be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Theorem 2 has been proved.

LITERATURE CITED

1. A. I. Mal'tsev, Algebraic Systems [in Russian], Nauka (1970).
2. Yu. L. Ershov, E. A. Palyutin, and M. A. Taitslin, Mathematical Logic [in Russian], Novosibirsk State Univ. (1973).
3. A. I. Abakumov, E. A. Palyutin, M. A. Taitslin, and Yu. E. Shishmarev, "Categorical quasivarieties," Algebra Logika, 11, No. 1, 3-38 (1972).
4. E. A. Palyutin, "On complete quasivarieties," Algebra Logika, 11, No. 6, 689-693 (1972).
5. E. A. Palyutin, "On categorical quasivarieties of arbitrary signature," Sib. Mat. Zh., 14, No. 6, 1285-1303 (1973).
6. E. A. Palyutin, "Description of categorical quasivarieties," Algebra Logika, 14, No. 2, 145-185 (1975).
7. A. Lachlan, "Complete varieties of algebras," Notices Am. Math. Soc., 19, No. 5, 598 (1972).
8. J. T. Baldwin and A. H. Lachlan, "On universal Horn classes categorical in some infinite power," Algebra Universalis, 3, No. 1, 98-111 (1973).
9. S. Shelah, "Categoricity of uncountable theories," Proc. Tarski Symposium (Proc. Symp. Pure Math., Vol. XXV, Univ. Calif., Berkeley, CA, 1971), Amer. Math. Soc., Providence, RI (1974), pp. 187-204.
10. S. Shelah, "Stability, the f.c.p. and superstability; model-theoretic properties of formulas in first-order theory," Ann. Math. Logic, 3, 271-362 (1971).

NILPOTENCY IN JORDAN AND RIGHT ALTERNATIVE ALGEBRAS

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It is well known that the structure of a right alternative (in particular, an alternative or associative) algebra A is determined in many respects by the structure of the associated Jordan algebra $A^{(+)}$. In the same way, there is a close connection between the properties of a special Jordan algebra J and its associative enveloping algebra J^*. These connections play an important role in the study of right alternative and special Jordan algebras. For example, the use of these connections can be seen in [8, 9, 10]. In the present paper we continue the study of these connections.

In Sec. 1 we study the connection between various nilpotency conditions on a right alternative algebra A and its associated algebra $A^{(+)}$. We prove that $A^{(+)}$ is nilpotent if and only if A is right nilpotent, and that $A^{(+)}$ is locally nilpotent if and only if A is locally right nilpotent in the sense of Shirshov.

In Sec. 2 we establish a relation between the locally finite (in the sense of Shirshov) radical $\mathcal{L}_z(J)$ of a special Jordan algebra J with the corresponding radical of its associative enveloping algebra J^*. Namely, we prove that

$$\mathcal{L}_z(J) = \mathcal{L}_z(J^*) \cap J.$$

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In particular, when $Z=\{0\}$ we obtain this equality for the locally nilpotent radical $\mathcal{L}(J)=\mathcal{L}_{\{0\}}(J)$. This result in the case of the locally nilpotent radical generalizes the results of [7, 12], which deal, respectively, with the case where $J=A^{(+)}$ for some alternative algebra A and $J=H(A,*)$ is the Jordan algebra of symmetric elements of an associative algebra A relative to the involution $*$. It follows from the relation connecting the locally nilpotent radicals of J and J^*, that the quotient algebra $J/\mathcal{L}_Z(J)$ is also a special algebra. Therefore, in a special Jordan algebra J we have the inclusion $M(J)\subseteq \mathcal{L}(J)$, where $M(J)$ is the McCrimmon radical of J and $\mathcal{L}(J)$ is the locally nilpotent radical. It was known earlier [5] that $M(J)\neq 0$ implies $\mathcal{L}(J)\neq 0$. As another consequence of our result we prove that an element x of an alternative algebra A lies in $\mathcal{L}_Z(A)$ if and only if the right multiplication operator R_x lies in the analogous radical of the right multiplication algebra.

In the last section we consider Jordan and right alternative algebras satisfying certain minimum conditions. In particular, we prove that every quasiregular ideal of a right alternative algebra with minimum condition for quadratic ideals is right nilpotent. This result answers a question of Thedy [9] and completes the description of right alternative algebras with minimum condition for quadratic ideals: Any such algebra is an extension of a right nilpotent algebra by a finite direct sum of matrix algebras over division rings and Cayley–Dickson algebras.

1. Right Nilpotency in Right Alternative Algebras

Let Φ denote a commutative associative ring with unity 1. All algebras considered in this paper are Φ-algebras. We will employ standard definitions and notation, used, e.g., in [1, 2].

By the operation $\langle\cdot\rangle$ we mean the Φ-linear operation in the free right alternative algebra $RAlt[X]$ on the set of generators $X=\{x_1,x_2,x_3,\dots,x_n,\dots\}$, that acts as follows: if $w=(x_1,x_2,\dots,x_n)_q$, where q is some arrangement of parentheses, then $\langle w\rangle = x_1x_2'x_3'\dots x_n'$, where $yx'=yx$.

In the sequel we will repeatedly use the following

Proposition 1 (Shirshov [2, 8]). For any j-polynomial $j(x)$ in the free right alternative algebra $RAlt[X]$ we have the relation $j(x)=\langle j(x)\rangle$.

We will first prove several simple auxiliary assertions.

In a right alternative algebra we define Φ-linear transformations $V_x, U_x, U_{x,y}$ as follows:

$$yV_x=y\circ x,\quad yU_x=(xy)x,\quad U_{x,y}=U_{x+y}-U_x-U_y.$$

LEMMA 1. In any right alternative algebra we have the following relations:

$$(1)\ w(x^2)'=-wU_x+wV_xx',$$

$$(1')\ w(x\circ y)'=-wU_{x,y}+wV_xy'+wV_yx',$$

$$(2)\ wx'y'+w(xy)'=-wV_yV_x+wV_yx'+wV_xy'+wV_{xy}.$$

The proof of these relations is trivial and amounts to applying the defining identities of right alternativity.

LEMMA 2. Let $RAlt[X_{n+1}]$ be the free right alternative algebra on the set of generators $X_{n+1}=\{x_0,x_1,x_2,\dots,x_n\}$. Then for any natural number m and arbitrary $y_1,y_2,\dots,y_m \in \{x_1,x_2,\dots,x_n\}$ we have

$$x_0 y_1' y_2' \dots y_m' = \sum j_i x_{i_1}' x_{i_2}' \dots x_{i_s}',$$

where $j_i=j_i(x_0,x_1,\dots,x_n)$ is a j-polynomial in X_{n+1}, that is linear with respect to x_0, and where $1\le i_1<i_2<\dots<i_s\le n$.

The proof is by induction on m. If $m=0,1$, then x_0 and x_0y_1 are already of the required form. Suppose the desired representation has been obtained for $m\le K$, and suppose $m=K+1$. Then, by the induction assumption, $x_0y_1'y_2'\dots y_{m-1}'=\sum j_i x_{i_1}'x_{i_2}'\dots x_{i_s}'$, where $1\le i_1<i_2<\dots<i_s$ and $s\le K$. If $s<K$, then, by the induction assumption, the element $j_i x_{i_1}'x_{i_2}'\dots x_{i_s}'x_t'$, where $x_t=y_m$, has a representation $j_i x_{i_1}'x_{i_2}'\dots x_{i_s}'x_t'=\sum v_p x_{p_1}'x_{p_2}'\dots x_{p_\tau}'$, where v_p is a j-polynomial in X_{n+1} that is linear with respect to x_0 and where $p_1<p_2<\dots<p_\tau$. If $t>i_s$, then the word $j_i x_{i_1}'x_{i_2}'\dots x_{i_s}'x_t'$ has the desired form.

Now observe that the words $j_i x_{i_1}'x_{i_2}'\dots x_{i_{p-1}}'(x_{i_p}\circ x_t)'x_{i_{p+1}}'\dots x_{i_s}'$ and $j_i x_{i_1}'x_{i_2}'\dots x_{i_{p-1}}'(x_{i_p}^2)'x_{i_{p+1}}'\dots x_{i_s}'$, where $1\le p\le s$, have a representation of the desired form because of the induction assumption. Indeed, as is easily seen from the linearized right alternativity identity, they can be represented as a sum of words of the form $j_i v' x_{k_1}'x_{k_2}'\dots x_{k_p}'$, where v is a j-monomial of degree at least 2 in $\{x_1,x_2,x_3,\dots,x_n\}$. Representing v as a product of j-monomials of smaller degree, we consider three cases:

$$1)\ v=u\circ w\ , \qquad 2)\ v=u^2\ , \qquad 3)\ v=(uw)u\ ,$$

where u, w are certain j-monomials. In the first two cases, applying (1) and (1') of Lemma 1, we obtain by induction that the word $j_i v' x_{k_1}'x_{k_2}'\dots x_{k_p}'$ has a representation of the form $\sum t_\tau x_{\tau_1}'x_{\tau_2}'\dots x_{\tau_k}'$, where t_τ is a j-polynomial in X_{n+1} that is linear with respect to x_0 and where $\tau_1<\tau_2<\dots<\tau_k$. If $v=(uw)u$, then, applying the obvious equality $v'=u'w'u'=-u^{2\prime}w'+((u\circ w)\circ u)'-(u\circ w)'u'$, we arrive at the conditions of the two considered cases. Therefore, interchanging in $j_i x_{i_1}'x_{i_2}'\dots x_{i_s}'x_t'$ the elements x_{i_p}' and x_t' until $i_p>t$, we obtain, by induction, a representation of the original word as a sum of words of the required form.

In right alternative algebras there are two definitions of right nilpotency [2].

Definition 1. An algebra A is called right nilpotent if there exists an n such that $x_1x_2'x_3'\dots x_n'=0$ for any $x_1,x_2,\dots,x_n\in A$.

Definition 2. An algebra A is called locally right nilpotent in the sense of Shirshov if for any finite set $R\subseteq A$ there exists an m such that $x_1x_2'x_3'\dots x_m'=0$ for $x_i\in R$.

It is well known that an alternative algebra is locally right nilpotent if and only if it is locally right nilpotent in the sense of Shirshov, and in this case it is locally nilpotent.

As Mikheev [4] showed, there exists a right alternative algebra on three generators that is locally right nilpotent in the sense of Shirshov, but not right nilpotent. Thus, Definitions 1 and 2 are already distinct in finitely generated, right alternative algebras. We give below a characterization of these nilpotencies in terms of the associated quadratic Jordan algebra $A^{(+)}$. The definition and properties of a quadratic Jordan algebra can be found in [9, 10].

Proposition 2. Suppose A is a right alternative algebra. Then A is locally right nilpotent in the sense of Shirshov if and only if the algebra $A^{(+)}$ is locally nilpotent.

Proof. Suppose A is locally right nilpotent in the sense of Shirshov, $R=\{a_1,a_2,\ldots,a_n\}\subseteq A$, and B is the subalgebra of $A^{(+)}$ generated by R. We will prove that $B^m=0$, where m is a natural number for which $x_1x_2'x_3'\ldots x_m'=0$, if $x_i\in R$. Indeed, it is easy to see that for any $v\in B^m$ in the free right alternative algebra $RAlt[X_n]$, where $X_n=\{x_1,x_2,\ldots,x_n\}$, there exists a j-polynomial w in X_n of degree at least m such that v is the image of w under the homomorphism $RAlt[X_n]\to A$, for which $x_i\mapsto a_i$. Since, by Proposition 1, $\langle w\rangle = w$, it follows that $v=0$.

Conversely, suppose the algebra $A^{(+)}$ is locally nilpotent, and R and B are as before. We will prove that $m+n$ is a number for R corresponding to Definition 2, where m is the index of nilpotency of B as a subalgebra of $A^{(+)}$. Indeed, in $RAlt[X_n]$, according to Lemma 2, we have $y_1y_2'y_3'\ldots y_{m+n}'=\sum j_i x_{i_1}'x_{i_2}'\ldots x_{i_s}'$, where $y_k\in X_n$, $k=1,2,\ldots,m+n$. Since $i_s\leq n$, the j-polynomial j_i has degree at least m. Therefore, passing to the algebra A under the homomorphism $x_i\mapsto a_i$, we obtain $y_1y_2'\ldots y_{m+n}'=0$ for any $y_1,y_2,\ldots,y_{m+n}\in R$.

The proposition is proved.

We denote by $(A^{(+)})^m$ the m-th power of the quadratic Jordan algebra $A^{(+)}$, and by I_m the right ideal of A generated by $(A^{(+)})^m$.

LEMMA 3. If $v\in(A^{(+)})^{m-1}$, then $2vx_1'x_2'x_3'x_4'\in I_m$ for any $x_1,x_2,x_3,x_4\in A$.

Proof. For elements $x,y\in A$ we will write $x\equiv y$ if $x-y\in I_m$. It follows from relations (1), (1'), (2) and Lemma 1 that $vx_1'x_2'\ldots x_s'(y\circ z)'\equiv 0$ and $vx_1'x_2'\equiv -v(x_1x_2)'$. Therefore,

$$vx_1'x_2'x_3'x_4'\equiv -v(x_1x_2)'x_3'x_4'\equiv vx_3'(x_1x_2)'x_4'\equiv -vx_3'x_4'(x_1x_2)'\equiv v(x_3x_4)'(x_1x_2)'\equiv -v(x_1$$

$$x_2)'(x_3x_4)'\equiv vx_1'x_2'(x_3x_4)'\equiv v(x_3x_4)'x_1'x_2'\equiv -vx_3'x_4'x_1'x_2'\equiv -vx_1'x_2'x_3'x_4', \text{ i.e., } 2vx_1'x_2'x_3'x_4'\in I_m.$$

The lemma is proved.

COROLLARY 1. Suppose A is a right alternative algebra without 2-torsion. Then A is right nilpotent if and only if $A^{(+)}$ is nilpotent.

Proof. If $A^{\langle m\rangle}=0$, then, by Proposition 1, $(A^{(+)})^m=0$. Conversely, if $(A^{(+)})^m=0$, then $I_m=0$, and it follows from Lemma 3 that $2^{4(m-1)}A^{\langle 4(m-n+1)\rangle}\subseteq I_m=0$, i.e., $A^{\langle 4m-3\rangle}=0$.

COROLLARY 2. An alternative algebra without 2-torsion is nilpotent if and only if the associated Jordan algebra is nilpotent.

The proof follows easily from Corollary 1 and the fact that an alternative algebra is nilpotent if and only if it is right nilpotent.

2. Locally Finite Radical of a Special Jordan Algebra

Let us fix some ideal Z of the operator ring Φ. We will use below the well-known definitions and notation found, e.g., in [2, 3].

A finitely generated Φ-algebra B is called finite over Z if there exists a finite set of elements $b_1, b_2, b_3, \ldots, b_m \in B$ such that for some natural number n we have $B^n \subseteq \sum_{i=1}^{m} b_i Z$. An algebra A is called locally finite over Z if every finitely generated subalgebra B of A is finite over Z.

As shown in [3], local finiteness over Z is a radical property in certain classes of algebras, in particular, in the classes of all alternative and Jordan algebras. In this section we study the connection between the locally finite radical of a special Jordan algebra J and that of its associative enveloping algebra J^*. For the existence of a locally finite radical we will assume that the operator ring Φ contains an element $\frac{1}{2}$ (see also the remark to Theorem 1).

Suppose A is an associative algebra, B a finitely generated subalgebra, and M a finitely generated Φ-submodule of A. In the sequel we will need the following assertions.

LEMMA 4. If $B \cup MB \subseteq M$, then the algebra B is finitely generated as a Φ-module.

LEMMA 5. If for some natural number n we have the inclusion $B^n \cup MB^n \subseteq MZ$, then the algebra B is finite over Z.

The proof of these lemmas repeats almost verbatim, with the natural changes, the proofs of Lemmas 7 and 8 of [2, p. 23], hence we omit it here (see also Lemmas 1 and 2 of [3]).

Let $Ass[X_n \cup \{x_0\}]$ be the free associative algebra on the set of generators $X_n \cup \{x_0\}$, where $X_n = \{x_1, x_2, \ldots, x_n\}$. We will denote by $\langle t, X_n \rangle$ the Φ-module generated by the set of monomials of the form $t, t x_{i_1} x_{i_2} \ldots x_{i_k}$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, $k = 1, 2, 3, \ldots, n$, and $x_{i_k} \in X_n = \{x_1, x_2, \ldots, x_n\}$.

Let W_{2n} denote the set of j-monomials that are linear with respect to x_0 and such that the degree relative to the set X_n is at most $2n$.

LEMMA 6. In the algebra $Ass[X_n \cup \{x_0\}]$ we have the inclusion $\langle t, X_n \rangle \langle x_0, X_n \rangle \subseteq \langle t W_{2n}, X_n \rangle$, where $t \in Ass[X_n \cup \{x_0\}]$.

Proof. Consider an element of the form $t x_{i_1} x_{i_2} \ldots x_{i_k} x_0 x_{j_1} x_{j_2} \ldots x_{j_m}$ in $\langle t, X_n \rangle \langle x_0, X_n \rangle$. As is easily seen, it can be represented as a sum of words of the form $t v x_{s_1} x_{s_2} \ldots x_{s_\ell} x_{j_1} x_{j_2} \ldots x_{j_m}$, where v is a j-monomial in $\{x_0, x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ such that $d_{x_0}(v) = 1$ and $d_{X_n}(v) + \ell = k$. Applying Lemma 2, we obtain a representation of the word $t v x_{s_1} x_{s_2} \ldots x_{s_\ell} x_{j_1} \ldots x_{j_m}$ as a sum of words of the form $t_{j_i} x_{i_1} x_{i_2} \dots x_{i_\rho}$, where $i_1 < i_2 < i_3 < \dots < i_\rho$ and j_i is some j-monomials in $X_n \cup \{x_0\}$ such that $d_{X_n}(j_i) + \rho = \kappa + m \leqslant 2n$. Therefore, $j_i \in W_{2n}$

THEOREM 1. Suppose J is a special Jordan algebra, J^* is its associative enveloping algebra, and $\mathcal{L}_Z(J)$ and $\mathcal{L}_Z(J^*)$ are the locally finite over Z radicals of J and J^*, respectively. Then $\mathcal{L}_Z(J) = \mathcal{L}_Z(J^*) \cap J$.

Proof. The inclusion $\mathcal{L}_Z(J^*) \cap J \subseteq \mathcal{L}_Z(J)$ is obvious. Let us prove that $\mathcal{L}_Z(J) \subseteq \mathcal{L}_Z(J^*)$. Take an arbitrary element $\ell \in \mathcal{L}_Z(J)$ and consider the ideal $I = \ell\Phi + \ell J^*$ of J^*. We will prove that I is locally finite over Z. Obviously, it is enough to show that for any finite set $T = \{a_1, a_2, \dots, a_n\}$, $T \subseteq J$, the subalgebra B generated by the set $<\ell, T>$ in J^* is finite over Z, where $<\ell, T>$ is the image of the Φ-module $<x_0, X_n>$ under the homomorphism $\varphi: Ass[X_n \cup \{x_0\}] \longrightarrow J^*$ such that $\varphi(x_0) = \ell$ and $\varphi(x_i) = a_i$, $i = 1, 2, \dots, n$.

Consider the set $\overline{W}_{2n} = \varphi(W_{2n})$. Since $\ell = \varphi(x_0) \in \mathcal{L}_Z(J)$, it follows that $\overline{W}_{2n} \subseteq \mathcal{L}_Z(J)$. Since the associative enveloping algebra of a locally finite over Z special Jordan algebra is locally finite over Z (this follows immediately from Theorem 4 of [2]), for the algebra A generated by the set $\overline{W}_{2n}$ in J^*, there exist a natural number N and a finite set $\{w_1, w_2, \dots, w_m\} \subseteq A$ such that $A^N \subseteq \sum_{i=1}^{m} w_i Z$.

Consider the Φ-module $M = \sum_{i=1}^{m} <w_i, T> \subseteq J^*$. It is finitely generated, since the Φ-module $<x_0, X_n>$ is finitely generated. From Lemma 6 we obtain that $<\ell, T>^N = \varphi(<x_0, X_n>)^N \subseteq \varphi(<x_0 W_{2n}, X_n><x_0, X_n>^{N-2}) \subseteq \dots \subseteq \varphi(<x_0 W_{2n}^{N-1}, X_n>) \subseteq <A^N, T> \subseteq <\sum_{i=1}^{m} w_i Z, T> \subseteq MZ$. Since $<w_i, T><\ell, T>^N \subseteq <w_i \overline{W}_{2n}, T><e, T>^{N-1} \subseteq \dots \subseteq <w_i \overline{W}_{2n}^N, T> \subseteq <A^{N+1}, T> \subseteq MZ$, we have $B^N \cup MB^N \subseteq MZ$. By Lemma 5, the algebra B is finite over Z.

The theorem is proved.

Remark. In using the theory of Jordan algebras it is sometimes useful to consider a somewhat more general case, namely, where we do not impose on the ring Φ the condition $\frac{1}{2} \in \Phi$. For this purpose we introduce the so-called quadratic Jordan algebras, which in the case $\frac{1}{2} \in \Phi$ are easily transformed into linear Jordan algebras. The theory of these algebras is the same in many respects as the theory of Jordan Φ-algebras where $\frac{1}{2} \in \Phi$. For example, in quadratic Jordan algebras, as proved in [11], we have an analog of Zhevlakov's theorem for linear algebras which says that every finitely generated, solvable Jordan algebra is nilpotent. Therefore, in particular, in quadratic Jordan algebras we can define the locally nilpotent radical, i.e., the largest locally nilpotent ideal $\mathcal{L}(J)$ such that the quotient algebra $J/\mathcal{L}(J)$ contains no locally nilpotent ideals. Moreover, using the methods of proof of [3], we can introduce in the class of these algebras the locally finite over Z (where $Z \triangleleft \Phi$) radical $\mathcal{L}_Z(J)$. It is easy to see that Theorem 1 is also true, i.e., we have the relation $\mathcal{L}_Z(J) = \mathcal{L}_Z(J^*) \cap J$, where $\mathcal{L}_Z(J^*)$ is the corresponding radical of the associative enveloping algebra J^* of the quadratic Jordan algebra J.

COROLLARY 1. Suppose J is a special Jordan algebra and $\mathcal{L}_Z(J)$ its locally finite over Z radical. Then the quotient algebra $J/\mathcal{L}_Z(J)$ is special.

Proof. Let I denote the algebra of J^*, generated by $\mathcal{L}_Z(J)$. Then, by Theorem 1, $I \cap J = \mathcal{L}_Z(J)$. Therefore,

$$J/\mathcal{L}_Z(J) \cong J+I^{(+)}/I^{(+)} \subsetneq (J^*)^{(+)}/I^{(+)} \cong (J^*/I)^{(+)},$$

i.e., $J/\mathcal{L}_Z(J)$ is special.

COROLLARY 2. Suppose J is a special Jordan algebra, $M(J)$ is the McCrimmon radical [6] of J, and $\mathcal{L}(J)$ is the locally nilpotent radical of J. Then $M(J) \subseteq \mathcal{L}(J)$.

Proof. By Corollary 1, when $Z=\{0\}$ the quotient algebra $J/\mathcal{L}(J)$ is special, hence, by Theorem 1 of [5], $Z(J/\mathcal{L}(J)) \subseteq \mathcal{L}(J/\mathcal{L}(J)) = 0$, where $Z(J/\mathcal{L}(J))$ is the ideal generated by the absolute zero-divisors of $J/\mathcal{L}(J)$. Therefore, $M(J) \subseteq \mathcal{L}(J)$.

COROLLARY 3. Suppose A is a right alternative Φ-algebra and $R(A)$ its right multiplication algebra. Then the conditions $a \in \mathcal{L}_Z(A^{(+)})$ and $a' \in \mathcal{L}_Z(R(A))$ are equivalent.

Proof. Let $R_A = \{a' \mid a \in A\}$. Then R_A is a quadratic Jordan subalgebra of $(R(A))^{(+)}$, and $R_A \cong A^{(+)}/\mathfrak{Z}_r(A)$, where $\mathfrak{Z}_r(A) = \{a \in A \mid Aa = 0\}$. If $a \in \mathcal{L}_Z(A^{(+)})$, then $a' \in \mathcal{L}_Z(R_A)$, since R_A is a homomorphic image of $A^{(+)}$. By Theorem 1, $a' \in \mathcal{L}_Z(R(A))$. Now suppose $a' \in \mathcal{L}_Z(R(A))$, hence $a' \in \mathcal{L}_Z(R_A)$. Since $(\mathfrak{Z}_r(A))^2 = 0$ and $A^{(+)}/\mathfrak{Z}_r(A) \cong R_A$, it follows that $a \in \mathcal{L}_Z(A^{(+)})$.

The corollary is proved.

COROLLARY 4. In an alternative algebra A, the conditions $a \in \mathcal{L}_Z(A)$ and $R_a \in \mathcal{L}_Z(R(A))$ are equivalent.

Proof. We use Corollary 3 and the fact that in an alternative algebra we have $\mathcal{L}_Z(A) = \mathcal{L}_Z(A^{(+)})$. This equality can also be obtained from Theorem 1.

3. Rings with Minimum Condition

In this section we assume that the operator ring Φ contains an element $\frac{1}{2}$.

Proposition 3. Suppose B is a Jordan algebra and $R(B)$ its right multiplication algebra. Then for any natural numbe n we have the inclusion $R(B^{<2n>}) \subseteq (R(B))^n$.

The proof is by induction on n. If $n=1$, then $R(B^2) \subseteq R(B)$. Assume that for some n we have $R(B^{<2n>}) \subseteq R^n(B)$. Take arbitrary elements $u_{2n} \in B^{<2n>}$ and $b, c \in B$; then $R_{(u_{2n}b)c} = -R_b R_c R_{u_{2n}} - R_{u_{2n}} R_c R_b + R_{u_{2n}b} R_c + R_{u_{2n}c} R_b + R_{bc} R_{u_{2n}}$, i.e., $R(B^{<2(n+1)>}) \subseteq R^{n+1}(B)$. The proposition is proved.

THEOREM 2. Suppose B is a locally nilpotent Jordan algebra. If for some natural number m the algebra B satisfies the minimum condition for ideals contained in B^m then B is nilpotent.

Proof. Consider the sequence of ideals $\{B^{<n>}\}$ of B. Since $B^{<m>} \subseteq B^m$ it follows from the conditions on B that there exists n such that $B^{<n>} = B^{<n>}B = B^{<n+1>}$. Assume that $B^{<n>} \neq 0$, hence the set of ideals J of B, such that $J=JB\neq 0$, is nonempty. Fix some minimal ideal J of B such that $J=JB\neq 0$.

Note that for any ideal I of B such that $I\subsetneqq J$, there exists S for which $IR^s(B)=0$. Indeed, consider the sequence of ideals $I_0=I, I_1=I_0B, \ldots, I_t=I_{t-1}B, \ldots$. Since $I\subsetneqq J\subseteq B^m$, there exists S such that $I_S=I_{S+1}=I_SB$. Therefore, since $I_S\subsetneqq J$, we obtain $I_S=0$. Thus, $IR^s(B)=0$.

We will prove that $JB^{<n>}=0$, where n is such that $B^{<n>}=B^{<n>}B$. Take $z\in J$, $z\neq 0$; then $I=zR(B)$ is an ideal of B and $I\subseteq J$. Since B is locally nilpotent, it follows from Zhevlakov's theorem [1] that the algebra $R(B)$ is also locally nilpotent, hence $z\bar{\in} I$. Therefore, in view of the remark, there exists S such that $IR^s(B)=0$. By Proposition 3, $R(B^{<2(s+1)>})\subseteq R^{s+1}(B)$, hence $zB^{<2(s+1)>}=0$. By the choice of n we have the inclusion $B^{<n>}\subseteq B^{<2(s+1)>}$, hence $zB^{<n>}=0$. Consequently, $JB^{<n>}=0$.

Fix K such that $JB^{<K>}=0$ and $JB^{<K-1>}\neq 0$, and choose $\omega_0\in B^{<K-1>}$ such that $J\omega_0\neq 0$. Let s be such that $(J\omega_0)B^{<s>}\neq 0$ and $(J\omega_0)B^{<s+1>}=0$. Consider the following three cases.

1) Suppose $s\geqslant 2$. Consider the set $I=(J\omega_0)P_0$, where $p_0\in B^{<s>}$ is such that $(J\omega_0)p_0\neq 0$. We will prove that $I\lhd B$ and $I\subseteq J$, and that for any k we have $I=IR^k(B)$. Indeed, for $b_1, b_2\in B$ and $z\in J$ we have $(z\omega_0)(b_1b_2)=(z(b_1b_2))\omega_0+(z(\omega_0 b_1))b_2-(zb_2)(\omega_0 b_1)+(z(\omega_0 b_2))b_1-(zb_1)(\omega_0 b_2)=(z(b_1b_2))\omega_0$. Consequently, $(z\omega_0)p_0=(zp_0)\omega_0$. Since $((zb_1)\omega_0)p_0=-((zp_0)\omega_0)b_1-z((p_0b_1)\omega_0)+(z(p_0\omega_0))b_1+(z(p_0b_1))\omega_0+(z(\omega_0 b_1))p_0=-((zp_0)\omega_0)b_1$, it follows that $I\lhd B$ and $I\subseteq IB$. Therefore, $0\neq I\subseteq IR^k(B)$ for any k. Thus, as mentioned, $I=J$. But since $2((z\omega_0)p_0)\omega_0=-z(\omega_0^2p_0)+2(z(\omega_0p_0))\omega_0+(z\omega_0^2)p_0=0$, we have $J\omega_0=0$, despite the choice of the element ω_0, hence $s<2$.

2) Suppose $s=1$. Consider the set $I=(J\omega_0)B+J\omega_0$ Since $((zb_1)b_2)\omega_0=-((z\omega_0)b_2)b_1-z((\omega_0b_1)b_2)+(z(\omega_0b_1))b_2+(z(\omega_0b_2))b_1+(z(b_1b_2))\omega_0=-((z\omega_0)b_2)b_1$, we have $I\lhd B$ and $I\subseteq J$. Also, for any k we have $IR^k(B)\neq 0$, since $J\omega_0\subseteq JR^k(B)$. Thus, $I=J$. As in case 1), $((J\omega_0)B)\omega_0=0$. If $K\geqslant 3$, then $\omega_0\in B^2$, hence $(J\omega_0)\omega_0=0$. Thus, $J\omega_0=0$. Contradiction.

3) Suppose $K=2$ and there exists an element $\omega_0\in B$ such that $(J\omega_0)\omega_0\neq 0$, for otherwise the desired contradiction follows from 2). Now consider $I=(J\omega_0)\omega_0$. As before, $I\lhd B$ and $I\subseteq J$, and for any k we have $I=IR^k(B)$ Therefore, $I=J$. But since $2((z\omega_0)\omega_0)\omega_0=-z\omega_0^3+3(z\omega_0^2)\omega_0=0$, we obtain $J\omega_0=0$. Thus, $K=1$ and $J=JB=0$. But this means that $B^{<n>}=B^{<n>}B=0$.

The theorem is proved.

<u>COROLLARY.</u> Suppose A is a Jordan algebra with minimum condition for quadratic ideals and B is a locally nilpotent ideal of A. Then B is nilpotent.

<u>Proof.</u> We will show that B satisfies the minimum condition for ideals contained in B^2. Indeed, suppose $I\lhd B$ and $I\subseteq B^2$. We will prove that I is a quadratic ideal of A. Take $i=\sum b_1^{(\kappa)}b_2^{(\kappa)}\in I$ and $a\in A$. Then $\{iai\}=(ai)i+(a,i,i)=\sum_\kappa(a,i,b_1^{(\kappa)}b_2^{(\kappa)})+(ia)i=(ia)i+\sum_\kappa[(ab_1^{(\kappa)},i,b_2^{(\kappa)})+(ab_2^{(\kappa)},i,b_1^{(\kappa)})]\in I$.

The corollary is proved. (For the case of algebras over a field this is proved in [5]).

COROLLARY 2. In a special Jordan algebra J satisfying the minimum condition for quadratic ideals, the quasiregular radical $\mathcal{J}(J)$ is nilpotent.

Proof. Use the scheme of the proof of the corresponding theorem for the case of algebras over a field [5, Theorem 4], along with Theorems 1 and 2.

THEOREM 3. Suppose A is a right alternative algebra with minimum condition for quadratic ideals. Then the quasiregular radical $\mathcal{J}(A)$ of A is right nilpotent, and the quotient algebra $A/\mathcal{J}(A)$ is a semisimple Artinian alternative algebra.

Proof. As Thedy showed [9], A contains a nil ideal $Z(A)$ such that $A/Z(A)$ is a semisimple Artinian alternative algebra. By Corollary 2 of Theorem 2, the Jordan algebra $(Z(A))^{(+)}$ is nilpotent, hence, by Corollary 1 of Lemma 3, $Z(A)$ is right nilpotent.

The theorem is proved.

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LITERATURE CITED

1. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, Jordan Algebras [in Russian], Novosibirsk (1976).
2. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, Alternative Algebras [in Russian], Part 1, Novosibirsk (1976).
3. K. A. Zhevlakov and I. P. Shestakov, "Local finiteness in the sense of Shirshov," Algebra Logika, 12, No. 1, 41-73 (1973).
4. I. M. Mikheev, "Right nilpotency in right alternative rings," Sib. Mat. Zh., 17, No. 1, 225-227 (1976).
5. A. M. Slin'ko, "The Jacobson radical and absolute zero-divisors in special Jordan algebras," Algebra Logika, 11, No. 6, 396-402 (1972).
6. A. M. Slin'ko, "Radicals of Jordan rings," Algebra Logika, 11, No. 2, 206-215 (1972).
7. A. M. Slin'ko, "Radicals of Jordan rings associated with alternative rings," Mat. Zametki, 16, No. 1, 135-140 (1974).
8. A. I. Shirshov, "On certain nonassociative nil rings and algebraic algebras," Mat. Sb., 41, No. 3, 381-394 (1957).
9. A. Thedy, "Right alternative rings with minimal condition," Math. Z., 155, No. 3, 277-286 (1977).
10. A. Thedy, "Right alternative rings," J. Algebra, 37, No. 1, 1-43 (1975).
11. K. McCrimmon, "Solvability and nilpotence for quadratic Jordan algebras," Script. Math., 29, Nos. 3-4, 467-483 (1973).
12. M. Rich, "The Levitzki radical in associative and Jordan algebras," J. Algebra, 40, No. 1, 97-104 (1976).