AUTOSTABILITY OF MODELS

S. S. Goncharov and V. D. Dzgoev UDC 517.15

A sufficient condition is found for nonautostability of models which has made it possible to prove in a unified way criteria for autostability for distributive structures with relative complements, and for linear orders, and has also made it possible to prove nonautostability of certain structures. The proof of Proposition 1 was obtained by the authors jointly; the remaining results of this work were obtained by V. D. Dzgoev.

A countable model $\mathfrak{N}$ is called constructivizable if there exists a numeration $\nu: N \to |\mathfrak{N}|$ such that the fundamental predicates and functions of $\mathfrak{N}$ become recursive. A constructivizable model $\mathfrak{N}$ is said to be autostable if given any two constructivizations ν and μ there exists an α -automorphism of $\mathfrak{N}$ and a recursive function f such that $\nu f = \alpha\mu$ The relevant definitions are in [8].

If $\varphi(x_0,\ldots,x_n)$ is a formula and $\mathfrak{M}$ a model, then

$$\varphi(\mathfrak{M}) \rightleftharpoons \{\langle a_0,\ldots,a_n\rangle \mid \mathfrak{M} \models \varphi(a_0,\ldots,a_n)\}.$$

If $M_0 \subseteq |\mathfrak{M}|$, and $\sigma_0 \subseteq \sigma$, where $|\mathfrak{M}|$ is the basic set of the model $\mathfrak{M}$ and σ is the signature of $\mathfrak{M}$, then $\mathfrak{M} \upharpoonright M_0$ will denote the submodel of $\mathfrak{M}$ with basic set M_0 , and $\mathfrak{M} \upharpoonright \sigma_0$ is the restriction of $\mathfrak{M}$ to the signature σ_0 . If $\mathfrak{M}_0$ is a submodel of $\mathfrak{M} \upharpoonright \sigma_0$ $(\mathfrak{M}_0 \subseteq \mathfrak{M} \upharpoonright \sigma_0)$, then we also write $\mathfrak{M}_0 \subseteq \mathfrak{M}$. The domain of definition and the range of a partial function will be denoted by δf and ρf , respectively. If $n_0,\ldots,n_k$ is a set of numbers, then $\langle\bar{n}\rangle = \langle n_0,\ldots,n_k\rangle$ denotes the index of this set in some effective numeration of all such sets, and $\bar{n}$ denotes an ordered set of these elements. If f is a function with a single variable then $f(\bar{n})$ denotes the ordered (k + 1)-tuple $f(n_0),\ldots,f(n_k)$. The set of natural numbers is denoted by N.

We introduce some definitions which are useful in the description which follows. Let $(\mathfrak{M},\nu)$ be a constructive model. It is then easy to see directly from the definition that there exist a strictly computable chain of signatures $\sigma_0 \subseteq \ldots \subseteq \sigma_n \subseteq \ldots;\ \sigma = \bigcup_{n \geq 0} \sigma_n$, where σ is the signature of the model $\mathfrak{M}$, a strictly computable chain of finite models $\mathfrak{M}_n^\nu$, $n \in N$, with signatures σ_n, $n \in N$ and a general-recursive function f^ν such that

(i) $M^\nu = \bigcup |\mathfrak{M}_n^\nu|$ is a recursive set;

(ii) $\mathfrak{M}_n^\nu \subseteq \mathfrak{M}_{n+1}^\nu$ and $\mathfrak{M}_n^\nu = \overline{0, m_n}$ for every $n \in N$;

(iii) $\nu f^\nu(M^\nu) = |\mathfrak{M}|$ and $\nu f^\nu \upharpoonright |\mathfrak{M}_n| : \mathfrak{M}_n \to \mathfrak{M}$ is an isomorphic imbedding.

We call such a sequence of models $\mathfrak{M}_n$, $n \in N$, together with the function f^ν, a representation of the constructive model $(\mathfrak{M},\nu)$.

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Let $\psi = \bigwedge_{i\in N} \varphi_i$ be an infinite conjunction of $\forall$ -formulas and let $\Pi_\nu = \langle\{\mathfrak{M}^\nu_n, n\in N\}, f^\nu\rangle$ be some representation of the constructive model $(\mathfrak{M},\nu)$. We define

$$B^m_\psi \rightleftharpoons \{\langle m_0,\ldots,m_\ell\rangle \mid \mathfrak{M} \models \psi(\nu f^\nu(m_0),\ldots,\nu f^\nu(m_\ell)),$$

and for every sequence of sets $\bar{m}_0,\ldots,\bar{m}_d$ such that $\bar{m}_d = \langle m_0,\ldots,m_\ell\rangle$, there exist infinitely many t and isomorphic imbeddings $\varphi_t : \mathfrak{M}_t \to \mathfrak{M}_{t+1}$ which have for some $i \leq d$ the following two properties:

(i) $\mathfrak{M}_{t+1} \models \neg\,\psi^{t+1}(\varphi(m_0),\ldots,\varphi(m_\ell))$, where $\psi^{t+1} = \bigwedge_{i=0}^{t+1} \varphi_i$;

(ii) φ is the identity on $|\mathfrak{M}_m|$ and on the elements of the sets $\bar{m}_j$ for $j<i$.

We now introduce the notion of branching found by S. S. Goncharov.

<u>Main Definition.</u> A representation $\Pi_\nu = \langle\{\mathfrak{M}^\nu_n, n\in N\}, f^\nu\rangle$ branches for an $\forall$ -formula if for every ψ the set B^m_ψ is nonempty and $\psi(\mathfrak{M})\setminus B^m_\psi$ is finite.

A model $\mathfrak{M}$ has branching if there exist an $\forall$ -formula ν and a constructivization ψ such that the representation $\Pi_\nu = \langle\{\mathfrak{M}^\nu_n, n\in N\}, f^\nu\}$ of the constructivization ν of $\mathfrak{M}$ branches for the $\forall$ -formula ψ .

A class of constructivizations of the model $\mathfrak{M}$ is called effectively infinite if given any computable class of constructivizations of $\mathfrak{M}$ we can effectively construct a constructivization which is nonautoequivalent to every constructivization in the given class.

<u>Proposition 1.</u> If a constructivizable model $\mathfrak{M}$ has branching, then the class of constructivizations of $\mathfrak{M}$ is effectively infinite.

<u>Proof.</u> Let (S,γ) be some computable class of constructive models isomorphic to $\mathfrak{M}$. We consider a constructivization $\mathfrak{M}$ for ν , a computable sequence of infinite $\forall$ -formulas, and a representation Π_ν which branches for ψ . We will construct in steps a constructivization μ and a representation $\Pi_\mu = \langle\{\mathfrak{M}^\mu_n, n\in N\}, f^M\rangle$ for μ having the desired properties. We also consider representations for the family (S,γ) . We denote by $\langle\{\mathfrak{M}^m_n, n\in N\}, f^m\rangle$ a representation for $(\mathfrak{M}_m, \lambda n\, \gamma(n,m))$ which is constructed effectively in terms of m.

At each step t we will construct a finite model $\mathfrak{M}^\mu_t$, a partial numeration μ^t, a partial map $\bar{\mu}^t$, and an auxiliary function τ . During the construction we will also use markers $\langle m,k\rangle$, where $m,k\in N$. Let $x_0,\ldots,x_\ell$ be all the variables in the formula ψ . We order all sets of numbers $\langle m_0,\ldots,m_\ell\rangle$ by their indices in some fixed computable numeration. We will also affix markers $\langle m_0,\ldots,m_\ell\rangle$ to the markers $\langle m,k\rangle$.

In the construction we will use the p.r.f. $\lambda n x\, f_n(x)$ which is universal for the class of one-place partially recursive functions. We denote by $f^t_n(x)$ the value of a function if it has been computed in fewer than t steps. Otherwise, the value is undefined.

We now turn to a direct description of the construction.

<u>Step 0.</u> $\mathfrak{M}^\mu_0 = \emptyset$, $\bar{\mu}^0 = \mu^0 = \emptyset$, $t(0,n) = 0$ for all n.

Step t + 1. We check whether there exists a marker $\langle m,\kappa\rangle \leq t+1$ such that f_κ^{t+1} is defined on $\{0,\dots,\tau(t,\langle m,\kappa\rangle)\}$ and $\gamma(m,\rho f_\kappa^{t+1}) \supseteq \{\gamma(m,0),\dots,\gamma(m,\tau(t,\langle m,\kappa\rangle))\}$, whether such a marker is nowhere found, or else whether it appears somewhere but with a marker $\langle m_0^*,\dots,m_\ell^*\rangle$; we also check whether one of the following two properties holds:

either (i) $\lambda n\gamma(m,f_\kappa^{t+1}(n))$ is not an isomorphic imbedding of $\mathfrak{M}_t^\mu \restriction \delta f_\kappa^{t+1}$ into $\mathfrak{M}_m$;

or (ii) condition (i) does not hold and there exist numbers $\rho,n_0,\dots,n_\ell$ not greater than t + 1 and an isomorphic imbedding $\varphi:\mathfrak{M}_t^\nu \longrightarrow \mathfrak{M}_{t+1}^\nu$ such that

$$\mathfrak{M}_t^\nu \models \psi^t(\gamma(m,f_\kappa^t(n_0)),\dots,\gamma(m,f_\kappa^t(n_\ell))$$

and

$$\mathfrak{M}_{t+1}^\nu \models \neg\,\psi^t(\varphi(\bar\mu^t(n_0)),\dots,\varphi(\bar\mu^t(n_\ell))),$$

but $\varphi\restriction G = id_G$, where $(\bar\mu^t)^{-1}(G)$ contains all numbers to which markers are affixed which are less than $\langle m,\kappa\rangle$; and for all sets $m_0,\dots,m_\ell$ such that

$$\langle m_0,\dots,m_\ell\rangle < \langle f_\kappa(n_0),\dots,f_\kappa(n_\ell)\rangle,$$

if all the $m_i, i\leq\ell$, belong to $\mathfrak{M}_{t+1}^m$ and $\mathfrak{M}_{t+1}^m \models \psi^{t+1}$, then all the $f_\kappa^{-1}(m_i)$ for $i\leq\ell$ belong to M and

$$\langle m_0^*,\dots,m_\ell^*\rangle > \langle f_\kappa(n_0),\dots,f_\kappa(n_\ell)\rangle$$

or else $\mathfrak{M}_{t+1}^m \models \neg\,\psi^t(m_0^*,\dots,m_\ell^*)$.

If no such marker exists then we pass to part A (cf. below), which concludes step t + 1. If such markers exist, we choose the marker with the smallest index. Let it be $\langle m,\kappa\rangle$. If condition (i) holds for $\langle m,\kappa\rangle$ then we affix this marker to all elements in $\mathfrak{M}_t^\mu$, all markers larger than $\langle m,\kappa\rangle$ are removed, and we remove all markers from all markers not smaller than $\langle m,\kappa\rangle$. If condition (ii) holds for $\langle m,\kappa\rangle$ then we choose the smallest set $\langle n_0,\dots,n_s\rangle$ such that (ii) holds. We remove all markers from all markers not less than $\langle m,\kappa\rangle$, and all markers not less than $\langle m,\kappa\rangle$ are also removed. We relabel the elements of the model $\mathfrak{M}_{t+1}^\nu$ so that the indices of the elements $\bar\mu^t\varphi(0),\dots,\bar\mu^t\varphi(\kappa_t)$ become respectively the numbers $0,1,\dots,\kappa_t$, where $\overline{0,\kappa_t} = |\mathfrak{M}_t^\mu|$, and we fix this numeration $\bar\mu^{t+1}:|\mathfrak{M}_{t+1}^\nu| \to |\mathfrak{M}_{t+1}^\nu|$ We now define on the set $|\mathfrak{M}_{t+1}^\nu|$ the predicates and constants induced by the map $\bar\mu^{t+1}$ from $\mathfrak{M}_{t+1}^\nu$, and we denote the resulting model by $\mathfrak{M}_{t+1}^\mu$. We affix a marker $\mathfrak{M}_{t+1}^\mu$ to the elements of $\langle m,\kappa\rangle$, and on $\langle m,\kappa\rangle$ we affix the marker $\langle n_0,\dots,n_\ell\rangle$, where $\langle 1,\bar n\rangle$ is minimal satisfying property (ii). We put

$$\mu^{t+1} \leftrightharpoons \nu f^\nu \bar\mu^{t+1},\ \tau(t+1,\langle m,\kappa\rangle) = \|\mathfrak{M}_{t+1}^\nu\|$$

the number of element of the model $\mathfrak{M}_{t+1}^\nu$, and for $n \neq \langle m,\kappa\rangle$ we put $\tau(t+1,n) = \tau(t,n)$ and go to the next step.

A. We extend the map $\bar\mu^t:|\mathfrak{M}_t^\mu| \to |\mathfrak{M}_t^\nu|$ to a map

$$\bar\mu^{t+1}:|\mathfrak{M}_{t+1}^\nu| \xrightarrow[\text{onto}]{1-1} |\mathfrak{M}_{t+1}^\nu|$$

and put $\mathfrak{M}^{\mu}_{t+1} = (\bar{\mu}^{t+1})^{-1}(\mathfrak{M}^{\nu}_{t+1}), \mu^{t+1} = \nu f^{\nu}_{\nu}\bar{\mu}^{t+1}, \tau(t+1,n) = \tau(t,n)$ for all n; we then go to the next step, defining in $\mathfrak{M}^{\mu}_{t+1}$ the predicates induced by $\bar{\mu}^{t+1}$ from the model $\mathfrak{M}^{\nu}_{t+1}$.

We now consider the sequence of models $\mathfrak{M}^{\mu}_0 \subseteq \mathfrak{M}^{\mu}_1 \subseteq \ldots \subseteq \mathfrak{M}^{\mu}_n \subseteq \ldots$ and put $\mathfrak{M}^{\mu} \rightleftharpoons \bigcup \mathfrak{M}^{\mu}_n$. By Lemma 5, we can define the map

$$\bar{\mu}(n) = \lim_{t\to\infty} \bar{\mu}^{t}(n),$$

which is defined for all n. Since for each n there exists a t_n such that $\bar{\mu}^{t}(n) = \bar{\mu}^{t_n}(n)$ for all $t \geqslant t_n$, the restriction of $\bar{\mu}$ to every $\mathfrak{M}^{\mu}_n$ is an isomorphic imbedding of $\mathfrak{M}^{\mu}_n$ into $\bigcup_{n\geqslant 0} \mathfrak{M}^{\nu}_n$, and hence by the definition of the representation for ν , we obtain that $\nu f_{\nu}\bar{\mu}$ is an isomorphism of $\mathfrak{M}^{\mu}$ onto $\mathfrak{M}$. We now put $\mu(n) \rightleftharpoons \nu f_{\nu}\bar{\mu}(n)$ for all $n \in N$. By Lemma 6, the numeration μ is a constructivization, and by Lemma 7 it is nonautoequivalent to every constructivization in S. This concludes the proof of the theorem.

<u>Remark 1.</u> The marker $\langle n_0, \ldots, n_e \rangle$ is removed from $\langle m, k \rangle$ if a marker less than $\langle m, k \rangle$ is affixed, or else

$$\mathfrak{M} \models \neg \psi : (\gamma(m, n_0), \ldots, \gamma(m, n_{t_i})).$$

<u>Remark 2.</u> If a marker smaller than $\langle m, k \rangle$ is only affixed a finite number of times, then the marker $\langle n_0, \ldots, n_e \rangle$ is also affixed only finitely many times to $\langle m, k \rangle$.

<u>LEMMA 1.</u> If the marker $\langle m, k \rangle$ is affixed infinitely many times, then the function f_k is everywhere defined and $\gamma(m, f_k(N)) = |\mathfrak{M}_m|$.

The proof follows from the definition of the function γ and part (i) of the above construction.

<u>LEMMA 2.</u> If all markers less than $\langle m, k \rangle$ are affixed only finitely many times, then $\langle m, k \rangle$ is also affixed only finitely many times.

<u>Proof.</u> Assume the contrary. Then by Lemma 1, $\gamma(m, f_k(N)) = |\mathfrak{M}_m|$ and f_k is an everywhere defined function.

Let t_0 be a step after which no marker smaller than $\langle m, k \rangle$ remains. We choose p such that $|\mathfrak{M}^{\mu}_p|$ contains all numbers to which markers smaller than $\langle m, k \rangle$ are affixed at step $t_0 + 1$. Let $\psi(\mathfrak{M}) \setminus B^{p}_{\psi}$ contain d elements. Since B^{p}_{ψ} is nonempty and $\psi(\mathfrak{M}^{e}) \setminus B^{p}_{\psi}$ contains exactly d elements, there exist in $\mathfrak{M}$ at least d + 1 sets of elements $\bar{m}_0, \ldots, \bar{m}_d$ such that $\mathfrak{M} \models \psi(\bar{m}_j)$, where $j \leqslant d$.

We consider sets of indices $\bar{m}'_0, \ldots, \bar{m}'_d$ contained in ρf_k such that $\gamma(m, \bar{m}'_0) = \bar{m}_0, \ldots, \gamma(m, \bar{m}'_d) = \bar{m}^{k}_d$, and a step t_0 after which these sets are contained in $\rho f^{t_0}_k$. From these sets we select the set $\bar{m}^{*}_0$ with largest index. After step t_0 each set $\langle m_0, \ldots, m_e \rangle$ can be affixed to the marker $\langle m, k \rangle$ only finitely many times.

Consider a step $t_1 > t_0$ such that all the numbers in these sets are contained in $\rho f^{t_1}_k$ and $\mathfrak{M}^{t_1}_m \models \neg \psi^{t_1}$ holds for those sets for which $\mathfrak{M} \not\models \psi$. Thus, there exist sets $\bar{m}'_0, \ldots, \bar{m}'_d$, such that $\mathfrak{M} \models \psi(\gamma(m, f^m(\bar{m}_i')))$ for $i \leq d'$ and $d' \geq d$, and for all sets $\bar{m}$ with indices smaller than the index $\bar{m}_0^*$ either $\bar{m} = \bar{m}_i'$ for some $i \leq d'$ or $\mathfrak{M} \models \neg \psi(\gamma(m, f^m(\bar{m}))$.

Consider a step $t_2 > t_0$ such that all the numbers in these sets which are contained in $\bigcup_{t \geq 0} \mathfrak{M}_t^m$ are already contained in $\mathfrak{M}_{t_2}^m$ and if $\mathfrak{M} \models \neg \psi(\gamma(m, f^m(\bar{m})))$ then $\mathfrak{M}_{t_2}^m \models \neg \psi(\bar{m})$. We choose the sets of numbers $m_0'', \ldots, m_{d'}''$ such that $f(\bar{m}_0'') = m_0', \ldots, f(\bar{m}_{d'}'') = \bar{m}_{d'}'$ and a step $t_3 > t_2$ after which all the numbers in these sets lie in $\mathfrak{M}_{t_3}^\mu$. Since after step t_0 each marker $\langle m, \kappa \rangle$ can be affixed to the marker $\langle m_0, \ldots, m_\rho \rangle$ only finitely many times, there exists a step $t_4 > t_3$ after which only markers larger than $\bar{m}^\kappa$ are affixed to $\langle m, \kappa \rangle$. But in this case, after step μ^t, t_4 will not change on the sets $\bar{m}_0'', \ldots, \bar{m}_d''$. If for some set $j \leq d'$ the formula $\neg \psi(\mu^{t_4}(\bar{m}_j''))$ holds in the model, then we consider a step $t_5 > t_4$ for which $\mathfrak{M}_\gamma^{t_5} \models \neg \psi^{t_5}(\bar{\mu}^{t_4}(\bar{m}_j''))$ and for which a marker $\bar{m}_j'$ is affixed to the marker $\langle m, \kappa \rangle$ and is thenceforth not removed. If for all j we have $\mathfrak{M} \models \psi(\mu^t(\bar{m}_j''))$, then at least one of these sets belongs to B_ψ^ρ. But then there exists a step $t_5 \geq t_4$ at which the condition of our step is satisfied for the marker $\langle m, \kappa \rangle$ of the above set and a marker $\langle m_j' \rangle$, where $j \leq d'$, is affixed to $\langle m, \kappa \rangle$. This contradicts our hypothesis.

COROLLARY. Every marker $\langle m, \kappa \rangle$ is affixed only finitely many times.

LEMMA 3. There are infinitely many markers $\langle m, \kappa \rangle$ which at some step are affixed and thenceforth not removed.

Proof. Assume the contrary. Let there exist a step t_0 after which all markers, if they have been affixed, are removed. Consider the smallest marker which is affixed after the steps t_0 and $t_1 \geq t$. It is clear from the construction and part (ii) above that this marker can only be removed if the marker affixed to it is changed, and consequently another marker will be affixed to it, which marker is again affixed; since all markers are affixed only finitely many times, at some step the marker is permanently affixed. In order to conclude the proof of the lemma, we show that after step t_0 at least one marker is affixed. Assuming this is false, we consider a number m larger than all markers present at step t_0 and a recursive rearrangement f_m. Since $\gamma(0, f_m(\kappa))$ gives an isomorphic imbedding of $\bigcup_{t \geq 0} \mathfrak{M}_t^\kappa$, into $\mathfrak{M}$, there exists a step as in Lemma 2 to which the marker $\langle m, 0 \rangle$ should be affixed. If this is not the case, then the conditions of (i) above holds for the marker $\langle m, \kappa \rangle$ at some step and it will be affixed, contrary to assumption.

LEMMA 4. If after step t_0 a marker is permanently affixed to the element n, then for all $t \geq t_0$ we have $\bar{\mu}^t(n) = \bar{\mu}^{t_0}(n)$.

The proof follows directly from the construction.

LEMMA 5. The value $\bar{\mu}(n) = \lim_{t \to \infty} \bar{\mu}^t(n)$ is defined for all n and $\bar{\mu} : N \xrightarrow[\text{onto}]{} N$.

The proof of the lemma follows from Lemmas 3 and 4 and the fact that $\bar{\mu}^t$ gives a rearrangement of $|\mathfrak{M}_t^\nu|$ for every t.

LEMMA 6. The numeration $\mu(n) = \nu f_\nu \bar{\mu}(n)$ is a constructivization of the model $\mathfrak{M}$.

The proof is obtained from Lemma 5 and the definition of the relations on $\mathfrak{M}_t^\mu$.

LEMMA 7. The constructivization μ is nonautoequivalent to every constructivization in S.

Proof. Assume the contrary and let μ be autoequivalent to a constructivization $\lambda_n \gamma(m,n)$. Consider a general-recursive function f_K such that

$$\mu(n)=\gamma(m, f_K(n)) \quad \text{and} \quad \gamma(m, f_K(N))=\mathfrak{M}_m$$

for all $n \in N$ and the marker $\langle m,K\rangle$. Using the construction in Lemma 2 and the corollary of Lemma 2, it can be shown that the marker $\langle m,K\rangle$ is permanently affixed at some step (not subsequently removed), since otherwise it would be affixed infinitely often, which contradicts the corollary of Lemma 2.

Proposition 1 is proved.

The effectiveness of Proposition 1 can be demonstrated by employing it to prove a number of results. Throughout the remainder of this section, the set B_ψ^m is taken as in the definition of branching, except that in each specific case, the formula ψ will be chosen in a suitable way.

We call a lattice a structure with relative complements if for any $a \leq x \leq b$ there exists a y such that $x \cap y = a,\ x \cup y = b$.

THEOREM 1. The following conditions are equivalent for a constructivizable distributive structure with relative complements and zero:

(1) the structure is nonautostable,

(2) the structure has infinitely many atoms,

(3) the class of all constructivizations of the structure is effectively infinite.

Proof. Let (D, ν) be a constructive distributive structure with relative complements and zero. The nontrivial part of the proof involves showing that D has branching. Put $\Psi(x) \rightleftharpoons \forall y\,(y \subseteq x \longrightarrow (x=y \vee y=0))$, i.e., $\Psi(x)$ selects the atoms of the structure. Consider a representation for D obtained by putting D^m equal to the substructure generated by $\{\nu 0, \nu 1, \ldots, \nu m\}$. We relabel the elements so that their indices form an initial segment of N.

We show that the set B_ψ^m contains element a having the following property:

(*) a is an atom of D which is contained in an atom z of D^m such that D_z is an infinite Boolean algebra, or else the atom a in D is not contained in D^m and there exist infinitely many elements of D not intersecting elements of D^m. (For $z \in D$ the structure $D_z \rightleftharpoons \{x \in D \mid x \subseteq z\}$ is the ideal generated by the element z. It is obvious that for distributive **structures with relative** complements and zero, D_z is always a Boolean algebra.)

Indeed, assume that a satisfies condition (*) and $a \in D_z$; D_z is an infinite Boolean algebra and z is an atom in D^m. We show that there exist infinitely many t such that there exists a φ-isomorphic imbedding of D^t into D^{t+1} and $\varphi(a)$ is not an atom in D^{t+1}. Assume that at some step $t > m$ the atom b has been computed in $D^{t+1} \setminus D^t$ and $b \subset c \subset z$, where c is an atom in D^t.

For $d \in D^t$ we define φ as follows:

$$\varphi(d) = \begin{cases} d \cup b & \text{, if } \quad d \supseteq a; \\ d \setminus b & \text{, if } \quad d \not\supseteq a. \end{cases}$$

It is verified directly that φ is an isomorphism and $\varphi(a)$ is not an atom in D^{t+1}. Since by hypothesis D_z is an infinite Boolean algebra, there obviously exist infinitely many such t and isomorphisms φ.

We consider analogously the case when $a \notin D^m$, a an atom in D, and there exist infinitely many elements in D not intersecting with elements in D^m. It is only necessary to wait until the step t at which an atom b in D^{t+1} has been computed and b lies under some atom of the algebra D^t. We define the imbedding φ as before, i.e., for $d \in D^t$ we have

$$\varphi(d) \rightleftharpoons \begin{cases} d \cup b & \text{, if } \quad d \supseteq a; \\ d \setminus b & \text{, if } \quad d \not\supseteq a. \end{cases}$$

Our hypothesis again guarantees the existence of infinitely many such steps t and imbeddings φ. It is clear from this that B^m_ψ contains all elements which satisfy condition (*). It is easy to see that only finitely many atoms in D fail to lie in B^m_ψ. This completes the proof. We note that Theorem 1 was announced in [5].

As a corollary of Theorem 1, we obtain a well-known result of Goncharov [1] concerning autostability of Boolean algebras.

Before proving the following theorem, we give a definition. We will say that two elements $\langle a, b \rangle$ in a linear order L are adjacent if $a < b$ and $\forall c \in L\, ((a \leq c \leq b) \to (a = c \vee b = c))$.

THEOREM 2 (cf. [6]). Let L be a constructive linear order. Then the following conditions are equivalent:

(1) L is nonautostable;

(2) L has infinitely many pairs of adjacent elements;

(3) the class of all constructivizations of L is effectively infinite.

Proof. As in the preceding assertion, it suffices to show that a constructive linear order (L, ν) containing infinitely many pairs of adjacent elements has branching. We consider a representation for (L, ν) obtained by putting L_n equal to the finite linear order $\langle \{\nu 0, \nu 1, \ldots, \nu n\}, \leq \rangle$.

We define $\psi(x, y) \rightleftharpoons x < y \,\&\, \forall z\, (x \leq z \leq y \to (x = z \vee z = y))$. We fix $m \in N$ and show that B^m_ψ consists of the set of pairs $\langle a, b \rangle$ of adjacent elements of L such that either these elements are positioned between adjacent elements of L_m, between which there also lie infinitely many elements of L, or else a, b are larger than the largest element of L_m and there exist in addition infinitely many elements of L with the same properties, or else the elements of the pair $\langle a, b\rangle$ are less than the smallest element in L_m and there also exist , infinitely many such pairs of elements. In fact, let $a<b$, $\langle a,b\rangle$ a pair of adjacent elements in L , $n<m$ a pair of adjacent elements in L_m with $n<a<b<m$, and there let exist infinitelymany elements of L laying between $\langle n,m\rangle$. Assume that at some step $t>m$ an element c has been computed in L_{t+1} and $n<a_1<\ldots<a_i<c<a_{i+1}<\ldots<a_k<a<b<b_1<\ldots<b_s<m$, where $a_1,\ldots,a_k$, $b_1,\ldots,b_s$ are all elements of L_t lying between n and m.

We define $\varphi : L_t \longrightarrow L_{t+1}$ by:

$$\varphi(x) = \begin{cases} c & \text{, if } x = a_{i+1}; \\ a_{i-1} & \text{, if } x = a_{i+2}, \ldots, a_k; \\ a_k & \text{, if } x = a; \\ x & \text{otherwise.} \end{cases}$$

It is obvious that φ is an isomorphic imbedding, and this situation can be repeated infinitely many times.

We intentionally do not consider all the possible cases, but hope that if desired the reader will treat them himself using an analogous method. It is also easily seen that $\psi(L)\setminus B_\psi^m$ is finite.

<u>THEOREM 3 [6].</u> There exists an autostable model $\mathfrak{A}$ such that the cartesian product $\mathfrak{A}^2$ is nonautostable.

<u>Proof.</u> Consider some one-valued constructivization ν_0 of the linearly ordered set $2+\eta$, where $\eta=\langle\{\frac{n}{m} \mid 0<n<m\},<\rangle$ and $2=\langle\{a,b\},<\rangle$ such that in terms of the ν_0-index of $\hat{n}$ we can decide whether $\hat{n}$ is equal to a or b , and if not, then effectively find numbers n and m such that $\nu_0(\hat{n}) = \frac{n}{m}$; and conversely, in terms of which we can now define a numeration ν of the partially ordered set $(2+\eta)\cdot(2+\eta)$ by putting: $\nu(n) = (\nu_0(\ell(n)), \nu_0(r(n)))$, where c, ℓ, r are Cantor functions [10] enumerating pairs of natural numbers. We now define

$$A_0 \leftrightharpoons \nu(\{\langle a,b\rangle, \langle b,a\rangle, \langle a,a\rangle, \langle b,b\rangle\}),$$

$$A_t \leftrightharpoons A_0 \cup \nu^{-1}\left(\left\{\langle x,\tfrac{p}{q}\rangle, \langle \tfrac{p}{q},y\rangle, \langle \tfrac{p}{q},\tfrac{p'}{q'}\rangle \,\middle|\, \begin{array}{l} x,y\in\{a,b\} \\ p,p',q,q'\le t \\ 0<p<q\ 0<p'<q' \end{array}\right\}\right).$$

We now consider the models $\mathfrak{A}_0,\ldots,\mathfrak{A}_t,\ldots$ with basic sets $A_0,\ldots,A_t,\ldots$ respectively, and the partial order induced on them from the partially ordered set $\mathfrak{A}$ via the numeration ν It is clear that $\langle\{\mathfrak{A}_t \mid t\in N\}, id_N\rangle$ is a representation of the constructivization ν . We remark that the set $\psi(\mathfrak{A}) \leftrightharpoons \{\langle x,y\rangle \mid \mathfrak{A} \models \psi(x,y)\}$ is equal to the set

$$\{\langle\langle a,\tfrac{p}{q}\rangle,\langle b,\tfrac{p}{q}\rangle\rangle \mid p<q\in N\} \cup \{\langle\langle \tfrac{p}{q},a\rangle,\langle \tfrac{p}{q},b\rangle\rangle \mid p<q\in N\} \cup \{\langle\langle a,b\rangle,\langle a,b\rangle\rangle, \langle\langle b,a\rangle,\langle b,a\rangle\rangle\}.$$

Consider the set B_ψ^m . We show that

$$B_\psi^m \supseteq B_m \leftrightharpoons \{\langle\langle a,\tfrac{p}{q}\rangle,\langle b,\tfrac{p}{q}\rangle\rangle \mid p,q$$

relatively prime $q>m,\ 0<p<q\} \cup \{\ll \frac{p}{q}, a>, <\frac{p}{q}, b\gg \mid p,q$ relatively prime $0<p<q, q>m\}$.

We consider in L_t the natural ordering on the rational numbers and elements a, b in L_t which are smaller than all the rational numbers in the ordering, with $a<b$. Then $L_t \times L_t \cong \mathfrak{A}_t$. The collection L_t is a linearly ordered set, and we therefore consider its elements in the order:

$$a \prec b \prec x_0 \prec x_1 \prec \ldots \prec x_{d_t}.$$

Now let $<a_0, b_0>, \ldots, <a_k, b_k>$ be a sequence of pairs in $\psi(\mathfrak{A})$ and $<a_k, b_k> = \ll a, \frac{p_0}{q_0}>, <b, \frac{p_0}{q_0}\gg$. Consider among the pairs $<a_i, b_i>$, $i \leq k$, those such that $a_i \notin L_m$ and $b_i \notin L_m$ and let i_0 be the smallest of all indices of pairs with this property. Such pairs exist, since by hypothesis $<a_k, b_k>$ is such a pair. Assume for definiteness that

$$a_{i_0} = <a, \frac{p_1}{q_1}>, \quad b_{i_0} = <b, \frac{p_1}{q_1}>,$$

p_1 and q_1 relatively prime.

Consider a step $t>m$. Assume that the element $\frac{p'}{q'}$ directly follows $\frac{p_1}{q_1}$ in L_t.

Consider the first step $t'>t$ such that $L_{t'}$ contains an element $\frac{p''}{q''}$ such that $\frac{p_1}{q} < \frac{p''}{q''} < \frac{p'}{q'}$. Then in this step we can consider the isomorphic imbedding $\varphi: \mathfrak{A}_{t'-1} \to \mathfrak{A}_{t'}$ defined as follows:

$$\varphi(<x,y>) = \begin{cases} <x, y>, & \text{if } y \neq \frac{p_1}{q_1} \vee x=a; \\ <x, \frac{p''}{q''}> & \text{otherwise} \end{cases}$$

It is easy to see that φ satisfies conditions (i) and (ii) in the definition of B^m_φ, and therefore since the sequence $<a_0, b_0>, \ldots, <a_k, b_k>$ and step t were chosen arbitrarily, the existence of branchings has been proved. Using Proposition 1, we now conclude our proof.

In conclusion, we give an example; the idea of its proof is very closely related to the ideas in this section. The variant of the proof which we present uses Theorem 3 directly.

Let $(\mathfrak{M}, \nu)$ be a constructive model. We denote by $Aut\,\mathfrak{M}$ the group of automorphisms of the model $\mathfrak{M}$. We say that an automorphism $\varphi \in Aut\,\mathfrak{M}$ is recursive for the constructivization ν if there exists a recursive function f such that $\varphi\nu = \nu f$. It is obvious that the set of recursive automorphisms forms a group. We denote it by $Aut(\mathfrak{M}, \nu)$.

Proposition 2. There exist a constructivizable model $\mathfrak{M}$ having continuum many automorphisms, and a constructivization ν such that the group of recursive automorphisms $Aut(\mathfrak{M}, \nu)$ is trivial.

Proof. We construct the constructivization ν of the linear order $\omega \cdot \eta$ (here $\bullet$ denotes the product of the linear orders, not the cartesian product). We fix a constructive ordering of N of the rational numbers type $\leq_\eta$, where $\leq$ is the usual ordering of N of ω type. We define on N^2 a linear ordering $\preccurlyeq$ of type $\omega \cdot \eta$:

$$<a,b> \preccurlyeq <c,d> \overset{df}{\Longleftrightarrow} a \leq_\eta c \text{ or } (a=c \text{ and } b \leq d).$$

By Theorem 3 there exists a computable set $\{\nu_i\}_{i\in N}$ of pairwise inequivalent constructivizations of ω. The desired constructivization of $\omega\cdot\eta$ is

$$\nu(n) = \langle l(n), \nu_{l(n)}(r(n))\rangle .$$

Indeed, let f be a recursive function, φ an automorphism of the linear order $\omega\cdot\eta$ such that

$$\nu f = \varphi\nu . \qquad (*)$$

We remark that since ω has no nontrivial automorphisms, any automorphism of η induces an automorphism of $\omega\cdot\eta$, and conversely. Let $n_0, n_1, \ldots$ be an effective enumeration of the ν indices of some copy of ω. For definiteness, we assume that these are the indices of the elements $\langle a_0, 0\rangle, \langle a_0, 1\rangle, \ldots, \langle a_0, n\rangle, \ldots$ (cf. the definition of the ordering $\preccurlyeq$). Then the automorphism φ takes these elements into the elements $\langle b, 0\rangle, \langle b, 1\rangle, \ldots$. Relation (*) permits us to effectimely find the ν indices of the images under φ. This last in fact means that the constructivizations of the corresponding copies of ω are equivalent, which contradicts their choice. Proposition 2 was announced previously in [2].

Examples of models with finite groups of recursive automorphisms can be constructed analogously to the above construction.

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