since by Lemma 8 there can be no proper inclusions among the elements of S, we have $\varkappa(\ell) = \gamma_j(g(\ell))$. If Subcase 3.4 or 3.5 holds for ℓ then $\ell \neq \ell_{j'}$. If a marker $\boxed{\varkappa'}$ is permanently affixed to $[j', i_{j'}]$ and $\lambda t M^t_{\varkappa}(j', i_{j'})$ stabilizes, then $[j', i_{j'}]$ is no longer used from step $t''_{j'}$ on in steps of type $5t+1$ (Case 3). Therefore $\varkappa^t(\ell)$ does not change after step $t''_{j'}$ and then as before $g(\ell)$ is defined and $\varkappa(\ell) = \gamma_j g(\ell)$. If on the other hand $\lambda t M^t_{\varkappa}(j', i_{j'})$ does not stabilize but the marker $\boxed{\varkappa'}$ is permanently affixed, one sees easily from the construction that all the elements in $M^t_{\varkappa}(j', i_{j'})$ apart from $\ell_{j'}$ are renewed after a certain time. Therefore there exists a step $t' > t'_\ell$ such that $\ell \notin M^{t'}_{\varkappa}(j', i_{j'})$; but then ℓ is no longer used in the construction and $\varkappa^t(\ell)$ does not change. Consequently, there exist t'' and d_ℓ such that $\varkappa^{t'}(\ell) \subseteq \gamma_j^{t''}(d_\ell)$; but then, as before, we remark that $g(\ell)$ is defined and $\varkappa(\ell) = \gamma_j g(\ell)$.

Let us consider Case B. By Lemma 5, in this case the pairs $[j, i]$ where $i < i_0$ are only used finitely many times in the construction. Consider a step t_0 after which none of these pairs is used in the construction. Consider all the steps $t_1 < t_2 < \ldots < t_k < \ldots$ at which the marker changes on $[j, i_0]$ and $t_0 < t_1$. We denote by $\varkappa_t$ a value in $\{\nu, \mu\}$ such that at step t the marker $\boxed{\varkappa_t}$ appears on $[j, i_0]$.

Consider a number d such that

$$\boxed{\varkappa_{t_1}}^{t_1}_{j,i_0}(S^{t_1}_{\varkappa_{t_1}}(j, i_0)) = d.$$

We show that

$$\boxed{\varkappa_t}^{t}_{j,i_0}(s^t_{\varkappa_t}(j,i_0)) = d, \ t \geq t_1.$$

Take the smallest $t > t_1$ for which the above condition is false. Then $\boxed{\varkappa_{t-1}}^{t-1}_{j,i_0}(s^{t-1}_{\varkappa_{t-1}}(j,i_0)) = d$. If $t_{\kappa-1} < t < t_\kappa$ then since the marker $\boxed{\varkappa_{t-1}}$ remains on $[j,i_0]$ in step t, we have $s^t_{\varkappa_t}(j,i_0) = s^{t-1}_{\varkappa_{t-1}}(j,i_0)$, $\varkappa_t = \varkappa_{t-1}$ and $\boxed{\varkappa_t}^t_{j,i_0}(s^{t-1}_{\varkappa_{t-1}}(j,i_0))$ does not change its value. Therefore our assumption is false, and so $t = t_\kappa$ and the marker $\boxed{\varkappa_{t-1}}$ changes to $\boxed{\hat{\varkappa}_{t-1}}$ at step t_κ. But in this case $s^{t-1}_{\varkappa_t}(j,i_0) = s^t_{\varkappa_t}(j,i_0)$ and

$$\boxed{\varkappa_t}^t_{j,i_0}(s^t_{\varkappa_t}(j,i_0)) = \boxed{\varkappa_{t-1}}^{t-1}_{j,i_0}(s^{t-1}_{\varkappa_{t-1}}(j,i_0)) = d.$$

By virtue of our construction, if a marker at $[j,i_0]$ changes at step t and a marker $\boxed{\varkappa}$ is affixed, then

$$\gamma_j^{t+1}(\boxed{\varkappa}^t_{j,i_0}(s^t_\varkappa(j,i_0))) \supseteq \varkappa^t(s^t_\varkappa(j,i_0)).$$

Thus $\gamma_j(d) \supseteq \varkappa^{t_i}(s^{t_i}_{\varkappa_{t_i}}(j,i_0))$ for all $i > 0$. Since all the $\varkappa^0(\ell)$, $\ell \in N$ are pairwise distinct and nonempty, by Lemma 3 $\gamma_j(d)$ is an infinite set. Since γ_j is a numeration of S there exist ℓ_1 and ℓ_2 such that $\gamma_j(d) = \nu(\ell_1) = \mu(\ell_2)$. But $\nu(\ell_1)$ can be infinite only if the marker $\boxed{\varkappa}$ is permanently affixed to the pair $[j',i']$ starting at some step t' and $\ell_1 = \lim\limits_{t\to\infty} d^t_\nu(j',i')$ for $\varkappa = \nu$ and $\ell_1 = \lim\limits_{t\to\infty} s^t_\nu(j',i')$ for $\varkappa = \mu$. If $\varkappa = \mu$, then

$$\ell_2 = \lim_{t\to\infty} s^t_\mu(j',i'),$$

and in addition, $[j',i']$ is used in Case 3 of steps of type $5t+1$ infinitely often. Therefore, these two cases are symmetric and we consider only the one when $\ell_1 = \lim\limits_{t\to\infty} s^t_\nu(j',i')$. Consider $\ell = \lim\limits_{t\to\infty} s^t_\nu(j',i')$ and a step t_0 after which $s^t_\nu(j',i')$ does not change further, the marker $\boxed{\nu}$ is permanently affixed to $[j',i']$, and $p(t_0,j',i') > 0$. Now take a step $t_1 > t_0$ such that the marker $\boxed{+}$ is placed on $\langle n^*, i^*\rangle$ at step t_1 and $[j',i'] \in \Pi^{i^*}_{n^*}(t_1)$. For every

$$\ell' \notin \bigcup_{t\geq 0} M^t_\nu(j',i')$$

we consider elements $a_{\ell'} \in \varkappa^{t_1}(\ell')$ and $a_{\ell'} \notin \varkappa^{t_1}(\ell'')$ for all $\ell'' \neq \ell'$. We prove that for all $t \geq t_1$ the following conditions hold:

(1) $a_{\ell'} \notin \nu^t(s^t_\nu(j',i')) \cup \nu^t(\ell^{t'}_\nu(j',i'))$.

(2) If there do not exist steps $t'_1 < t'_2$ such that (*) $t'_1 \leq t < t'_2$ and at step t'_1 the marker $\boxed{+}$ is placed on $\langle n^{**}, i^{**}\rangle$ and $[j',i'] \in \Pi^{i^{**}}_{n^{**}}(t'_1)$, while at step t'_2 for $[j',i']$ the conditions of Case 2 of a step of type $5t+2$ are satisfied on the pair $\langle n^{**}, i^{**}\rangle$, then $a_{\ell'} \notin \nu^t(r^t_\nu(j',i')) \cup \nu^t(d^t_\nu(j',i'))$.

(3) If steps $t'_1 < t'_2$ satisfying condition (*) exist but $p(t,j',i') \neq 1$, then $a_{\ell'} \notin \nu^t(d^t_\nu(j',i'))$.

We prove the last assertion by induction on t. Assume the result has already been proved for all $t' < t$; we prove it for t. If there do not exist $t'_1 < t'_2$ such that (*) holds, or they do exist but $t_1' < t < t_2'$, then $M_\nu^t(j',i') = M_\nu^{t-1}(j',i')$ and for all $x \in M_\nu^t(j',i')$ we have $\nu^t(x) = \nu^{t-1}(x)$. Consequently, by the induction hypothesis all the conditions also hold for t.

If there exist $t_1' < t_2'$ such that (*) holds and $t_1' = t$, then by a property of our construction, for $t-1$, there exists no pair $t_1'' < t_2''$ such that (*) holds, and therefore $a_{\ell'} \notin x^{t-1}(x)$ for all $x \in M_\nu^{t-1}(j',i')$. But at step t_1', when a marker $\boxplus$ is affixed the values

$$s_\nu^{t_1'}(j'',i''),\ d_\nu^{t_1'}(j'',i''),\ r_\nu^{t_1'}(j'',i''),\ \ell_\nu^{t_1'}(j'',i'')$$

remain unchanged for all pairs $[j'',i'']$, and

$$\nu^t(\ell_\nu^t(j',i')) = \nu^{t-1}(\ell_\nu^t(j',i')) \cup \nu^{t-1}(s_\nu^t(j',i')) \cup \{a\},$$

$$\nu^t(s_\nu^t(j',i')) = \nu^{t-1}(s_\nu^t(j',i')) \cup \nu^{t-1}(r_\nu^t(j',i')) \cup \{b\},$$

where the numbers $a \neq b$ still are not contained in any set. Therefore condition (1) is satisfied. Condition (2) also holds, since its condition is false.

Let us prove condition (3). If $p(t_1,j',i') \neq 1$, then $d_\nu^t(j',i')$ is equal either to $s_\nu^t(j',i')$ for $p(t,j',i') = 2$ or to $d_\nu^{t-1}(j',i')$ for $p(t,j',i') = 2$ & $x^t(d_\nu^{t-1}(j',i')) = x^t(d_\nu^{t-1}(j',i'))$. The foregoing and the induction assumption imply that in all cases $a_{\ell'} \notin x^t(d_\nu^t(j',i'))$ and the assertion is proved. This result implies that for all $t \geq t_1$ and $\ell' \notin \bigcup_{t \geq 0} M_\nu^t(j',i')$ we have $\nu^t(\ell') \nsubseteq \nu^t s_\nu^t(j',i')$, and since $\ell_1 = \lim s_\nu^t(j',i')$, we have $\nu^t(\ell') \nsubseteq \nu(\ell_1)$. But for all K

$$x^{t_K}(s_{x_{t_K}}^{t_K}(j,i_0)) \subseteq \gamma_j(d) = \nu^t(\ell_1),$$

and therefore there exists a K_0 such that for all $K \geq K_0$ with $x_{t_K} = \nu$ we have $s_\nu^{t_K}(j,i_0) \in \bigcup_{t \geq 0} M_\nu^t(j',i')$. By Lemma 1, the sets $M_{j,i}^\nu$ for different pairs are disjoint, and therefore $j' = j$ and $i' = i_0$. But by hypothesis the marker $\boxed{\nu}$ is permanently affixed to $[j',i']$ starting at some step, while the same is not true for $[j,i_0]$. This contradiction shows that Case B cannot hold.

We now consider Case C. We show that it cannot be satisfied either. To this end we consider a step t_0 after which no pair $[j,i]$ with $i \in N$ is used in the construction. We observe that after step $5j+5$ there always exists a pair $[j,i]$ without the marker $\boxminus$ if Z_j does not occur beginning at some step. If, starting at step t_1, Z_j appears on $\langle x,\ell,\ell_1,\ell_2 \rangle$ but

$$x^{t_1}(\ell) \subseteq \gamma_j^{t_1+1}(\ell_1) \text{ and } x^{t_1}(\ell) \subseteq \gamma_j^{t_1+1}(\ell_2)$$

and the number ℓ is not subsequently used in the construction, then $x^t(\ell) = x^{t_1}(\ell)$ for all $t \geq t_1$. Thus, $x(\ell) \subseteq \gamma_j(\ell_1)$ and $x(\ell) \subseteq \gamma_j(\ell_2)$; but γ_j is a numeration of S, and there can be no proper inclusions in S (by Lemma 8). Therefore, $\gamma_j(\ell_1) = \gamma_j(\ell_2) = x(\ell)$. But $\ell_1 \neq \ell_2$, and by assumption γ_j is a univalent numeration, and therefore Z_j does not occur at any step. Take a pair $[j,i_0]$ such that the marker $\boxminus$ is not affixed to it and let i_0 be maximal with this property. For there exists no pair $\langle n^*,i^* \rangle$ such that

$[j,i_0] \in \Pi^{i^*}_{n^*}(t_0)$, since otherwise a larger pair with the marker $\boxminus$ would be defined; since χ_j does not appear, the marker $\boxminus$ can only be added because of a pair $[j,i]$ with a smaller second coordinate — but its second coordinate must be greater than i_0, since otherwise the marker $\boxminus$ would be affixed to $[j,i_0]$ also.

We now take smallest $i_0' \leq i_0$ such that there is no marker $\boxminus$ on $[j,i_0']$ and $[j,i_0'] \notin \Pi^{i^*}_{n^*}(t_0)$ for all pairs $\langle n^*,i^*\rangle$. If there exists a t such that $\boxed{\varkappa}^t_{j,i_0'}$ is completely defined on $D\boxed{\varkappa}^{t_0}_{j,i_0'}$, then either a step of type $5t+3$ holds for the pair $[j,i_0']$ after step t_0, or else there exists an $i_0'' < i_0'$ such that (**) no marker $\boxminus$ appears $[j,i_0'']$, $[j,i_0''] \in \Pi^{i^*}_{n^*}(t_0)$ for some pair $\langle n^*,i^*\rangle$, and function $\boxed{\varkappa}^t_{j,i_0''}$ is not completely defined on the $\langle \varkappa,n^*,i^*\rangle$-list, where the marker $\boxed{\varkappa}$ occurs on $[j,i_0'']$. Since the pair $[j,i]$ is not used after step t_0, only the second case remains. We take the smallest i_0'' satisfying condition (**).

Consider a step $t_1 > t_0$ such that the set $\Pi^{i^{**}}_{m^{**}}(t)$ for $\langle m^{**},i^{**}\rangle \leq_{lex} \langle n^*,i^*\rangle$ no longer changes after step and $\langle m^{**},i^{**}\rangle$ is not subsequently used in the construction. This is possible by Lemma 2. We remark that $\boxed{\varkappa}^{t_1}_{j,i_0''}$ is completely defined on $L^{t_1}_{\langle \varkappa,m^{**},i^{**}\rangle}$ for every pair $\langle m^{**},i^{**}\rangle <_{lex} \langle n^*,i^*\rangle$. Indeed, if this is not so consider the smallest pair $\langle m^{**},i^{**}\rangle$ such that $\boxed{\varkappa}^{t_1}_{j,i_0''}$ is not completely defined on $L^{t_1}_{\langle \varkappa,m^{**},i^{**}\rangle}$.

Consider all the elements $\ell_1,\ldots,\ell_k$ in $L^{t_1}_{\langle \varkappa,m^{**},i^{**},j,i_0''\rangle}$. If the value $\boxed{\varkappa}^{t_1}_{j,i_0''}(\ell_i)$, where $1 \leq i \leq k$ is defined, then by Remarks 11 and 12

$$\varkappa^{t_1}(\ell_i) \subseteq \gamma_j(\boxed{\varkappa}^{t_1}_{j,i_0''}(\ell_i)) \quad \text{and} \quad \varkappa^{t}(\ell_i) = \varkappa^{t_1}(\ell_i)$$

for all $t \geq t_1$, $1 \leq i \leq k$. Since γ_j is a numeration of S, for all ℓ_i there exists a d_i such that $\varkappa^{t_1}(\ell_i) \subseteq \gamma_j(d_i)$. Consider a step $t_2 \geq t_1$ such that $\gamma_j^{t_2}(d_i) \supseteq \varkappa^{t_1}(\ell_1)$ and consider the step $T = 5t+2$, where $\ell(t) = j$ and $r(t) > t_2$. Then the conditions of Case 3 or Case 1 hold for $[j,i_0'']$ at this step. But χ_j cannot be affixed, and therefore $\boxed{\varkappa}^T_{j,i_0''}$ is defined on $L^{t_1}_{\langle \varkappa,m^{**},i^{**},j,i_0''\rangle}$. But this contradicts the fact that no pair $[j,i]$ can participate in the construction any longer. This contradiction proves our assertion. If there is no marker $\boxplus$ on $\langle n^*,i^*\rangle$ then we can show as above that $\boxed{\varkappa}^{t_1}_{j,i_0''}$ is completely defined on $L^{t_1}_{\langle \varkappa,n^*,i^*\rangle}$. It thus remains only to consider the case when the marker $\boxplus$ is present on $\langle n^*,i^*\rangle$.

Let $\ell_1 \prec \ell_2 \prec \ldots \prec \ell_k$ be all the elements in $L^{t_1}_{\langle \varkappa,n^*,i^*\rangle}$ and consider a step $t_2 \leq t_1$ at which a marker $\boxplus$ is affixed to $\langle n^*,i^*\rangle$. Assume for definiteness that $\varkappa = \mu$. In this case

$$\ell_1 = r^{t_1}_{\mu}(j,i_0''),\ \ \ell_2 = s^{t_1}_{\mu}(j,i_0''),\ \ \ell_3 = \ell^{t}_{\mu}(j,i_0'').$$

We remark that by the definition of Case 3 of a step of type $5t+1$,

$$\mu^{t_2-1}(\ell_{i+1}) \subseteq \mu^{t_2}(\ell_{i+1}) \quad \text{and} \quad \mu^{t_2-1}(\ell_{i+1}) \subseteq \mu^{t_2}(\ell_i),$$

where $0 < i < k$ and $\mu^t(\ell_i)$ for all $i \leq k$ are no longer changed after step t_2. Since $\boxplus$ appears on $\langle n^*,i^*\rangle$ and $[j,i_0''] \in \Pi^{i^*}_{n^*}(t_2)$, we have $\boxed{\mu}^{t_2-1}_{j,i_0''}(\ell_i)$ is defined for all i and

$$\mu^{t_2-1}(\ell_i) \subseteq \gamma_j^{t_2}(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i)).$$

Since after step t_2 the μ-indices $\ell_1, \ell_2, \ldots, \ell_K$ are no longer used in the construction, there are no $\ell \notin \{\ell_1, \ldots, \ell_K\}$ such that $\mu^t(\ell) \supseteq \mu^{t_2-1}(\ell_i)$, where $1 < i \leq K$ and $t \geq t_2$, and therefore the set $\gamma_j(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i))$ for $1 < i \leq K$ can contain only one of the two elements $\text{æ}(\ell_i)$ or $\text{æ}(\ell_{i-1})$. If for all $0 < i \leq K$ we have $\mu^t(\ell_i) \subseteq \gamma_j(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i))$, then we find d_0 and $t_3 > t_1$ such that

$$\mu^{t_2}(\ell_i) \subseteq \gamma_j^{t_3}(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i))$$

for $1 < i \leq K$ and $\mu^{t_2}(\ell_1) \subseteq \gamma_j^{t_3}(d_0)$. Then at step $T = 5c(j, max\{d_0, t_3\}) + 2$, Case 2 holds for the pair $[j, i_0'']$ and the construction will be carried out for some pair $[j, i']$, which is impossible. If on the other hand for some i, $1 < i \leq K$ we have

$$\mu^{t_2}(\ell_{i-1}) \subseteq \gamma_j(\boxed{\text{æ}}_{j,i_0''}^{t_2}(\ell_i)),$$

then we consider the smallest such i. If $i > 2$, then

$$\gamma_j(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i)) = \mu^{t_2}(\ell_{i-1})$$

and

$$\gamma_j(\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_{i-1})) = \mu^{t_2}(\ell_{i-1}).$$

But

$$\boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_i) \neq \boxed{\text{æ}}_{j,i_0''}^{t_2-1}(\ell_{i-1}),$$

which contradicts the fact that γ_j is a univalent numeration. Hence $i = 2$.

But we then consider a $t_3 \geq t_1$ such that

$$\mu^{t_2}(\ell_1) \subseteq \gamma_j^{t_3}(\boxed{\text{æ}}_{j,i_0''}^{t_2}(\ell_2)).$$

Case 2 holds for $[j, i_0'']$ at step $5c(j, t_3) + 2$, and the pair $[j, i']$ will be used in the construction. But this is impossible by the choice of t_0. This contradiction completes the proof of the lemma.

We now conclude the proof of the main theorem. We define $S = \{\nu(n) | n \in N\}$. By Lemma 9, ν and μ are univalent computable numerations of the family S. By Lemma 10, they are not equivalent. Consider a univalent computable numeration ξ of S. By Lemma 11, there exists a recursive function g such that $(\forall n)(\nu(n) = \xi g(n))$ or $(\forall n)(\mu(n) = \xi g(n))$. Consequently, $\nu \leq \xi$ or $\mu \leq \xi$, and since the numeration ξ is univalent, it is minimal [1] and hence $\xi \equiv \mu$ or $\xi \equiv \nu$.

The theorem is proved.

The following corollary can be proved if we make the construction more complicated by introducing K numerations $\nu_1, \ldots, \nu_K$, K markers $\boxed{\nu_1}, \ldots, \boxed{\nu_K}$, and K-tuples of functions $\ell_K^t(j,i)$, $s_K^t(j,i)$, $z_K^t(j,i)$, $d_K^t(j,i)$ and functions $\varphi_1^t, \varphi_2^t, \ldots, \varphi_{K-1}^t$.

COROLLARY 1. For every $\kappa \in N$ there exists a family S of recursively enumerable sets which has up to equivalence precisely κ univalent computable numerations.

A further improvement of the above construction enables us to prove:

COROLLARY 2. There exists a family S such that the family $\hat{S}$ of all univalent computable numerations (up to equivalence) of S is computable, but such that $\hat{S}$ contains infinitely many inequivalent numerations.

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