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PRODUCT OF TWO GROUPS WITH NILPOTENT SUBGROUPS OF INDEX AT MOST 2

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Introduction

Suppose a finite group G is the product of its subgroups A and B, i.e., $G=AB$. Assume that A has a nilpotent subgroup H of index $\leqslant 2$ and B has a nilpotent subgroup K of index $\leqslant 2$. Is G then solvable?

Solvability of such a group has been proved in certain special cases, namely; if H and K are cyclic, G is solvable by Theorem 1 of [1]; if B is cyclic or H and B are Dedekind, solvability of G follows from [2]. The Wielandt–Kegel theorem [3, p. 674] can also be regarded as a special case of the above situation.

In the present paper we investigate the genral situation. In §1 we establish a number of properties of a hypothetical nonsolvable group of least order admitting the indicated factorization. On the basis of these properties we prove in §§2, 3, and 4 the following

MAIN THEOREM. Suppose a finite group G is the product of its subgroups A and B. Assume that A has a nilpotent subgroup H of index $\leqslant 2$, and B has a subgroup K of index $\leqslant 2$. Then G is solvable in each of the following cases: 1) K is primary; 2) K is cyclic; 3) B is Dedekind; 4) $A=H$ and all subgroups of K are B-invariant.

The above-mentioned results of [1, 2] and also Theorems 13.6.1, 13.6.2, and 13.10.1 in Scott [4] are special cases of our theorem. In §5 of this present paper we prove a proposition which, with the aid of a result of [5], enables us to generalize Theorem 13.10.3 of [4].

We will need the following notation. Suppose Y is a subgroup of a finite group X. Then:

Y_X is the largest X-invariant subgroup contained in Y;
Y^X is the smallest X-invariant subgroup containing Y;
X_p is a Sylow p-subgroup of X, where p is a prime;
$X_{p'}$ is a Sylow p-complement of X;
$Z(X)$ is the center of X.

We call a subgroup Y quasicentral in X if all subgroups of Y (including Y itself) are X-invariant (cf. the definition in [4, p. 396]).

The remaining notation and definitions are clear from the text and when necessary can be found in [6]. We will frequently use the following lemmas.

LEMMA A [1, Lemma 2]. Suppose a finite group G is the product of two subgroups A and B. If A has even order and a cyclic Sylow 2-subgroup and $A\cap B=1$, then G contains a subgroup of index 2.

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LEMMA B [1, Lemma 3]. Suppose G is a doubly transitive group of permutations and H is the stabilizer of some point. Then all involutions in the center of H are contained in $O_{2',2}(G)$.

LEMMA C [5]. Suppose a finite group $G=AB$, where A and B are p-closed subgroups. If $A_p \subseteq B_p^G$, then A_p^G is a p-group.

§1. A Study of the General Situation

In this section the object of study is a finite group G, satisfying the following conditions:

F1. G is the product of two subgroups A and B;

F2. A contains a nilpotent subgroup H of index ≤ 2, and B contains a nilpotent subgroup K of index ≤ 2;

F3. K possesses a group-theoretic property θ inherited by subgroups and factor groups;

F4. G is nonsolvable, but each nontrivial subgroup and factor group of G satisfying F1-F3 is solvable.

Without specifying the property θ, we establish in §1 some properties of a group satisfying F1-F4. Note that if property θ is trivial, i.e., if no restrictions are imposed on K in F3, then a group satisfying F1-F4 is a counterexample of least order to the general problem posed at the beginning of this paper.

Thus, we suppose in this section that G is a finite group satisfying F1-F4. Let A_2 and B_2 denote Sylow 2-subgroups of A and B, respectively, which are permutable and whose product A_2B_2 is a Sylow 2-subgroup of G (see [3, p. 676]). Let N be a minimal G-invariant subgroup and N_2 a Sylow 2-subgroup of N contained in A_2B_2.

LEMMA 1.1. $R(G)=1$.

Proof. Let $R=R(G)$ be the product of all solvable G-invariant subgroups. Since $G/R=AR/R \cdot BR/R$, and $AR/R \cong A/A\cap R$, and $BR/R \cong B/B\cap R$, it follows that AR/R contains a nilpotent subgroup $HR/R \cong H/H\cap R$ of index ≤ 2, and BR/R contains a nilpotent subgroup $KR/R \cong K/K\cap R$ of index ≤ 2. Property θ is inherited by factor groups, hence KR/R possesses property θ. If $R\neq 1$, then, by F4, G/R is solvable, hence so is G. Contradiction.

LEMMA 1.2. Subgroup A is not a 2-group and neither is B.

Proof. If, for example, B is a 2-group, then $G=AB=A_{2'}(A_2B_2)$ is solvable by the Wielandt-Kegel theorem.

LEMMA 1.3. If C is a proper subgroup of G containing A, then C is solvable and $(C\cap B)_B=1$. If D is a proper subgroup of G containing B, then D is solvable and $(D\cap A)_A=1$.

Proof. Since $C=A(C\cap B)$ and $C\cap K$ is a nilpotent subgroup possessing property θ and having index ≤ 2 in $C\cap B$, it follows that C is solvable. Also, $G=CB$, and if $X=(C\cap B)_B\neq 1$, then $X^G=X^{BC}=X^C\subseteq C$, hence X^G is a nonidentity G-invariant solvable subgroup. This contradicts Lemma 1.1. Thus, $(C\cap B)_B=1$.

Subgroup D is handled in a similar way.

LEMMA 1.4. $(|A|,|B|)=2^m$.

Proof. Assume the contrary. Suppose p is an odd prime dividing the order of A and the order of B. A Sylow p-subgroup A_p of A is contained in H and is A-invariant. If $A_p^G A \neq G$, then, by Lemma 1.3, A^G is solvable, which contradicts Lemma 1.1. Thus, $A_p^G A = G$ and $G/A_p^G \cong A/A \cap A_p^G$ is a p'-group. A Sylow p-subgroup B_p of B is different from the identity and is B-invariant. Clearly, $B_p \subseteq A_p^G$. By Lemma C, B_p^G is a p-group, which contradicts Lemma 1.1. Thus, the orders of A and B have no common odd divisors different from unity.

LEMMA 1.5. $AN=G=BN$ and G is isomorphic to some group of automorphisms of N. In addition, the order of G/N divides 2^m and $\langle A_{2'},B_{2'}\rangle \subseteq N$.

Proof. Since $R(G)=1$, it follows that N is nonsolvable and is the direct product of isomorphic simple groups. By Lemma 1.3, the subgroups AN and BN coincide with G, hence $A/A\cap N \cong G/N \cong B/B\cap N$. By Lemma 1.4, G/N is a 2-group of order dividing 2^m. Thus, $A_{2'} \subseteq N$ and $B_{2'} \subseteq N$.

The centralizer C of N is G-invariant and, since the center of N is equal to 1, we have $C\cap N=1$. Therefore, C is isomorphic to a subgroup of G/N, hence $C=1$ and G is isormorphic to some group of automorphisms of N.

LEMMA 1.6. The Sylow 2-subgroup N_2 of N is nondihedral.

Proof. Assume that N_2 is dihedral. Then N is simple. If N is isomorphic to A_7, then either $A_{2'}$ or $B_{2'}$ is a nonprimary Hall subgroup of A_7, which contradicts Hall's theorem (see [3, p. 177]). By a theorem of [7], N is isomorphic to $PSL(2,p^n)$, where p is an odd prime and $p^n > 5$.

The order of $PSL(2,p^n)$ is equal to $\frac{1}{2}p^n(p^n-1)(p^n+1)$, and $PSL(2,p^n)$ contains a dihedral subgroup X of order p^n+1. A Sylow p-subgroup P of $PSL(2,p^n)$ is elementary Abelian, self-centralizing, and its normalizer M has order $\frac{1}{2}p^n(p^n-1)$, and M/P is cyclic. The remaining Sylow subgroups of $PSL(2,p^n)$ of odd orders are cyclic (see [3, Chap. II]).

We may assume, without loss of generality, that $P\subseteq H$. Then $B_{2'}$ is cyclic. Since P is A-invariant, it follows that $N_G(P)$ contains A, and, by Lemma 1.3, $N_G(P)\cap B_{2'}=1$. Subgroup H centralizes P, hence $N_G(P)/C_G(P)$ is a 2-group.

Since $M\subseteq N_G(P)$ and $M\cap C_G(P)\subseteq N\cap C_G(P)=C_N(P)=P$, it follows that M/P is also a 2-group. Let the order of M/P be 2^k. Then $p^n=2^{k+1}+1$ and $|X|=2(2^k+1)$. A Sylow 2-

subgroup of M is not a Sylow subgroup of N, hence $M \cap X = 1$ and $MX = N$. But this contradicts Lemma A.

§2. Subgroup K is is Primary

The aim of this section is to prove the following theorem.

THEOREM 2.1. If A contains a nilpotent subgroup H of index $\leqslant 2$, and B contains a primary subgroup K of index $\leqslant 2$, then the finite group $G = AB$ is solvable.

Proof. We assume that Theorem 2.1 is false and apply the results of §1. Suppose θ is the property of being primary and G is a group satisfying F1-F4. Then G is a counter-example of least order to Theorem 2.1. Group G possesses all of the properties listed in Lemmas 1.1-1.6, in particular, it follows from Lemma 1.2 that the order of K is odd. Let K be a p-group, where p is an odd prime.

Assume that $AK = G$. Since the order of H is even, $H_2 \cap Z(A_2)$ is different from the identity and the centralizer of a nonidentity element of this intersection has index equal to a power of p in G. By Burnside's lemma (see [3, p. 491]), G is nonsimple. By Lemma 1.5, N contains K, hence $N = (N \cap A)K$ and N is solvable. This contradicts Lemma 1.1.

Therefore, AK does not coincide with G. Hence, $A \cap B = 1$ and $|B : K| = 2$. By Lemma A, G contains a subgroup of index 2, and, since $(|A|, |B|) = 2$, it follows from Lemma 1.5 that $|G : N| = 2$.

Assume that $H_2 \cap N_2 \cap Z(N_2) = 1$. This means that $H_2 \cap N_2$ contains no N_2-invariant nonidentity subgroups. Therefore, the representation of N_2 by permutations on the cosets modulo the subgroup $H_2 \cap N_2$ is faithful of degree $|N_2 : H_2 \cap N_2|$. The index of H_2 in $A_2 B_2$ is equal to 2 (if $A = H$) or 4. If $H_2 \subseteq N_2$, then $H_2 \cap Z(N_2) = 1$ implies $|N_2| = 2$. Contradiction. If H_2 is not contained in N_2, then $|A_2 B_2 : H_2| = |N_2 : N_2 \cap H_2|$, i.e., $N_2 \cap H_2$ has index 2 or 4 in N_2. Now, N_2 can be isomorphically embedded in the symmetric group on four letters. Since a Sylow 2-subgroup of the latter is dihedral, this contradicts Lemma 1.6.

Thus, $H_2 \cap N_2 \cap Z(N_2)$ is different from the identity. The centralizer $C = C_N(i)$ of an involution i of this intersection contains the subgroup $\langle H_2, N_2 \rangle$, hence the index of C in N is equal to a power of p. By Burnside's lemma, N is nonsimple. But N is a direct product of isomorphic simple groups, hence $i = xy$, where $x \neq 1$ is an element of a simple group X, that is a direct factor of N, and y is an element of a complement Y to X in N. Clearly, $C_X(x) = C_N(i) \cap X$. Thus, $C_X(x)$ contains $\langle H_2 \cap X, N_2 \cap X \rangle$, and the index of $C_X(x)$ in X is a power of the prime p. Contradiction.

Theorem 2.1 is proved.

§3. Subgroup B is Dedekind

THEOREM 3.1. If A contains a nilpotent subgroup H of index ≤ 2, and B is Dedekind, then the finite group $G = AB$ is solvable.

Proof. Assume the theorem is false. Let θ denote the following property: K is Dedekind and $K=B$. Then a group G satisfying F1-F4 is a counterexample of least order to Theorem 3.1. Group G possesses the properties listed in Lemmas 1.1-1.6. Let us study this group in greater detail.

LEMMA 3.1. Subgroup A is maximal in G, $A \cap B = 1$ and $|A : H| = 2$.

Proof. Let M be a maximal subgroup of G containing A. Then M is solvable and $M \cap B = 1$, by Lemma 1.3. This means that $A = M$ and $A \cap B = 1$. It follows from the Wielandt-Kegel theorem that $A \neq H$.

LEMMA 3.2. The order of B is odd.

Proof. Assume the contrary, i.e., $B_2 \neq 1$. Assume that H has even order. Let S be a 2-subgroup of G that contains A_2 as a subgroup of index 2. If $H_2 \cap Z(S) \neq 1$, then the centralizer of an involution of this intersection properly contains A, which contradicts Lemma 3.1. Thus, $H_2 \cap Z(S) = 1$. The representation of S by permutations on the cosets module H_2 is faithful of degree $|S : H_2| = 4$. Therefore, S is a subgroup of the symmetric group on four letters and $|H_2| = 2$. Now B_2 has index 4 in A_2B_2. Since A_2B_2 is nondihedral by Lemma 1.6, we have $B_2 \cap Z(A_2B_2) \neq 1$. Therefore, the centralizer D of an involution of $B_2 \cap Z(A_2B_2)$ contains B and H_2, hence $H_2 \subseteq (D \cap A)_A$, which contradicts Lemma 1.3. Thus, H has odd order.

By Lemma A, G contains a subgroup of index 2. It follows from Lemma 1.5 that $|G : N| = 2$. Now, H is a Hall subgroup of G and nonprimary by Theorem 2.1. If P is any Sylow subgroup of H, then $N_N(P) = N_G(P) \cap N = A \cap H = H$. By a theorem of Wielandt (see [3, p. 444]), N contains a normal complement to H. But this is impossible in N.

Lemma 3.2 is proved.

LEMMA 3.3. G is simple.

Proof. Since the orders of A and B are relatively prime, Lemma 1.5 implies that G is simple.

LEMMA 3.4. Subgroup $\mathcal{B}$ is nonprimary and contains no nonidentity cyclic Sylow subgroup.

Proof. Subgroup $\mathcal{B}$ is nonprimary by Theorem 2.1. Since H_2 is a nonidentity A-invariant subgroup, the center of A has even order. The representation of G by permutations on the cosets modulo A is faithful of degree $|G:A|=\mathcal{B}$. Since A is maximal in G, it follows that G is a primitive permutation group. The subgroup $\mathcal{B}$ is regular, hence if $\mathcal{B}$ contains a cyclic Sylow subgroup $\neq 1$, then, by Theorem 25.4 of [8], G is doubly transitive. Then all involutions in the center of A are contained in $R(G)$ (see Lemma B). Contradiction.

LEMMA 3.5. If X is a subgroup containing $H_{2'}$, then X is a π-group, where $\pi=\pi(A)$ and is solvable.

Proof. Assume that a prime p divides the order of X and the order of $\mathcal{B}$. We may assume, without loss of generality, that a Sylow p-subgroup X_p of X is contained in $\mathcal{B}$, otherwise we can replace $\mathcal{B}$ by a conjugate subgroup. Group X contains the subgroup $Y=\langle H_{2'}, X\cap\mathcal{B}\rangle$, and, by Lemma IV.4.12 of [3], the normalizer of Y is factorizable. Since the orders of A and $\mathcal{B}$ are relatively prime, $Y=(Y\cap A)(Y\cap\mathcal{B})$. By induction, Y is solvable, hence there exists a Hall subgroup $H_{2'}Q$ of Y, where Q is a Sylow p-subgroup of Y. Now $Q\subseteq\mathcal{B}^g$ for some $g\in G$ and Q is $\mathcal{B}^g$-invariant. Since $H_{2'}$ is A-invariant and $G=A\mathcal{B}^g$, it follows from a lemma of Kegel (see [3, p. 677]) that G is nonsimple. This contradicts Lemma 3.3. Thus, X is a π-group. Since $H_{2'}\subseteq X$, the Wielandt-Kegel theorem implies that X is solvable.

LEMMA 3.6. Subgroup $H_{2'}$ is not quasicentral in X.

Proof. Assume that all subgroups of $H_{2'}$ are A-invariant. Since $H_2\cap Z(A_2)\subseteq Z(A)$, we have $Z(A)\neq 1$ and, by a theorem of Ito [9], there is an element $b\neq 1$ in $\mathcal{B}$ such that $\mathcal{D}=C_G(b)$ properly contains $\mathcal{B}$. Clearly, $\mathcal{D}\cap H_{2'}=1$. Assume that $\mathcal{D}\cap H\neq 1$. Then the centralizer of an involution of $\mathcal{D}\cap H$ contains $H_{2'}$ and b, which contradicts Lemma 3.5. Thus, $\mathcal{D}\cap H=1$. But then $G=H\mathcal{D}$ is nonsimple by Lemma A, which contradicts Lemma 3.3.

LEMMA 3.7. If X is a subgroup containing $H_{2'}$, then $X\subseteq A$.

Proof. Assume that $V=O_p(X)\neq 1$, where p is an odd prime. Then V is H'-invariant and $H\subseteq N_G(V)$. By Lemma 3.5, $N_G(V)$ is a π-group, where $\pi=\pi(A)$, hence $|N_G(V):H|=1$ or 2 and H is $N_G(V)$-invariant. Consequently, $X\subseteq N_G(V)\subseteq N_G(H)=A$.

Now suppose $O_p(X)=1$. Since X is solvable, it follows that $\mathcal{F}=O_2(X)\neq 1$ and $\mathcal{F}$ is the

Fitting subgroup of X. Therefore, $\mathcal{F} \supseteq C_X(\mathcal{F})$ (see [3, p. 277]).

Assume that the order of $C_G(\mathcal{F})$ is divisible by some odd prime $p \in \pi(A)$. Since $X \subseteq N_G(\mathcal{F})$ and $C_G(\mathcal{F})$ is $N_G(\mathcal{F})$-invariant, $XC_G(\mathcal{F})$ is a subgroup. Since the Sylow p-subgroups of $XC_G(\mathcal{F})$ and X have the same order, it follows from the equality $|XC_G(\mathcal{F})||C_X(\mathcal{F})| = |X||C_G(\mathcal{F})|$ that p divides the order of $C_X(\mathcal{F})$. Contradiction. Therefore, the order of $C_G(\mathcal{F})$ is not divisible by the odd primes in $\pi(A)$.

The 2-group $\mathcal{F}$ is contained in some Sylow 2-subgroup A_2^g, $g \in G$, and $\mathcal{F}$ is not contained in H_2^g. Therefore, a nonidentity element $x \in \mathcal{F} - H_2^g$ induces an automorphism of order 2 on $H_{2'}^g$. If $yx = xy$ for some $y \in H_{2'}^g$, then $y \in C_G(\mathcal{F})$. Contradiction. Thus, x induces a regular automorphism on $H_{2'}^g$. But then $H_{2'}^g$ is quasicentral in A^g (see [4, Theorem 12.6.8]). This contradicts Lemma 3.6.

Lemma 3.7 is proved.

LEMMA 3.8. The centralizers of all involutions are 2-nilpotent.

Proof. If i is an involution of H, then $C_G(i) \supseteq H_{2'}$ and, by Lemma 3.7, $C_G(i)$ is contained in A. Consequently, the group $C_G(i)$ has a normal 2-complement. Since G is simple, the involutions of $A - H$ are conjugate to involutions of H (see [10, p. 265]).

Since a Sylow 2-subgroup of G is nondihedral, it follows from a theorem of Gorenstein [11] that G is isomorphic to $PSL(3, 4)$ or $Sz(2^n)$, $n \geq 3$. The order of $PSL(3, 4)$ is equal to $2^6 3^2 5 \cdot 7$, and all Sylow subgroups of $Sz(2^n)$ of odd orders are cyclic. Since B is nonprimary, B has a cyclic Sylow subgroup $\neq 1$, which contradicts Lemma 3.4.

Theorem 3.1 is completely proved.

COROLLARY. If A is nilpotent and K contains a quasicentral subgroup B of index $\leqslant 2$, then the finite group $G = AB$ is solvable.

Proof. Suppose A is nonmaximal in G. If M is a maximal subgroup of G containing A, then $M \cap K = 1$, $|M \cap B| = 2$, and $|M \cap A| = 2$. Since $G = MK$, solvability of G follows from Theorem 3.1.

Now suppose A is maximal in G. By Theorem 2.1, A is nonprimary, and, by a theorem of Thompson (see [3, p. 445]), the order of A is even. If the Sylow 2-subgroup A_2 of A is G-invariant, then G is solvable by induction. If A_2 is not G-invariant, then A_2 is a Sylow subgroup of G and $N_G(A_2) = A$. Then G possesses a normal 2-complement (see [3, p. 444]), i.e., G is again solvable. The corollary is proved.

§4. Subgroup K is Cyclic

In this section we prove the following

THEOREM 4.1. If A contains a nilpotent subgroup H of index ≤ 2, and B contains a cyclic subgroup K of index ≤ 2, then the finite group $G = AB$ is solvable.

Proof. Assume that Theorem 4.1 is false. Suppose θ denotes the property of being cyclic and G satisfies F1-F4. Then G is a counterexample of least order to Theorem 4.1. Group G possesses all of the properties listed in Lemmas 1.1-1.6, in particular, it follows from Lemma 1.5 that a minimal G-invariant subgroup N contains a nonidentity cyclic Hall subgroup $B_{2'}$. Therefore, N is simple. Let us continue the investigation of G.

LEMMA 4.1. Suppose C is a proper subgroup of G containing A. Then C is solvable and $C \cap K = 1$. If $G = CK$, then C is maximal in G and the center of C has odd order.

Proof. By Lemma 1.3, C is solvable, and, since K is quasi-central in B, we have $C \cap K = 1$.

Suppose $G = CK$. Any proper subgroup of G containing C intersects K in the identity, hence C is maximal in G. Subgroup C contains no nonidentity G-invariant subgroups, hence the representation of G by permutations on the cosets modulo C is faithful of degree $|K|$. It follows from the minimality of C that G is a primitive permutation group. Subgroup K is regular in this representation. If the order of K is a prime, then G is either solvable or doubly transitive (see [3, p. 609]). If the order of K is not a prime, then G is doubly transitive (see [8, p. 65]).

Thus, in both cases G is doubly transitive. By Lemma B, all involutions of the center of C are contained in $R(G)$. Therefore, the center of C must have odd order.

LEMMA 4.2. If P is a Sylow subgroup of A of odd order, then $N_G(P)/C_G(P)$ has order 1, 2, or 4.

Proof. Clearly, $A \subseteq N_G(P)$. If $N_G(P) \neq A$, then $N_G(P) \cap K = 1$ and $|N_G(P) \cap B| = 2$. Since $H \subseteq C_G(P)P$, it follows that $N_G(P)/C_G(P)P$ has order 1, 2, or 4.

LEMMA 4.3. A_2 is not a Sylow subgroup of G.

Proof. Assume that A_2 is a Sylow subgroup of G. Then B_2 is contained in A_2 and $G = AK$. Since H_2 is different from the identity, it follows that $H_2 \cap Z(A_2)$ is different from the identity and is contained in the order of A. This contradicts Lemma 4.1.

LEMMA 4.4. The order of H_2 is at most 2. If $|H_2|=2$, then A_2 is elementary Abelian of order 4 and there exists a dihedral subgroup S of order 8 containing A_2.

Proof. Since A_2 is not a Sylow subgroup of G, A_2 is contained as a subgroup of index 2 in some 2-group S.

Assume that $H_2 \cap Z(S) \neq 1$, and let x be an involution in this intersection. Then $C= C_G(x) \supseteq \langle A, S \rangle \neq A$. Since $G=CK$, we have a contradiction to Lemma 4.1. Thus, $H_2 \cap Z(S)=1$.

Suppose $H_2 \neq 1$. Then H_2 is not S-invariant and H_2 contains no nonidentity S-invariant subgroups. In particular, H_2 is a proper subgroup of A_2 and $|S:H_2|=4$. The representation of S by permutations on the cosets modulo H_2 is faithful of degree 4, hence S is a subgroup of the symmetric group on four letters. Thus, S is dihedral of order 8 and $|H_2|=2$. If A_2 is cyclic, then $Z(S)=H_2$. Contradiction. Therefore, A_2 is elementary Abelian.

LEMMA 4.5. $A \cap B=1$, $|A:H|=2$, and $|B:K|=2$.

Proof. Since $A \cap K=1$, we have $|A \cap B| \leq 2$. If $G=AK$ or $G=HB$, then G is solvable by Theorem 3.1 or its corollary. Therefore, $A \cap B=1$ and $|A:H|=2$, $|B:K|=2$.

LEMMA 4.6. N_2 is non-Abelian.

Proof. Suppose N_2 is Abelian. Then N_2 is elementary Abelian. It follows from Lemma 4.4 that $H_2=1$. Now $(|A|,|B|)=2$ and N has index 2 in G.

If $N_2 \cap K=1$, then $|K_2|=2$ and $|G_2|=8$. In this case N_2 is dihedral of order 4. This contradicts Lemma 1.6.

Since K_2 is cyclic, we have $|N_2 \cap K|=2$ and $|K_2| \leq 4$, and also $|G_2| \leq 16$. Consequently, $|N_2|=8$. The centralizer of an involution in $N_2 \cap K$ contains B, hence it is solvable and $N \cong PSL(2,8)$ (see [12]). But the factor group $\mathrm{Aut}\,PSL(2,8)/PSL(2,8)$ has order 3, hence G cannot be embedded in $\mathrm{Aut}\,PSL(2,8)$. This contradicts Lemma 1.5.

LEMMA 4.7. The order of K is even, $K_2 \cap N \neq 1$, and the centralizer of an involution in $K_2 \cap N$ is solvable.

Proof. Assume that $K_2=1$. If $H_2=1$, then a Sylow 2-subgroup of G has order 4. If $H_2 \neq 1$, then the dihedral group S in Lemma 4.4 is a Sylow subgroup of G. In both cases we have a contradiction to Lemma 1.6. Therefore, the order of K is even.

Assume that $K_2 \cap N=1$. Then K_2 can be isomorphically embedded in G/N. Since $A_2/A_2 \cap N$ is an elementary Abelian group isomorphic to G/N, we have $|K_2|=2$. Now $N \neq G$

and $|N_2| \le 8$. Contradiction.

Thus, $K_2 \cap N \ne 1$. The centralizer of an involution in $K_2 \cap N$ contains B , hence is solvable.

LEMMA 4.8. N_2 is not semidihedral and is not wreathed.

Proof. Suppose N_2 is semidihedral or wreathed. Since N is simple, all involutions of N are conjugate (see [13, pp. 10-11]). The centralizer of an involution in $K_2 \cap N$ is solvable, hence N is isomorphic to $PSL(3,3)$, M_{11} , or $PSU(3,3^2)$.

The order of $PSL(3,3)$ is equal to $2^4 3^3 13$, and a Sylow 3-subgroup is non-Abelian. The normalizer of a Sylow 13-subgroup has order 3·13. If $N \cong PSL(3,3)$, then 3^3 divides the order of A and 13 divides the order of B , and also $|N \cap B| = 13$. This contradicts Lemma 4.7.

The order of M_{11} is equal to $2^4 3^2 5 \cdot 11$, and the normalizer of a Sylow 3-subgroup has order $2^4 3^2$, and the normalizer of a Sylow 5-subgroup has order 30. If $N \cong M_{11}$, then 11 divides neither $|A|$ nor $|B|$. Contradiction.

The order of $PSU(3,3^2)$ is equal to $2^5 3^3 7$ and a Sylow 7-subgroup is self-centralizing. If $N \cong PSU(3,3^2)$, then $|N \cap B| = 7$, which contradicts Lemma 4.7.

LEMMA 4.9. $|N_2| \ge 2^6$.

Proof. If $|N_2| \le 2^5$, then N_2 is Abelian, dihedral, semidihedral, or wreathed [14]. Contradiction.

LEMMA 4.10. $K_2 \cap Z(A_2 B_2) = 1$.

Proof. Assume the contrary. Then the centralizer of an involution $i \in K_2 \cap Z(A_2 B_2)$ is solvable and contains a Hall subgroup $A_2 B$. Clearly, $G = H(A_2 B)$, and it follows from Lemma 1.3 that the order of H is odd. Therefore, $|A_2| = 2$ and $|G:N| = 2$.

Since N contains H, we have $N = H(N \cap A_2 B)$. Subgroup N is simple; hence $N \cap A_2 B$ contains no cyclic subgroups of index ≤ 2 . Thus, N does not contain K , and $N \cap K$ has index 4 in $N \cap A_2 B$.

By Theorem 2.1 H is nonprimary, and, by Wielandt's theorem (see [3, p. 444]), H has a Sylow subgroup P, whose normalizer $X = N_N(P)$ properly contains H . Since $A_2 \cap N = 1$, and also $X = N_G(P) \cap N$ and $A \subseteq N_G(P)$, it follows that $|X:H| = 2$.

Let X_2 be a Sylow 2-subgroup of X contained in N_2 . If X_2 is not contained in $N_2 \cap B$, then $N_2 = X_2 \leftthreetimes (N_2 \cap B)$ and $N = X(N \cap B)$ is solvable by induction. Contradiction. Therefore,

$X_2 \subseteq N_2 \cap B$. If $X_2 \subseteq N_2 \cap K$, then $X_2 = \langle i \rangle$, and since $X_2 \subseteq N_2 \cap B$ and $X \cap N \cap A_2B = \langle i \rangle \subseteq Z(N \cap A_2B)$ we have $\langle i \rangle^N = \langle i \rangle^X \subseteq X$ and $R(N) \neq 1$. Contradiction. Thus, $N_2 \cap B = X_2 \leftthreetimes (N_2 \cap K)$ and $N_2 \cap B = X_2 \leftthreetimes (N \cap K)$.

Group B_2 contains a cyclic subgroup K_2 of index 2, hence all maximal subgroups of B_2 are either Abelian or dihedral (see [10, p. 191]). If $N_2 \cap B$ is dihedral, of order >4, then $|N_2 \cap K| \geqslant 4$, and since all elements of $N_2 \cap B - N_2 \cap K$ have order 2, it follows that $N_2 \cap K$ is a characteristic subgroup of N_2, hence $N_2 \cap K$ is N_2-invariant.

Thus, either a Sylow 2-subgroup N_2 of the simple group N contains an Abelian maximal subgroup or the commutator subgroup of N_2 is cyclic. By a theorem of [15], N_2 is Abelian, dihedral semidihedral, or wreathed. Contradiction.

Lemma 4.10 is proved.

<u>LEMMA 4.11</u>. The order of H_2 is equal to 2, and a Sylow 2-subgroup of G has order $\leqslant 2^6$.

<u>Proof</u>. Since $K_2 \cap Z(A_2B_2) = 1$, it follows that K_2 contains no nonidentity A_2B_2-invariant subgroups. The representation of A_2B_2 by permutations on the cosets modulo K_2 is faithful of degree $|A_2B_2 : K_2|$. If $|H_2| = 1$, then $|A_2B_2 : K_2| = 4$ and A_2B_2 is dihedral of order 8. This contradicts Lemma 1.6.

Thus, $H_2 = 1$ and $|A_2B_2 : K_2| = 8$. Therefore, A_2B_2 can be isomorphically embedded in the symmetric group of degree 8. Since a Sylow 2-subgroup of the latter has exponent 2^3, we have $|K_2| \leqslant 2^3$ and $|A_2B_2| \leqslant 2^6$.

It follows from Lemma 4.9 that G is simple with a Sylow 2-subgroup of order 2^6. Since K_2 is cyclic of order 8, we obtain from [16], using a theorem of Fong (see [14]), that G is isomorphic to M_{12}. In M_{12} the normalizer of Sylow 3-subgroup has order $3^3 \cdot 4$, and the normalizer of a Sylow 5-subgroup has order $2^3 \cdot 5$. Since the order of M_{12} is equal to $2^6 \cdot 3^3 \cdot 5 \cdot 11$, it follows that 11 divides neither $|A|$ nor $|B|$. Contradiction.

Theorem 4.1 is completely proved.

Theorems 2.1, 3.1, 4.1 and the corollary of Theorem 3.1 taken together constitute a proof of the main theorem of this paper, which was stated in the introduction.

§5. Product of Groups with Quasicentral Subgroups of Odd Indices

<u>LEMMA 5.1</u>. Suppose p is the smallest prime divisor of the order of a group G and P is a Sylow p-subgroup of G. If P is quasicentral in G, then G is p-decomposable with a Dedekind Sylow p-subgroup. In particular, if $p>2$, then P is contained in the center of G.

The *proof* is by induction on the order of G. Suppose x is an element of order p in P. Then $\langle x\rangle$ is G-invariant and so is $C_G(\langle x\rangle)$. If $C_G(\langle x\rangle)\neq G$, then $G/C_G(\langle x\rangle)$ is isomorphic to a group of automorphisms of $\langle x\rangle$. Since $Aut\langle x\rangle$ is cyclic of order $p-1$ and is the smallest number dividing the order of G, we have a contradiction. Therefore, $C_G(\langle x\rangle)=G$, and is contained in the center of G. The factor group $G/\langle x\rangle$, is, by induction, p-decomposable, hence G is also p-decomposable. Since P is Dedekind and, when $p>2$, Abelian, it follows that P is contained in the center of G when $p>2$.

Now, applying [5], we obtain

<u>COROLLARY 1</u>. If A and B contain quasicentral subgroups of odd indices, then the group is solvable.

<u>COROLLARY 2</u>. If A is 2-decomposable with a modular Sylow 2-subgroup, and B contains a quasicentral subgroup of odd index, then the group $G=AB$ is solvable.

Corollary 2 generalizes Theorem 13.10.3 in Scott [4].

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