PROPERTIES OF THE JOIN OF VARIETIES OF ALGEBRAS

G. V. Dorofeev

UDC 519.48

By the *join of varieties* $\mathfrak{M}$ and $\mathfrak{N}$ of algebras over a commutative associative ring Φ we mean the smallest variety $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}$, containing $\mathfrak{M}$ and $\mathfrak{N}$. This operation was considered in [1-3], where it was shown, in particular, that the varieties of accessible and generalized accessible algebras, and also the varieties of standard and generalized standard algebras studied by A. Albert, R. Schafer, and E. Kleinfeld with coauthors, can be obtained by means of the join operation from the varieties of associative, alternative, commutative, and Jordan algebras.

The representation of some variety $\mathfrak{P}$ as a join of varieties $\mathfrak{M}$ and $\mathfrak{N}$ solves in a specific sense the identity problem in a free algebra $F\mathfrak{P}$ of the variety $\mathfrak{P}$: An identity f is satisifed in the algebra $F\mathfrak{P}$ if and only if it is satisfied in the free algebras $F\mathfrak{M}$ and $F\mathfrak{N}$ of the varieties $\mathfrak{M}$ and $\mathfrak{N}$.

There naturally arises the question of the possibility of carrying over other properties of varieties to their join. A simple argument, given below, shows that the join of varieties having finite basis rank also has finite basis rank. At the same time, the join of varieties of finite axiomatic rank can have infinite axiomatic rank. Also, the join of Spechtian varieties is Spechtian. On the other hand, the join of varieties, each of which is defined by a finite number of identities, need not have a finite number of defining identities.

In addition, we show in this paper that certain theorems involving relations among different types of nilpotency carry over to the join of varieties.

1°. Suppose $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}$ are varieties of algebras over a ring Φ, $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}$ and A_k, B_k, C_k are free algebras of these varieties on k generators. Let $\mathfrak{M}_k, \mathfrak{N}_k, \mathfrak{P}_k$ denote the varieties generated by these algebras. Clearly, $\mathfrak{M}_k \subset \mathfrak{P}_k$, $\mathfrak{N}_k \subset \mathfrak{P}_k$, hence the T-ideals of these varieties are related by the inclusion $T(\mathfrak{P}_k) \subset T(\mathfrak{M}_k) \cap T(\mathfrak{N}_k)$; we will show that the reverse inclusion also holds.

Indeed, suppose an identity $f(x_1,\ldots,x_n)$ is satisifed in $\mathfrak{M}_k$ and $\mathfrak{N}_k$; this means that for any elements $h_1(x_1,\ldots,x_k),\ldots,h_n(x_1,\ldots,x_k)$ of a free nonassociative algebra the varieties $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the identity $f(h_1(x_1,\ldots,x_k),\ldots,h_n(x_1,\ldots,x_k))$. But this identity is then satisfied in $\mathfrak{P}$, hence the identity $f(x_1,\ldots,x_n)$ is satisfied in $\mathfrak{P}_k$. In other words, $T(\mathfrak{M}_k) \cap T(\mathfrak{N}_k) \subset T(\mathfrak{P}_k)$, hence $T(\mathfrak{P}_k)=T(\mathfrak{M}_k) \cap T(\mathfrak{N}_k)$. Thus, $\mathfrak{P}_k=\mathfrak{M}_k+\mathfrak{N}_k$.

Recall that the *basis rank* of a variety $\mathfrak{M}$ is the smallest number $\tau=\tau_b(\mathfrak{M})$, for which $\mathfrak{M}_\tau=\mathfrak{M}$ [4]. From the above argument we at once obtain

<u>THEOREM 1</u>. $\tau_b(\mathfrak{M}+\mathfrak{N}) \leq max\{\tau_b(\mathfrak{M}), \tau_b(\mathfrak{N})\}$.

As is well known, a variety $\mathfrak{M}$ of algebras is called *Spechtian* if every algebra of this variety possesses a finite basis of identities. In other words, every ascending sequence

Translated from Algebra i Logika, Vol. 16, No. 1, pp. 24-39, January-February, 1977. Original article submitted November 4, 1976.

0002-5232/77/1601-0017$07.50 © 1978 Plenum Publishing Corporation

of T -ideals of a free algebra of this variety stabilizes. We will show that this property carries over to the join of varieties.

THEOREM 2. The join of two Spechtian varieties is Spechtian.

Proof. Suppose the varieties $\mathfrak{M}$ and $\mathfrak{N}$ are Spechtian, $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}$, and in a free algebra C of the variety $\mathfrak{P}$ there is given an increasing sequence

$$T_1 \subset T_2 \subset T_3 \subset \dots \tag{1}$$

of T -ideals. The algebra C is a subdirect sum of free algebras A and B of $\mathfrak{M}$ and $\mathfrak{N}$, hence in C there are T -ideals I and K such that $I \cap K = 0$, $C/I \simeq A$, $C/K \simeq B$.

By hypothesis, the ascending sequence

$$(T_1+I)/I \subset (T_2+I)/I \subset (T_3+I)/I \subset \dots$$

of T -ideals of A stabilizes, so for some ℓ we have the equalities $(T_\ell+I)/I=(T_{\ell+1}+I)/I=\dots$, from which it follows that $T_\ell+I = T_{\ell+1}+I = \dots$

On the other hand, to the ascending sequence of T -ideals $T_1 \cap I \subset T_2 \cap I \subset \dots$ of C correspnds an ascending sequence

$$(T_1 \cap I + K)/K \subset (T_2 \cap I + K)/K \subset (T_3 \cap I + K)/K \subset \dots$$

of T -ideals of B, hence from some m on we have the equalities

$$(T_m \cap I + K)/K = (T_{m+1} \cap I + K)/K = \dots,$$

from which it follows that $T_m \cap I+K = T_{m+1} \cap I+K=\dots$. At the same time, for any i we have $(T_i \cap I) \cap K = 0$, so that from our equalities, in view of the modularity of the lattice of ideals, we obtain $T_m \cap I = T_{m+1} \cap I$.

Thus, we have shown that $T_n+I=T_{n+1}+I=\dots$, $T_n \cap I = T_{n+1} \cap I = \dots$ for $n=\max\{\ell,m\}$, hence, using once again the modularity of the lattice of ideals, we obtain $T_n = T_{n+1} = \dots$, which means that the sequence (1) stabilizes.

The theorem is proved.

2°. In this section we will show that the join of the variety $\mathcal{C}omm$ of commutative algebras and the variety $\mathcal{S}ol_2$ of solvable algebras of index 2 cannot be defined by a finite system of identities. Recall that an algebra is called *solvable of index 2* if it satisfies the identity $(xy)(zt)=0$.

THEOREM 3. Suppose a variety $\mathfrak{P}$ of algebras over a ring Φ is defined by a system of identities

$$[xy, zt] = 0, \tag{2}$$

$$[(xy)(zt)T_{u_1} \dots T_{u_m}, v] = 0 \quad (m=0,1,2,\dots), \tag{3}$$

$$([x,y]R_{z_1}\dots R_{z_m})(tu)=0 \quad (m=0,1,2,\dots), \tag{4}$$

where T is either a right or left multiplication by an element of the algebra.

Then $\mathfrak{P}=\mathrm{Comm}+\mathrm{Solv}^2$.

Proof. First note that $\mathfrak{P}$ satisfies the system of identities

$$([x,y]T_{z_1}\dots T_{z_m})(tu)=0 \quad (m=0,1,2,\dots). \tag{5}$$

Indeed, if, for example, $T_{z_i}=L_{z_i}$, then

$$([x,y]T_{z_1}\dots T_{z_{i-1}}L_{z_i}T_{z_{i+1}}\dots T_{z_m})(tu)=$$
$$=([x,y]T_{z_1}\dots T_{z_{i-1}}R_{z_i}T_{z_{i+1}}\dots T_m)(tu)-([[x,y]T_{z_1}\dots T_{z_{i-1}},z_i]T_{z_{i+1}}\dots T_{z_m})(tu),$$

and the proof of the identities (5) is easily completed, using (4) and (2), by induction on the number of left multiplications in the product.

Let A be a free algebra of $\mathfrak{P}$ with set of free generators X. Regarding X as linearly ordered, consider in the free commutative algebra $\bar{A}=A/A'$, where A' is the ideal of A generated by all commutators, the set $\bar{B}$ of regular (in the sense of [5]) words lying in the ideal generated by the set $\bar{A}^2\bar{A}^2$. Let B denote the set of monomials of A sent into a word of $\bar{B}$ under the homomorphism $\xi: A\to A/A'$, which is identical on X. We will prove that B generates as a Φ-module the ideal C of A generated by the set A^2A^2.

Suppose w is an arbitrary monomial of the ideal C and $w=w_1w_2$. We will show by induction on the degree n of w that $w\in B$. For $n=4$ we have $w_1=x_ix_j$, $w_2=x_kx_\ell$, and the identities (4) (with $m=0$) and (2) allow us to permute the factors in w_1 and in w_2, and also to permute w_1 and w_2 themselves. By means of these permutations the element w can be reduced to a form w' such that $w'\xi$ is a regular word of $\bar{A}$. That is to say, $w'\xi\in\bar{B}$, hence $w'\in B$, i.e., $w\in B$.

Now suppose $n\geq 4$; assume that any monomial in C of degree $\leq n$ belongs to B, and suppose the monomial $w=w_1w_2$ has degree $n+1$. If the degree of w_1 is 1, then $w_2\in C$ and, by the induction assumption, $w_2\in B$; we may obviously assume that $w_2\xi\in\bar{B}$. But then the word $w_2\xi\cdot w_1\xi$ is regular, i.e., $w_2\xi\cdot w_1\xi\in\bar{B}$, hence $w_2w_1\in B$; using (3), we obtain $w=w_1w_2=w_2w_1\in B$. The case where w_2 has degree 1 is handled analogously.

Finally, if both w_1 and w_2 have degree greater than 1, then they belong to A^2, and then identities (5) and (2) allow us to permute the factors in each arbitrarily and also to permute them with each other. Clearly, by means of these transformations the element w can be reduced to a form w' such that $w'\xi\in\bar{B}$ and therefore, $w=w'\in B$.

Thus, every monomial $w\in C$ belongs to B, so that the set B indeed generates the ideal C as a Φ-module.

Now let s be an arbitrary element of $C\cap A'$; then $s=\sum_{i=1}^{n}\alpha_i b_i$ $(\alpha_i\in\Phi, b_i\in B)$ and $s\xi=\sum_{i=1}^{n}\alpha_i(b_i\xi)=0$. But the regular words $b_i\xi$ in $\bar{A}$ are linearly independent, hence $\alpha_i=0$, so that $i=0$. That is to say, $C\cap A'=0$, which means that $\mathfrak{P} = \mathit{Comm} + \mathit{Solv}^2$.

The theorem is proved.

Note that the system of identities (2)-(4) is not independent: it suffices, for example, to require the fulfillment of the identities (3) only for right multiplications, i.e., to replace the system of identities (3) by the system

$$[(xy)(zt)R_{u_1}\dots R_{u_m}, v]=0 \quad (m=0,1,2,\dots), \tag{6}$$

and derive the remaining identities (3) from (6).

Nevertheless, the system (2)-(4) is not equivalent to any finite subsystem. To prove this we construct the following example.

Let $X=\{x_1, x_2,\dots\}$ be a set, ordered in accordance with the increase of indices, containing more than one element, and let $n\geq 1$ be a fixed integer. Let B denote the set of associative words in alphabet X of length at most $n+4$, and define on the set A of linear combinations of elements of B with coefficients from a given ring Φ a multiplication $*$ by defining it on the basis B as follows.

Suppose $\xi,\eta\in B$ are words of length k and ℓ , respectively; we call a word $x_i x_j$ *regular* if $i\leq j$, and we write $\overline{x_i x_j}= x_j x_i$.

Put:

a) for $k+\ell\leq n+2$

$$\xi*\eta = \begin{cases} \xi\eta, & \text{if } k>1,\ \ell=1; \\ \eta\xi, & \text{if } k=1,\ \ell>1; \\ 0, & \text{if } k\geq 2,\ \ell\geq 2; \end{cases}$$

b) for $k+\ell=n+4$

$$\xi*\eta = \begin{cases} \eta\xi, & \text{if } k=2,\ \xi \text{ regular;} \\ \eta\bar{\xi}, & \text{if } k=2,\ \xi \text{ nonregular;} \\ \xi\eta, & \text{if } \ell=2,\ \eta \text{ regular;} \\ \xi\bar{\eta}, & \text{if } \ell=2,\ \eta \text{ nonregular;} \\ 0, & \text{if } k\neq 2,\ \ell\neq 2; \end{cases}$$

c) all other products equal to zero.

We will prove that the resulting algebra A satisfies identities (2) and (3) and the identities (4) with $m\neq n$. Note first that for any elements $x,y,z,t\in B$ for which the sum of the degrees is at most $n+2$, we have $(x*y)*(z*t)=0$, hence such elements satisfy all of the identities (2)-(4). Moreover, any product in which the sum of the degrees of the factors is equal to $n+3$ or greater than $n+4$ is equal to zero. It remains, therefore to consider elements for which the sum of the degrees is equal to $n+4$.

Suppose the sum of the degrees of $x,y,z,t\in B$ is equal to $n+4$. Assume these elements do not satisfy (2). Then $x*y\neq 0, z*t\neq 0$, and it follows from a) that at least one factor in each of these nonzero products has degree 1; furthermore, it follows from b) that another of the elements has degree 1, and then the remaining fourth element has degree greater than 1, since $n+4 \geq 5$. It also follows from a) that the elements x and y , as well as z and t , are equivalent, so we need only consider the single case $y = x_p$, $z = x_q$, $t = x_r$.

For $q \leq r$ we have

$$[x*y, z*t] = xx_p * x_q x_r - x_q x_r * xx_p = xx_p x_q x_r - xx_p x_q x_r = 0,$$

and similarly for $r < q$ we have

$$[x*y, z*t] = xx_p * x_q x_r - x_q x_r * xx_p = xx_p x_r x_q - xx_p x_r x_q = 0,$$

which contradicts our assumption. This contradiction shows that A satisfies identity (2).

That A satisfies the identities (3) for monomials of summary degree $n+4$ is obvious, since the first element of the commutator on the left-hand side has degree at most $n+3$, hence is equal to zero either by a) or by c).

Suppose, finally, that $m \neq n$ and assume that an element $v = [x,y] R_{z_1} \dots R_{z_m} * (t*u)$, in which the sum α of the degrees of the monomials $x, y, z_1, \dots, z_m, t, u$ is equal to $n+4$ is nonzero. Clearly, $m<n$, otherwise $\alpha \geq m+4 > n+4$. It follows from a) that $x, y, z_1, \dots, z_m$ have degree 1, and since $m+2 < n+2$, the degree of $t*u$ is greater than 2, hence $v=0$ by b), which, by assumption, is false. This contradiction shows that A satisfies the identities (4) for any $m \neq n$.

On the other hand,

$$\begin{aligned}[x_1, x_2] R_{x_3} \dots R_{x_{n+2}} * (x_{n+3} * x_{n+4}) &= \\ &= x_1 x_2 x_3 \dots x_{n+2} * x_{n+3} x_{n+4} - x_2 x_1 x_3 \dots x_{n+2} * x_{n+3} x_{n+4} = \\ &= x_1 x_2 x_3 \dots x_{n+2} x_{n+3} x_{n+4} - x_2 x_1 x_3 \dots x_{n+2} x_{n+3} x_{n+4} \neq 0.\end{aligned}$$

Thus, A satisfies all of the identities (2)-(4) except the identity (4) corresponding to $m=n$. This means that it is impossible to discard any of the identities (4) from the system (2)-(4), hence the given system is not equivalent to any finite subsystem.

Thus, the variety $\mathcal{C}omm + \mathcal{S}olv^2$ cannot be defined by a finite system of identities, even though each component variety is defined by a single identity.

We will now construct, for each natural number n , and example of an algebra such that every subalgebra with n generators belongs to the variety $\mathcal{P}$, but the algebra itself does not. It follows from the existence of such a series of examples that $\mathcal{P}$ has infinite axiomatic rank [4].

Let $X = \{x_1, x_2, \dots\}$ be a set, ordered in accordance with the increase of indices, and let $n \geq 1$ be a fixed integer. For each associative word ξ in alphabet X containing no repeated letters we denote by $|\xi|$ the word of the same composition in which the letters appear in increasing order; also, we write $x_i \notin \xi$, if x_i does not appear in the composition of ξ.

Denote by B_1 the set of associative words in alphabet X, in each of which all letters $x_i \in X$ are distinct and, from the third on, appear in increasing order, and by B_2 the set of words of the form $\xi(x_i x_j)$, where $\xi \in B_1$, the lenght of ξ is greater than 2, and $x_i, x_j \notin \xi$, $i<j$. On the set A of linear combinations of elements of the set $B = B_1 \cup B_2$ with coefficients from some ring Φ containing 1/2 we define a multiplication $*$ by defining it on the basis B as follows:

a) $x_i * x_j = x_i x_j$ for any distinct x_i and x_j;

b) $x_k * \xi = \xi * x_k = (-1)^t x_i x_j |\eta x_k|$, where t is the number of inversions of indices of the generators in the word ηx_k, for any $\xi = x_i x_j \eta \in B_1$ (where η can be empty) and any $x_k \notin \xi$;

c) $x_i x_j * \xi = \xi * x_i x_j = \xi(|x_i x_j|)$ for any $\xi \in B_1$ of length greater than $n+2$, and any $x_i, x_j \notin \xi$;

d) all other products are equal to zero.

Note first that the resulting algebra A satisfies the identities

$$[x*y, z] = 0, \tag{7}$$

$$[(x*y)*(z*t)]*u = u*[(x*y)*(z*t)] = 0, \tag{8}$$

from which (2) and (3) follow at once.

At the same time, identity (4) with $m = n+1$ does not hold in A: indeed, according to a), $[x_1 x_2] = x_1 x_2 - x_2 x_1$, $x_{n+4} * x_{n+5} = x_{n+4} x_{n+5}$, and, applying b) and c), we obtain

$$\begin{aligned}[x_1, x_2] R_{x_3} \dots R_{x_{n+3}} * (x_{n+4} * x_{n+5}) &= \\ = x_1 x_2 x_3 \dots x_{n+3} * (x_{n+4} x_{n+5}) - x_2 x_1 x_3 \dots x_{n+3} * x_{n+4} x_{n+5} &= \\ = x_1 x_2 x_3 \dots x_{n+3} (x_{n+4} x_{n+5}) - x_2 x_1 x_3 \dots x_{n+3} (x_{n+4} x_{n+5}).\end{aligned}$$

Thus, A does not belong to $\mathfrak{P}$.

We will prove that for $m \leqslant n$ identity (4) holds in A. Assume the contrary: suppose that for some set of monomials we have

$$f = [x, y] R_{z_1} \dots R_{z_m} * (t * u) \neq 0. \tag{9}$$

It follows at once from (7) and (8) that all of the elements $x, y, z_1, \dots, z_m$ belong to X, and when $m \neq 0$ the elements t and u also belong to X, otherwise the last multiplication in f is performed in accordance with d), which contradicts (9). However, $m+2 \leqslant n+2$, hence the last product in f corresponds to d) anyway, which again contradicts (9). Therefore, $m = 0$.

On the other hand, when $m=0$ we obtain from (8) that at least one of t, u belongs to X ; then we may assume that $t*u$ belongs to B_1 , and the last multiplication in f is performed either in accordance with c) or d), and in both cases $f=0$, which contradicts (9). Thus, when $m \leqslant n$ identity (4) indeed holds in A .

Now observe that the product

$$f = [x,y] R_{z_1} \ldots R_{z_m} * (t * u) \tag{10}$$

is a skew-symmetric function on A of the arguments $z_1,\ldots,z_m$. Indeed, if all of the elements $z_1,\ldots,z_m$ are monomials, then, in view of (7) and (8), $f=0$, if at least one of them does not belong to X ; but then it follows from b) that f is skew-symmetric on the set of monomials of A . But f is multilinear and is therefore skew-symmetric on all of A .

Finally, suppose C is a subalgebra of A having a generating system S consisting of n elements. We will prove that C satisfies all of the identities (4). Suppose the product (10) is nonzero for some set of monomials in the generators of C . Then, by what we have proved, $m>n$. If all of the elements $z_1,z_2,\ldots,z_m$ belong to S , then they are not all distinct, hence $f=0$ by skew-symmetry. If some z_i does not belong to S , then it is a product of at least two elements of S and therefore belongs to A^2 . But then $f=0$ in view of (8). Hence, C satisifes all of the identities (2)-(4), and so $C \in \mathfrak{P}$.

Thus, A does not belong to $\mathfrak{P}$, whicle each subalgebra having n generators does belong to this variety. In other words, $A \in \mathfrak{P}^n$ (see [4]). Consequently, each variety $\mathfrak{P}^n$ is different from $\mathfrak{P}$, hence $\mathfrak{P} = \mathit{Comm} + \mathit{Solv}^2$ has infinite axiomatic rank, while the component varieties Comm and Solv^2 have axiomatic ranks 2 and 4, respectively. Note also that the fact that $\mathfrak{P}$ has infinite axiomatic rank implies that this variety cannot be defined by a finite system of identities, which we proved earlier.

3°. A property φ of varieties of algebras will naturally be called *join-hereditary* if the join of two varieties possessing property φ also possesses property φ . In this section we will prove the heredity of certain properties of varieties involving various types of nilpotency of algebras.

We introduce two auxiliary notations. Suppose A is an algebra over a ring Φ, and X is a generating system of this algebra. For any subsets Q_1, $Q_2 \subset A$ we denote by $Q_1 * Q_2$ the Φ-submodule generated by the elements of A, obtained from elements $f(x_{i_1}, \ldots, x_{i_n}) \in Q_1$ by replacing one or several generators $x_{i_1},\ldots,x_{i_n}$ by any element $g(x_{j_1},\ldots,x_{j_k}) \in Q_2$.

Also, for any subset Q of the free nonassociative algebra F with generating system X we denote by $Q(A)$ the image of Q under the homomorphism $F \to A$, which is identical on X .

<u>LEMMA 1</u>. Suppose A,B,C are free algebras of the varieties $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{P} = \mathfrak{M} + \mathfrak{N}$ respectively, with generating system X , and let Q_1, Q_2 be two subsets of the free nonassociative algebra F with the same generating system and P a fully characteristic ideal of F .*

*In English, "fully invariant" is more common than "fully characteristic" — Translator.

Then the inclusions

$$Q_1(A) \subset P(A), \tag{11}$$

$$Q_2(B) \subset P(B) \tag{12}$$

imply

$$(Q_1 * Q_2)(C) \subset P(C). \tag{13}$$

Proof. Inclusions (11) and (12) are obviously systems of identities satisfied in the algebras A and B, respectively, i.e., in the varieties $\mathfrak{M}$ and $\mathfrak{N}$. We will assume that each of these identities is written in the form

$$f_s(x_{i_1},\ldots,x_{i_n}) - P_{1s}(x_{i_1},\ldots,x_{i_n}) = 0, \tag{14}$$

$$g_t(x_{j_1},\ldots,x_{j_k}) - P_{2t}(x_{j_1},\ldots,x_{j_k}) = 0, \tag{15}$$

where the polynomials f_s, g_t, P_{2s}, P_{2t} can be regarded as elements of F belonging to Q_1, Q_2, P and P, respectively.

Thus, the varieties $\mathfrak{M}$ and $\mathfrak{N}$ satisfy (14) and (15), and then on substituting into the left-hand side of (14) elements of the form $g_t - g_{st}$ in place of one or several arguments $x_{i_1},\ldots,x_{i_n}$ we obtain an identity in the variety $\mathfrak{P}$, or, equivalently, in the algebra C. Since P is a fully characteristic ideal of F, this substitution reduces, modulo P, to a substitution into the polynomial f_s of elements g_t, the result of which belongs to P. This means that inclusion (13) holds.

The lemma is proved.

Let us recall some definitions pertaining to various types of nilpotency of algebras. For a given algebra A we put $A^1 = A$, $A^{<1>} = A$, $A^{[1]} = A$, $A^{(0)} = A$, and define inductively the following subalgebras:

$$A^n = A^{n-1}A + A^{n-2}A^2 + \ldots + AA^{n-1},$$

$$A^{<n>} = A^{<n-1>}A, \quad A^{[n]} = AA^{[n-1]}, \quad A^{(n)} = (A^{(n-1)})^2.$$

An algebra A is called *nilpotent* (resp., *right nilpotent, left nilpotent, solvable*) if there exists n such that $A^n = 0$ (resp., $A^{<n>} = 0, A^{[n]} = 0, A^{(n)} = 0$). The smallest n for which this equality holds is called the *index* of the corresponding type of nilpotency. An algebra A is called a *nil algebra of bounded index* if all subalgebras with one generator are nilpotent and their indices of nilpotency are bounded in the aggregate.

LEMMA 2. In any algebra A we have, for any natural numbers s and t, the following inclusions:

$$A^{2^{s+t}} \subset A^{s+1} * A^{2^t}, \tag{16}$$

$$A^{<s+t>} \subset A^{<s+1>} * A^{<t>}, \tag{17}$$

$$A^{[s+t]} \subset A^{[s+1]} * A^{[t]}, \tag{18}$$

$$A^{(s+t)} \subset A^{(s)} * A^{(t)}. \tag{19}$$

Proof. Obviously, the lemma will be proved if we establish these inclusions in a free nonassociative algebra A with countable free generating set X.

Suppose $y \in A^{2^{s+t}}$ is a monomial in the generators of A; isolating the "last" multiplication, we write it in the form $y = y_1 T_{z_1}$, where $y_1 \in A^{2^{s+t-1}}$, and then, applying the same method to y_1, we obtain $y = y_2 T_{z_2} T_{z_1}$, where $y_2 \in A^{2^{s+t-2}}$. After s such transformations we have $y = y_s T_{z_s} \ldots T_{z_1}$ where $y_s \in A^{2^t}$. But then, taking a generator $x \in X$ not appearing in the composition of y, we can obtain y from the monomial $x T_{z_s} \ldots T_{z_1} \in A^{s+1}$ by replacing x by y_s. Consequently, $y \in A^{s+1} * A^{2^t}$, hence (16) holds.

To prove (17) it suffices to observe that any monomial of $A^{<s+t>}$ can be obtained from some monomial of $A^{<s+1>}$ by replacing the first generator occurring in it by a monomial of $A^{<t>}$. Inclusion (18) is proved analogously.

Finally, it follows from the obvious equality $A^{(s+t)} = (A^{(t)})^{(s)}$ that any monomial of $A^{(s+t)}$ can be obtained from some monomial of $A^{(s)}$ by replacing all generators occurring in it by monomials of $A^{(t)}$.

Let us now consider the following two properties of varieties of algebras: $\varphi_1(\mathfrak{M}) \Leftrightarrow$ "every right nilpotent algebra $A \in \mathfrak{M}$ is nilpotent," $\varphi_2(\mathfrak{M}) \Leftrightarrow$ " every solvable algebra $A \in \mathfrak{M}$ with a finite number of generators is nilpotent."

If a variety $\mathfrak{M}$ possesses property φ_1, then for any natural number n there exists $s = s(n)$ such that a free algebra A of this variety satisfies the inclusion

$$A^s \subset (A^{<n>}), \tag{20}$$

where $(A^{<n>})$ is the ideal of A generated by the set $A^{<n>}$. Similarly, in a free algebra B of a variety $\mathfrak{N}$ possessing property φ_1 we have for some $t = t(n)$ the inclusion

$$B^t \subset (B^{<n>}). \tag{21}$$

In a free nonassociative algebra F, putting $Q_1 = F^s$, $Q_2 = F^t$ and taking as P the fully characteristic ideal generated by the set $F^{<n>}$, we obtain from (20) and (21) that the algebras A and B satisfy conditions (11) and (12) of Lemma 1. Consequently, in a free algebra C of the variety $\mathfrak{P} = \mathfrak{M} + \mathfrak{N}$ we have $C^s * C^t \subset (C^{<n>})$ and, in view of (16), $C^{2^{s+t}}$

$\subset C^{s+1} * C^t \subset C^s * C^t \subset (C^{\langle n \rangle})$. But this means that the variety $\mathcal{P}$ possesses property φ_1, hence property φ_1 is join-hereditary.

Now suppose that varieties $\mathfrak{M}$ and $\mathfrak{N}$ possess property $\varphi_2, \mathcal{P} = \mathfrak{M} + \mathfrak{N}$, and A, B, C are free algebras of these varieties on k generators. Then for any n there exist $s = s(n, k)$ and $t = t(n, k)$, such that $A^s \subset (A^{(n)}), B^t \subset (B^{(n)})$, hence it is easy to obtain from Lemmas 1 and 2 that $C^{2^{s+t}} \subset (C^{(n)})$. Thus, property φ_2 is also join-hereditary.

By similar arguments, varying the subsets Q_1 and Q_2 and the ideal P, we can prove the heredity of several other properties of varieties; for example, the properties "every nil algebra of bounded index is solvable," "every nil algebra of bounded index with a finite number of generators is nilpotent," "every left nilpotent algebra is nilpotent," and "every anticommutative algebra is nilpotent" are join-hereditary.

If we apply our results to some specific varieties of algebras, we obtain the following assertions.

THEOREM 4. Every right nilpotent, generalized accessible algebra over a ring Φ containing 1/6 is nilpotent.

Indeed, the variety $\mathcal{GAcc}$ of generalized accessible algebras over a ring Φ containing 1/6 is the join of the variety $\mathcal{Comm}$ of commutative algebras and the variety $\mathcal{Alt}$ of alternative algebras [2], and in both of these varieties the assertion of the theorem holds [6].

THEOREM 5. Every solvable, generalized standard algebra with a finite number of generators over a ring Φ containing 1/6 is nilpotent.

Indeed, the variety $\mathcal{GSt}$ of generalized standard algebras over a ring Φ, containing 1/6 is the join of the variety $\mathcal{Jord}$ of Jordan algebras and the variety $\mathcal{Alt}$ of alternative algebras, and in both of these varieties the assertion of the theorem holds [6, 7].

This result was obtained by I. P. Shestakov for certain wider classes of algebras under the weaker restriction $\frac{1}{2} \in \Phi$ [8].

THEOREM 6. Every anticommutative, generalized accessible algebra over a ring Φ containing 1/6 is nilpotent.

Indeed, it is easy to see that every commutative and every alternative algebra satisfy the assertion of the theorem.

Note that it is proved in [8], under weaker restrictions, that if an algebra A satisfies the hypothesis of Theorem 6, then $A^7 = 0$.

In conclusion, we give an example showing that if in varieties $\mathfrak{M}$ and $\mathfrak{N}$ every right nilpotent algebra of index n is nilpotent of index N, then in the join of these varieties the index of nilpotency of a right nilpotent algebra of index n can be strictly larger than N.

Let A be an algebra with basis $\{a, b, c, d, e\}$ over some ring Φ and suppose only the following products of basis elements are nonzero: $a^2 = c$, $ab = d$, $bc = -ad = e$. It is easy to see that A is accessible and is right nilpotent of index 3, but its index of nilpotency is 4. At the same time, every commutative or associative algebra that is right nilpotent of index 3 is nilpotent of index 3.

This example also shows that the variety $RNilp^3$ of right nilpotent algebras of index 3 forms a nondistributive triple with the varieties $\mathcal{C}omm$ and $\mathcal{A}ss$, otherwise we would have the inclusion

$$RNilp^3 Acc = RNilp^3(Comm + Ass) = RNilp^3 Comm + RNilp^3 Ass \subset Nilp^3,$$

which contradicts the above example.

LITERATURE CITED

1. G. V. Dorofeev and S. V. Pchelintsev, "The varieties of standard and accessible algebras," Sib. Mat. Zh., 18, No. 5, 995-1001 (1977).
2. G. V. Dorofeev, "The varieties of generalized accessible and generalized standard algebras," Algebra Logika, 15, No. 2, 143-167 (1976).
3. G. V. Dorofeev, "The join of varieties of algebras," Algebra Logika, 15, No. 3, 267-291 (1976).
4. A. I. Mal'tsev, Algebraic Systems [in Russian], Nauka (1970).
5. A. I. Shirshov, "Subalgebras of free commutative and free anticommutative algebras," Mat. Sb., 34, No. 1, 81-88 (1954).
6. A. I. Shirshov, "On certain nonassociative nil rings and algebraic algebras," Mat. Sb., 41, No. 3, 381-394 (1957).
7. K. A. Zhevlakov, "Solvability and nilpotency of Jordan rings," Algebra Logika, 5, No. 3, 37-38 (1966).
8. I. P. Shestakov, "On certain classes of noncommutative Jordan rings," Algebra Logika, 10, No. 4, 407-448 (1971).

THE LATTICES OF VARIETIES OF LATTICE-ORDERED GROUPS AND LIE ALGEBRAS

N. Ya. Medvedev

UDC 519.46

Let $\mathcal{G}_o, \mathcal{G}_\ell, \mathcal{L}_o, \mathcal{L}_\ell$ be the categories of linearly ordered groups, lattice-ordered groups, linearly ordered Lie algebras, and lattice-ordered Lie algebras, respectively. As usual, concepts from the general theory of groups and Lie algebras attributed to these categories will be denoted by prefixes o– and ℓ– or by subscripts o and ℓ, for example: o-group, $\simeq_o$, ℓ-homomorphism, o-approximable ℓ-group, the ℓ-variety $\mathcal{A}_\ell$ of all Abelian ℓ-groups, and so on. The letters o and ℓ will not be used for other purposes.

It is well known that $\mathcal{A}_\ell$ is the smallest nontrivial element in the lattice of ℓ-varieties of ℓ-groups. In this present paper we will answer negatively Question 8 of [7] concerning the uniqueness of an element which is minimal over $\mathcal{A}_\ell$, namely, we will prove that the lattice of ℓ-varieties of solvable o-approximable ℓ-groups contains exactly three elements which are minimal over $\mathcal{A}_\ell$. On the other hand, we will prove that the lattice of ℓ-varieties of Lie ℓ-algebras contains exactly one element which is minimal over the ℓ-variety of Abelian Lie ℓ-algebras.

Recall that the class $\mathcal{G}_\ell$ of all ℓ-groups in the signature $\langle \cdot, ^{-1}, 1, \vee, \wedge \rangle$ and the class $\mathcal{L}_\ell$ of all Lie ℓ-algebras in the signature $\langle +, [\,], \cdot, 0, \vee, \wedge \rangle$ are varieties. The Cartesian (direct) product of ℓ-groups, in which $x \leq y$ if and only if $x_i \leq y_i$ for all com-

Translated from Algebra i Logika, Vol. 16, No. 1, pp. 40-45, January-February, 1977. Original article submitted November 12, 1976.

© 1978 Plenum Publishing Corporation