SCHREIER THEOREM FOR ANALYTIC MOUFANG LOOPS

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The study of Lie groups in the large is based to a significant extent on the Schreier theorem on the extension of local homomorphisms of a simply connected topological group to a homomorphism in the large. Appropriate analogs of this theorem for Moufang loops in different particular cases have been used in [5] for the construction of a simply connected analytic Moufang loop with a preassigned tangent Mal'tsev algebra. The analog of the Schreier theorem for analytic Moufang loops with solvable tangent Mal'tsev algebra, obtained in the same article, had enabled us to give a classification of these loops. In the present article the restrictions on the tangent Mal'tsev algebra are removed. As a consequence, we obtain a classification of connected analytic Moufang loops with an arbitrary preassigned tangent Mal'tsev algebra.

1. Let G be an analytic Moufang loop and $a \in G$. As usual, R_a and L_a are the operators of the right and the left multiplication by the element a,

$$xR_a = xa, \quad xL_a = ax.$$

A homomorphism $\varphi: G \to G$ is called a pseudoautomorphism if there exists an element $c \in G$ such that for arbitrary $x, y \in G$

$$c\cdot(xy)\varphi = (c\cdot x\varphi)\cdot y\varphi. \tag{1}$$

The element c is called the companion of the pseudoautomorphism φ. We know [6] that the operators $T_x = L_x^{-1} R_x$ and $R_{x,y} = R_x R_y R_{xy}^{-1}$ are pseudoautomorphisms of the Moufang loop G with the companions x^3 and $[y,x] = y^{-1}x^{-1}yx$, respectively.

If A is the tangent Mal'tsev algebra of the loop G, then to each subalgebra A_0 of the algebra A there corresponds a subloop U_0 of the local analytic Moufang Loop U, where U is a sufficiently small neighborhood of the identity of the loop G. Let us consider G_0 of finite products of elements of U_0 with arbitrary arrangement of parentheses and equip it with the intrinsic topology: A subset V of G_0 is open if and only if for each $x \in V$ there exists a neighborhood V_x of the identity of the local loop U_0 such that $xV_x \subset V$. Let us verify that the space G_0 forms an arcwise-connected topological Moufang loop with respect to the multiplication in G.

Let $W_1, \ldots, W_k$ be T-words in $x_1, \ldots, x_k \in U_0$ (see [5]). If $a = \{x_1W_1 \ldots x_kW_k\}$ is a word with a certain arrangement of parentheses, then $a = \langle x_1\overline{W}_1 \ldots x_k\overline{W}_k\rangle = x_1\overline{W}_1 R_{x_2\overline{W}_2} \ldots R_{x_k\overline{W}_k}$, where $\overline{W}_i$ are also T-words in $x_1, \ldots, x_k \in U_0$. We carry out induction on k. Let $a = \{x_1W_1 \ldots x_nW_n\} = bc$, where b and c are words of length less than n. Then $b = \langle x_1\overline{W}_1 \ldots x_k\overline{W}_k\rangle$, $c = \langle x_{k+1}\overline{W}_{k+1} \ldots x_n\overline{W}_n\rangle$, and it is sufficient to consider the case $n > k+1$. If $u = \langle x_{k+1}\overline{W}_{k+1} \ldots x_{n-1}\overline{W}_{n-1}\rangle$, $v = x_n\overline{W}_n$,

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then $a = b R_{u,v}^{-1} u \cdot v$. Since $R_{u,v}^{-1}$ is a T-word in $x_1, \ldots, x_n$ (see [5, Lemma 6]), it follows that $b R_{u,v}^{-1} = \langle x_1 \overline{W}_1' \ldots x_k \overline{W}_k' \rangle$, where $\overline{W}_1', \ldots, \overline{W}_k'$ are T-words in $x_1, \ldots, x_n \in U_0$ (see [5, Lemma 5]). The word $b R_{u,v}^{-1} u$ has length $n-1$ and the induction hypothesis is applicable to it. The induction is carried out. In particular, each element $a \in G_0$ can be represented in the form $a = \langle x_1 W_1 \ldots x_n W_n \rangle$, where $W_1, \ldots, W_n$ are T-words in $x_1, \ldots, x_n \in U_0$.

Let x and y be arbitrary elements of G_0 and V be a neighborhood of the identity of the local loop U_0. For the continuity of the multiplication in G_0 it is sufficient that there exists a neighborhood $V_{x,y}$ of the identity of the local loop U_0 such that $xV_{x,y} \cdot yV_{x,y} \subset xy \cdot V$. By what we have proved above, $x = \langle x_1 W_1 \ldots x_n W_n \rangle$ and $y = \langle x_{n+1} W_{n+1} \ldots x_k W_k \rangle$, where $W_1, \ldots, W_k$ are T-words in $x_1, \ldots, x_k \in U_0$. For arbitrary elements $a, b \in G_0$ we will have $xa \cdot yb = xy \cdot (a\overline{W}_1 \cdot b\overline{W}_2)\overline{W}$, where $\overline{W}, \overline{W}_1$, and $\overline{W}_2$ are T-words in $x_1, \ldots, x_k \in U_0$ (see [5, Lemma 11]). It is easily seen that for an arbitrary T-word W in $x_1, \ldots, x_k \in U_0$ and an arbitrary neighborhood V of the identity of the local loop U_0 there exists a neighborhood V_0 of the identity of the local loop U_0 such that $V_0 W \subset V$. Consequently, there exists a neighborhood $V_{x,y} \subset U_0$ of the identity such that $a, b \in V_{x,y}$ for $(a\overline{W}_1 \cdot b\overline{W}_2)\overline{W} \subset V$.

This is what was required to be proved.

Thus, G_0 is a topological Moufang loop. Now the arcwise connectedness of the space G_0 follows from the fact that G_0 is generated by neighborhood U_0 of the identity in it.

We will call G_0 the subloop of G corresponding to the subalgebra A_0.

LEMMA 1. Let G be an analytic Moufang loop, A be the tangent Mal'tsev algebra G, A_0 be a Lie subalgebra of A that is a direct sum of subalgebras: $A_0 = B \oplus C$, where B is a semisimple and C is a solvable Lie algebra. Then the subloop G_0, corresponding to the subalgebra A_0, is an arcwise connected group.

Proof. Let us consider a neighborhood U of the identity of the loop G such that a canonical coordinate system of the first kind [1] can be introduced in it. If V is the Lie subgroup in U that corresponds to the subalgebra A_0, then we can assume that for arbitrary $x, y, z \in V$ the elements $[x,y]$, $xy \cdot z$, $x \cdot yz \in U$. As shown above, each $x \in G_0$ has the form $x = \langle x_1 \ldots x_n \rangle$, where $x_1, \ldots, x_n \in V$. Moreover, if $x, y, z \in V$, then $xy \cdot z = x \cdot yz$ or $x R_{y,z} = x$. We show that $x R_{y,z} = x$ for arbitrary $x \in G_0$ and $y, z \in V$.

Let $x = \langle x_1 \ldots x_n \rangle$, where $x_1, \ldots, x_n \in V$, and suppose that the above statement is valid for words of length less than n. Since $c = [z, y]$ is the companion of the pseudoautomorphism $R_{y,z}$ it follows by the induction hypothesis and (1) that

$$x R_{y,z} = (\bar{x} x_n) R_{y,z} = c^{-1}[(c \cdot \bar{x} R_{y,z}) \cdot x_n R_{y,z}] = c^{-1}(c \bar{x} \cdot x_n),$$

where $\bar{x} = \langle x_1 \ldots x_{n-1} \rangle$. The element $c = [z, y] \in U$, and, consequently, there exists a $c_0 \in V$ such that $c = c_0^m$ for a certain natural number m. By the induction hypothesis, we have $\bar{x} R_{c_0, x_n} = \bar{x}$. By virtue of the Moufang theorem, the set $\{\bar{x}, c_0, x_n\}$ generates a subgroup in G (see [6]). Finally, we get $x R_{y,z} = c^{-1}(c \bar{x} \cdot x_n) = \bar{x} x_n = x$.

Now let $a, b \in G_0$ and x and y be arbitrary elements of V and set $c = [y, x]$. Then, by what we have proved

$$c \cdot ab = c\cdot(ab)R_{x,y} = (c\cdot aR_{x,y})\cdot bR_{x,y} = ca\cdot b,$$

whence it follows that $c \in N(G_0)$, where $N(G_0)$ is the associative center of the Moufang loop G_0.

Let us consider the decomposition $V = V_1V_2$ of the local Lie group V into a direct product of local normal divisors V_1 and V_2, corresponding to the decomposition of the algebra $A_0 = B \oplus C$. Since B is a semisimple algebra, the inclusion $[V, V] \subset N(G_0)$ implies that $V_1 \subset N(G_0)$. Let us denote the subloops of the Moufang loop G_0 that are generated by the sets G_1 and G_2 by V_1 and V_2, respectively. Then $G_1 \subset N(G_0)$. If $\tilde{G}_2$ is a simple Lie group with tangent algebra C, then a local homomorphism $\psi: \tilde{G}_2 \to G$ corresponds to the embedding $\varphi: C \to A_0 = B\oplus C \subset A$. Since the algebra C is solvable, it follows by [5] that ψ can be extended to a homomorphism $\tilde{\psi}$ in the large. It is clear that $\tilde{G}_2\tilde{\psi} = G_2$. Consequently, G_2 is a group. It follows from the equation $[V_1, V_2] = e$, where e is the identity of the loop G, and the inclusion $G_1 \subset N(G_0)$ that $[G_1, G_2] = e$. Hence $G_0 = G_1G_2$ is a group. It has already been observed that G_0 is arcwise connected.

The lemma is proved.

We know [8] that only two nonisomorphic Cayley–Dickson algebras exist over the field $\mathbb{R}$ of real numbers. These are the division algebra of Cayley numbers and the splittable Cayley–Dickson algebra. There exists only one (up to isomorphism) (splittable) Cayley–Dickson algebra over the field $\mathbb{C}$ of complex numbers. Let F denote either $\mathbb{R}$ or $\mathbb{C}$ and K be a Cayley–Dickson algebra over F. Then the commutator algebra $K^{(-)}$ is decomposed into the direct sum of the one-dimensional center and a seven-dimensional simple Mal'tsev subalgebra: $K^{(-)} = F \oplus V$, where $V = [K, K]$ is the linear space that is spanned by all the commutators of the algebra K. We define functions $t: K \to F$ and $n: K \to F$ on K such that for arbitrary $x, y \in K$

$$x^2 = t(x)x - n(x), \quad n(xy) = n(x)n(y),$$

where the quadratic form n(x) on K is nondegenerate.

If $x, y \in V$, then we will denote their product in the Mal'tsev algebra $x * y$ by V. We know [2] that a nondegenerate symmetric bilinear form (x, y) is defined on V such that for $x, y, z \in V$

$$(x*y)*y = -(y,y)x + (x,y)y, \tag{2}$$

$$(xy, z) = (x, yz), \tag{3}$$

$$(xy, xy) = (x,x)(y,y) - (x,y)(y,x). \tag{4}$$

Moreover, for $a = x_0 + x \in F\oplus V$ $(x_0 \in F, x \in V)$ we have

$$n(a) = x_0^2 + (x, x). \tag{5}$$

Let us set

$$G = \{x \in K \mid n(x) = 1\},$$

Then G forms an analytic Moufang loop with respect to the multiplication in the algebra K with the tangent algebra isomorphic to V; moreover, for $F = \mathbb{R}$ the space G is analytically isomorphic to S^7 or $S^3 \times \mathbb{R}^4$, and for $F = \mathbb{C}$ the algebra K may be considered as a 16-dimensional simple algebra over $\mathbb{R}$, and the space G is analytically isomorphic to $S^7 \times \mathbb{R}^7$ (see [5]).

We denote the subalgebra of K over the field F that is generated by arbitrary elements $a, b \in G$ by $K(a,b)$. Let us set $G(a,b) = \{x \in K(a,b) \mid n(x) = 1\}$. Since $K(a,b)$ is an associative algebra, it follows that $G(a,b)$ is a subgroup of the loop G; moreover $a, b \in G(a,b)$. In the sequel we will be interested in the topological structure of the group $G(a,b)$. For brevity we set $G_0 = G(a,b)$ and $K_0 = K(a,b)$.

We know that each subalgebra of a Cayley—Dickson algebra that is generated by two elements has dimension at most four. Consequently, the commutator algebra $K_0^{(-)}$ is decomposed into the direct sum of the one-dimensional center and a Lie subalgebra: $K_0^{(-)} = F \oplus Z$, where $\dim_F Z \leq 3$. Let us analyze the cases $\dim_F Z = 0, 1, 2, 3$ one by one. The expression $G_0 = \bar{G}_0 \cup (-1)\bar{G}_0$ means that G_0 has two connected components $\bar{G}_0$ and $(-1)\bar{G}_0$, where $1 \in \bar{G}_0$ and, consequently, $\bar{G}_0$ is a subgroup. The omitted computations can be easily restored:

a) $\dim_F Z = 0,\ G_0 = \{1, -1\}$;

b) $\dim_F Z = 1,\ Z = (e_1)_F,\ (e_1, e_1) = \alpha$, and for $x = x_0 + x_1 e_1 \in F \oplus Z$, by virtue of (5), we get $n(x) = x_0^2 + \alpha x_1^2$. The following cases are possible:

$F = \mathbb{R}$:

1) $\alpha = 1$; G_0 is homeomorphic to S^1;

2) $\alpha = 0, -1$; $G_0 = \bar{G}_0 \cup (-1)\bar{G}_0$; $\bar{G}_0$ is homeomorphic to $\mathbb{R}$.

$F = \mathbb{C}$:

3) $\alpha = 0$, $G_0 = \bar{G}_0 \cup (-1)\bar{G}_0$; $\bar{G}_0$ is homeomorphic to $\mathbb{R}^2$;

4) $\alpha = 1$; G_0 is homeomorphic to $S^1 \times \mathbb{R}$.

c) $\dim_F Z = 2,\ Z = (e_1, e_2)_F$.

1) Z is an Abelian Lie algebra, i.e., $e_1 \times e_2 = 0$. Then, by virtue of (2), we get

$$0 = (e_1 \times e_2) \times e_2 = -(e_2, e_2)e_1 + (e_1, e_2)e_2,$$

whence $(e_1, e_2) = (e_2, e_2) = 0$, and analogously $(e_1, e_1) = 0$. Thus, by virtue of (5), the norm $n(a) = x_0^2$ for each $a = x_0 + x \in F \oplus Z$. Consequently, $G_0 = \bar{G}_0 \cup (-1)\bar{G}_0$, and if $F = \mathbb{R}$, then $\bar{G}_0$ is homomorphic to R^2, and if $F = \mathbb{C}$, then $\bar{G}_0$ is homeomorphic to $\mathbb{R}^4$.

2) Z is a solvable non-Abelian Lie algebra. Then we can choose a basis such that $e_1 \times e_2 = e_1$. Using (2), we get

$$e_1 = (e_1 \times e_2) \times e_2 = -(e_2, e_2)e_1 + (e_1, e_2)e_2,$$

whence $(e_2, e_2) = -1$ and $(e_1, e_2) = 0$. In the same manner, from the equation

$$0 = (e_2 \times e_1) \times e_1 = -(e_1, e_1) e_2 + (e_1, e_2) e_1$$

we get $(e_1, e_1) = 0$. Consequently, the norm $n(x) = x_0^2 - x_2^2$ for $x = x_0 + x_1 e_1 + x_2 e_2 \in F \oplus Z$. If $F = R$, then $G_0 = \bar{G}_0 \cup (-1) \bar{G}_0$ and the space $\bar{G}_0$ is homeomorphic to R^2. But if $F = C$, then the group G_0 is connected and is homeomorphic to $S^1 \times R^3$.

d) $\dim_F Z = 3$, $Z = (e_1, e_2, e_3)_F$. Since the Lie algebra Z is generated by two elements, the basis e_1, e_2, e_3 can be chosen in such a way that $e_1 \times e_2 = e_3$. Moreover, without loss of generality, we can assume that $(e_1, e_2) = 0$. Further, by virtue of (3), we have $(e_1, e_3) = (e_2, e_3) = 0$. If $(e_1, e_1) = \alpha_1$ and $(e_2, e_2) = \alpha_2$, then by (4) we get $(e_3, e_3) = \alpha_1 \alpha_2$, and the norm $n(x) = x_0^2 + \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_1 \alpha_2 x_3^2$ for $x = x_0 + \sum_{i=1}^{3} x_i e_i \in F \oplus Z$. The following cases are possible:

$F = R$:

1) $\alpha_1 = 0$, $\alpha_2 = 0, -1$; $G_0 = \bar{G}_0 \cup (-1) \bar{G}_0$ and $\bar{G}_0$ is homeomorphic to R^3;

2) $\alpha_1 = 1$, $\alpha_2 = 0, -1$; G_0 is homeomorphic to $S^1 \times R^2$;

3) $\alpha_1 = \alpha_2 = 1$; G_0 is homeomorphic to S^3.

$F = C$:

1) $\alpha_1 = \alpha_2 = 0$; $G_0 = \bar{G}_0 \cup (-1) \bar{G}_0$; and $\bar{G}_0$ is homeomorphic to R^6;

2) $\alpha_1 = 0$, $\alpha_2 = 1$; G_0 is homeomorphic to $S^1 \times R^5$;

3) $\alpha_1 = \alpha_2 = 1$; G_0 is homeomorphic to $S^3 \times R^3$.

In the case $F = C$ the topological structure of the group G_0 is elucidated in the same manner as that of the whole group G in [5].

The following remarks will also be useful to us.

Remark 1. Let $x \in G_0$ be such that $x \neq \pm 1$. Then either x or −x belongs to a one-parameter subgroup of G_0.

Indeed, if K_x is the subalgebra in K generated by the element x, then $K_x \subset K_0$ and $K_x^{(-)} = F \oplus Z_x$, where $\dim_F Z_x = 1$. The subgroup $G_x = \{a \in K_x \mid n(a) = 1\}$ of G_0 is Abelian and contains x: Moreover, either G_x is connected or $G_x = \bar{G}_x \cup (-1) \bar{G}_x$, and the subgroup $\bar{G}_x$ is connected. In the latter case either x or $-x$ belongs to $\bar{G}_x$, and it is sufficient to use the following well-known proposition: Each element of a connected Abelian Lie group belongs to a one-parameter subgroup of this group.

Remark 2. Let G_0 be one of the connected subgroups, enumerated in the paragraphs b)-d). Then the element $1 \in G_0$ belongs to a certain one-parameter subgroup G_0.

Indeed, as usual, let $K_0^{(-)} = F \oplus Z$. From the classification given above it is obvious that G_0 is connected if and only if there exists an $x \in Z$ such that $n(x) = 1$. Considering the subalgebra K_x, generated by this element in the algebra K, and the subgroup G_x connected with it, we see that G_x is a connected Abelian group that contains −1. The proof is completed in the same way as in the preceding remark.

Remark 3. Let G_0 be one of the connected, but not simply connected, subgroups, enumerated in the paragraphs b)-d). The space G_0 has the form $S^1 \times R^k$; $k=0,1,2,3,5$, and contains a one-parameter subgroup $G_0^* = S^1 \times O$, where O is the origin of coordinates in the Euclidean space R^k. Then there exists a subgroup $\hat{G}_0$ of the loop G such that the space $\hat{G}_0$ is homeomorphic to the sphere S^3 and $G_0^* \subset \hat{G}_0$.

Indeed, it is easily seen that G_0^* is contained in a subalgebra K_0^* of the form $(1, e)_R$, where $(e, e) = 1$. The subalgebra K_0^* can be embedded in the subfield of quaternions $\hat{K}_0$ of the algebra K. We take the subgroup $\hat{G}_0 = \{x \in \hat{K}_0 \mid n(x) = 1\}$ as the desired subgroup.

Remark 4. G contains an element b such that the subgroup $a \in G$ is connected for each $G(a, b)$.

It suffices to take an element $b \in V$ such that $(b, b) = 1$.

Let us now consider the problem of extension of local homomorphisms of simply connected Moufang loops, solved earlier for loops with a solvable tangent Mal'tsev algebra, in another important particular case, where the tangent algebra is semisimple.

Let A be a semisimple Mal'tsev algebra over R. Then $A = A_0 \oplus \sum_{\alpha \in I} \oplus A_\alpha$, where A_0 is a semisimple Lie subalgebra and A_α are non-Lie simple Mal'tsev algebras. The analytic Moufang loop

$$G = G_0 \times \prod_{\alpha \in I} G_\alpha , \tag{6}$$

where G_0 is a simply connected Lie group with the tangent algebra A_0 and $G_\alpha = \{a \in K_\alpha \mid n(a) = 1\}$, K_α being a Cayley—Dickson algebra over $F = R$ or C such that $K_\alpha^{(-)} = F \oplus A_\alpha$, has a tangent algebra that is isomorphic to A. If $a \in G$, then we will denote the projection of a_α in a by G_α, $a_\alpha \in G_\alpha$. Let $a, b \in G$, $K_\alpha(a, b)$ be the subalgebra of K_α that is generated by the elements a_α and b_α; $G_\alpha(a, b) = \{x \in K_\alpha(a, b) \mid n(x) = 1\}$, and $\bar{G}_\alpha(a, b)$ be the connected component of the group $G_\alpha(a, b)$ that contains the identity, and $\bar{G}(a, b) = G_0 \times \prod_{\alpha \in I} \bar{G}_\alpha(a, b)$.

If G' is a connected analytic Moufang Loop and φ is a local homomorphism of Loop G into G', then the following lemma is valid.

LEMMA 2. Let a, and b be arbitrary elements of G and let $\varphi_{a,b}$ be the restriction of the local homomorphism φ to the subgroup $\bar{G}(a, b)$. Then $\varphi_{a,b}$ can be uniquely extended to a homomorphism $\tilde{\varphi}_{a,b}$ of the group $\bar{G}(a, b)$ into the loop G'.

Proof. Since the tangent algebra of the Lie group $\bar{G}_\alpha(a, b)$ is either simple or is a solvable Lie algebra, it follows that the tangent algebra $A(a, b)$ of the group $\bar{G}(a, b)$ has the form $A(a, b) = B \oplus C$, where B is a semisimple and C is a solvable Lie algebra. Let us consider the homomorphism $\bar{\varphi}$ of the algebra $A(a, b)$ into the tangent Mal'tsev algebra A' of the loop G' that is induced by the local homomorphism $\varphi_{a,b}$. If $A'(a, b) = [A(a, b)]\bar{\varphi}$, then $A'(a, b) = B' \oplus C'$, where $B' = B\bar{\varphi}$ is a semisimple and $C' = C\bar{\varphi}$ is a solvable Lie algebra.

Let $\bar{G}'(a,b)$ be the subloop of the loop G' that corresponds to the subalgebra $A'(a,b)$. By Lemma 1, $\bar{G}'(a,b)$ is an arcwise connected subgroup. It is obvious that $\varphi_{a,b}$ is a local homomorphism of $\bar{G}(a,b)$ onto $\bar{G}'(a,b)$.

For its extension to $\bar{G}(a,b)$ we use a modification of the method set forth in [7]. Let $x \in \bar{G}(a,b)$ and let $f(t)$ be a path in $\bar{G}(a,b)$ such that $f(0) = e$, where e is the identity of the loop G, and $f(1) = x$. As shown in [7], to the path $f(t)$ corresponds a path $f'(t)$ in $\bar{G}'(a,b)$ that satisfies the following conditions:

α) $f'(0) = e'$, where e' is the identity of the group $\bar{G}'(a,b)$.

β) If U is that neighborhood of the identity of the Loop G in which the local homomorphism φ is defined, then there exists an $\varepsilon > 0$ such that

$$f(t_1)^{-1} f(t_2) \in U \quad \text{and} \quad f'(t_1)^{-1} f'(t_2) = \varphi(f(t_1)^{-1} f(t_2)).$$

for $|t_1 - t_2| \leq \varepsilon$. Path $f'(t)$ is determined uniquely by these conditions.

Let us set $x\tilde{\varphi}_{a,b} = f'(1)$ and show that the so-defined element $x\tilde{\varphi}_{a,b} \in \bar{G}'(a,b)$ does not depend on the choice of the path f(t) that joins the elements e and x (notwithstanding the possibility that $\bar{G}(a,b)$ may not be arcwise connected). The idea of the proof consists in the replacement of the path f(t) by a path that is homotopic to it and passes through certain simply connected subspaces, lying in subgroups. We need the following lemma.

LEMMA 3. Let f be a path in a direct product of topological spaces $X \times Y$ such that $f(0) = (x_0, y_0)$ and $f(1) = (x_1, y_1)$. Then f is homotopic to the product of paths h·k $(f \sim h \cdot k)$, where h is a path in the subspace (X, y_0) that joins the points (x_0, y_0) and (x_1, y_0) and k is a path in the subspace (x_1, Y) that joins the points (x_1, y_0) and (x_1, y_1).

Proof. Let us represent the path f in the form of a pair $[f_1, f_2]$ and set for $0 \leq s \leq 1$

$$f^s(t) = \begin{cases} [f_1((1+s)t),\ f_2((1-s)t)], & 0 \leq t \leq t_s = \frac{1}{1+s}; \\ [f_1(1),\ f_2((1+s)t) - 2st_s)], & t_s \leq t \leq 1. \end{cases}$$

It is obvious that $f^0 = f$, $f^1 = h \cdot k$ and f^s is a homotopy that connects these two mappings.

Let us return to the proof of Lemma 2. Let g(t) be another path in $\bar{G}(a,b)$ such that $g(0) = e$ and $g(1) = x$, and let $g'(t)$ be the path in $\bar{G}'(a,b)$ that corresponds to it. We show that $f'(1) = g'(1)$. For this let us consider the set $I_1 = \{\alpha \in I \mid \bar{G}_\alpha(a,b)$ is a subgroup in $G_\alpha\}$ that is not simply connected. If $\alpha \in I_1$, then the space $\bar{G}_\alpha(a,b)$ can be represented in the form $S^1 \times R^\kappa$, $\kappa = 0,1,2,3,5$. In addition, the path $f_\alpha(t)$ has the origin at the point $f_\alpha(0) = 1 \in S^1 \times 0 = G^*_\alpha(a,b)$ and the end at the point $f_\alpha(1) = x_\alpha = (x^{(1)}_\alpha, x^{(2)}_\alpha)$. By Lemma 3 the path $f_\alpha(t)$ is homotopic in $\bar{G}_\alpha(a,b)$ to the product $h_\alpha \cdot k_\alpha$, where

$$h_\alpha(t) \in G^*_\alpha(a,b),\ h_\alpha(1) = (x^{(1)}_\alpha, 0), \quad k_\alpha(t) \in (x^{(1)}_\alpha, R^\kappa)$$

for $0 \leq t \leq 1$. If $\alpha \notin I_1$, then we set $h_\alpha(t) = f_\alpha(t)$ and $k_\alpha(t) = f_\alpha(1)$ for $0 \leq t \leq 1$. Then f(t) is homotopic in $\bar{G}(a,b)$ to the product $h \cdot k$, where $h(t)$ and $k(t)$ are paths in $\bar{G}(a,b)$ whose projections on G_α are equal to $h_\alpha(t)$ and $k_\alpha(t)$, respectively, $\alpha \in I \cup \{0\}$. Under a continuous deformation of the path $f(t)$ in $\bar{G}(a,b)$ without change of the end points, the image $f'(t)$ will also be continuously deformed in $\bar{G}'(a,b)$ without change of the end point [7]. Therefore the end points of the path f' and $(h\cdot k)'$ coincide. The path $(h\cdot k)'$ is the product of paths $h'\cdot k'$, where h' is the path in $\bar{G}'(a,b)$ that corresponds to the path h and k', by construction, depends only on the element $k'(0)=h'(1)$ and the path k. Analogously, the path g(t) is homotopic in $\bar{G}(a,b)$ to a product $m\cdot n$, where $m_\alpha(t)\in G_\alpha^*(a,b)$ and $n_\alpha(t)\in(x_\alpha^{(1)},\mathbb{R}^\kappa)$, $0\le t\le 1$, for $\alpha\in I_1$ and $h(1)=m(1)$ and $g'\sim m'\cdot n'$. Now, to prove the equality $f'(1)=g'(1)$ it is sufficient to show that $k'(1)=n'(1)$.

As a preliminary we prove that $h'(1)=m'(1)$. For this, using Remark 3, we embed the subspaces $G_\alpha^*(a,b)$ in simply connected subgroups $\hat{G}_\alpha(a,b)$ of the loop G_α, $\alpha\in I_1$. For $\alpha\notin I_1$ we set $\hat{G}_\alpha(a,b)=\bar{G}_\alpha(a,b)$ and consider the subgroup

$$\hat{G}(a,b) = G_0\times\prod_{\alpha\in I}\hat{G}_\alpha(a,b).$$

As in the subgroup $\bar{G}(a,b)$, in the loop G' there corresponds to it an arcwise connected subgroup $\hat{G}'(a,b)$ such that the restriction of the local homomorphism φ to $\hat{G}(a,b)$ is a local homomorphism of this subgroup onto $\hat{G}'(a,b)$. The paths h(t) and m(t) Lie in $\hat{G}(a,b)$. Defining the corresponds paths in $\hat{G}'(a,b)$ that satisfy the conditions α) and β), we obviosuly get the already considered paths h'(t) and m'(t). The group $\hat{G}(a,b)$ is, by construction, simply connected. Consequently, $h\sim m$ in $\hat{G}(a,b)$; whence $h'\sim m'$ in $\hat{G}'(a,b)$. In particular $h'(1)=m'(1)$.

To complete the proof of the lemma, it is now sufficient to observe that the paths k(t) and n(t) lie in a simply connected subspace of the space of the group $\bar{G}(a,b)$. Since these paths have a common origin and a common end, it follows that $k\sim n$ in $\bar{G}(a,b)$. Moreover, by what we have already proved, $k'(0)=h'(1)=m'(1)=n'(0)$; whence $k'(1)=n'(1)$.

Thus, we have shown that the element $x\tilde{\varphi}_{a,b}=f'(1)$ does not depend on the choice of the path f in the group $\bar{G}(a,b)$ that satisfies the conditions $f(0)=e$ and $f(1)=x$. It is proved, verbatim as in [7], that the so-defined mapping $\tilde{\varphi}_{a,b}:\bar{G}(a,b)\longrightarrow\bar{G}'(a,b)$ is a homomorphism that extends the local homomorphism $\varphi_{a,b}$ and also that it is unique.

This is what was required to be proved.

We pass to the construction of the homomorphism $\tilde{\varphi}$ of the loop G into G' that extends the local homomorphism φ. Let us consider the set

$$E=\{x\in e_0\times\prod_{\alpha\in I}G_\alpha\mid x_\alpha=\pm 1,\ \alpha\in I\},$$

where e_0 is the identity of the group G_0. For $x\in E$ we find $a,b\in G$ such that $x\in\bar{G}(a,b)$. Let us set $x\tilde{\varphi}=x\tilde{\varphi}_{a,b}$ and show that the so-defined element $x\tilde{\varphi}$ does not depend on the choice of $a,b\in G$. Indeed, let $c,d\in G$ be such that $x\in\bar{G}(c,d)$. If $\alpha\in I$ for a certain $x_\alpha=-1$, then $\bar{G}_\alpha(a,b)$ contains −1, and therefore the group $G_\alpha(a,b)$ is connected. By Remark 2 the element x belongs to a certain one-parameter subgroup h(t) of the group $\bar{G}(a,b)$. Analogously $x\in g(t)$, where g(t) is a one-parameter subgroup of the group

$\bar{G}(c,d)$. It is obvious from the proof of Remark 2 that the subgroups $h_\alpha(t)$ and $g_\alpha(t)$ are contained in two-dimensional subalgebras of the algebra K_α. Consequently, there exist $u,v \in G$ such that $h(t), g(t) \subset \bar{G}(u,v)$. If U is a neighborhood of the identity of the loop G in which the local homomorphism φ is defined and y and z are elements of U such that $y=h(\tau_1)$ and $z=g(\tau_2)$ and $x=y^m=z^n$ for certain natural numbers m and n, then $x\tilde{\varphi}_{a,b}=(y\varphi)^m = x\tilde{\varphi}_{u,v} = (z\varphi)^n = x\tilde{\varphi}_{c,d}$.

Let a, b, c, and d be arbitrary elements of G and let $u \in \bar{G}(a,b) \cap \bar{G}(c,d)$. Then $u\tilde{\varphi}_{a,b} = u\hat{\varphi}_{c,d}$. Indeed, $u=u_0\bar{u}$, where $u_0 \in G_0$ and $\bar{u} \in e_0 \times \prod_{\alpha\in I} G_\alpha$. By Remarks 1 and 2 there exists an $x \in E \cap \bar{G}(a,b) \cap \bar{G}(c,d)$ such that $x\bar{u}$ belongs to the one-parameter subgroups h(t) and g(t) of the groups $\bar{G}(a,b)$ and $\bar{G}(c,d)$, respectively. As in the preceding case let us consider the subgroup $\bar{G}(p,s)$ that contains the subgroups h(t) and g(t). Then we get

$$(x\bar{u})\tilde{\varphi}_{a,b} = (x\bar{u})\tilde{\varphi}_{p,s} = (x\bar{u})\tilde{\varphi}_{c,d},$$

whence $\bar{u}\tilde{\varphi}_{a,b} = (xx\bar{u})\tilde{\varphi}_{a,b} = x\tilde{\varphi}\cdot(x\bar{u})\tilde{\varphi}_{a,b} = x\tilde{\varphi}\cdot(x\bar{u})\tilde{\varphi}_{c,d} = (xx\bar{u})\tilde{\varphi}_{c,d} = \bar{u}\tilde{\varphi}_{c,d}$. It is obvious that the mappings $\tilde{\varphi}_{a,b}$ and $\tilde{\varphi}_{c,d}$ coincide on G_0, so that

$$u\tilde{\varphi}_{a,b} = u_0\tilde{\varphi}_{a,b}\cdot\bar{u}\tilde{\varphi}_{a,b} = u_0\tilde{\varphi}_{c,d}\cdot\bar{u}\tilde{\varphi}_{c,d} = u\tilde{\varphi}_{c,d}.$$

It is now natural to set $u \in \bar{G}(a,b)$ for an arbitrary $u\tilde{\varphi} = u\tilde{\varphi}_{a,b}$. By what we have proved above, this definition does not depend on the choice of suitable $a,b \in G$. By virtue of Remark 4, there exists an element $b \in G$ such that the group $a \in G$ is connected for each $G(a,b)$. Consequently, $a \in G(a,b) = \bar{G}(a,b)$, and the mapping $\tilde{\varphi}$ is defined on the whole space G. It remains to prove that $\tilde{\varphi}$ is a homomorphism.

<u>LEMMA 4.</u> For arbitrary $x \in E$ and $y \in G$

$$(xy)\tilde{\varphi} = x\tilde{\varphi}\cdot y\tilde{\varphi} = y\tilde{\varphi}\cdot x\tilde{\varphi}.$$

<u>Proof.</u> By Remark 4 there exists $z \in G$ such that $G_\alpha(y,z)$ is connected for each $\alpha \in I$. Then $G(y,z) = \bar{G}(y,z)$, $E \subset G(y,z)$ and the statement of the lemma follows from the equation $xy=yx$ and the fact that the mapping $\tilde{\varphi}$ coincides with the homomorphism $\tilde{\varphi}_{y,z}$ on $G(y,z)$. The lemma is proved.

<u>LEMMA 5.</u> Let a and b be arbitrary elements of G. Then the mapping $\tilde{\varphi}$ is a homomorphism on $G(a,b)$.

<u>Proof.</u> Each element $u \in G(a,b)$ can be represented in the form $u=xu_0$, where $x \in E$ and $u_0 \in \bar{G}(a,b)$. Moreover, $E \subset G(a,b)$, so that $G(a,b) = E\cdot\bar{G}(a,b)$. The mapping $G(a,b)$ acts as a homomorphism on the group $\tilde{\varphi} = \tilde{\varphi}_{a,b}$. Let x be an arbitrary element of E and u_0 and v_0 be arbitrary elements of $\bar{G}(a,b)$. Let us consider the pseudoautomorphism $T_{x\tilde{\varphi}}$ of the Loop G' with the companion $(x\tilde{\varphi})^3 = x^3\tilde{\varphi} = x\tilde{\varphi}$. Setting, for brevity, $z\tilde{\varphi}=z'$ for $z \in G(a,b)$ and using Lemma 4, we get

$$x'\cdot u_0'v_0' = x'\cdot(u_0v_0)' = x'\cdot(u_0v_0)'T_{x'} = (x'\cdot u_0'T_{x'})\cdot v_0'T_{x'} = x'u_0'\cdot v_0', \tag{7}$$

i.e., the triple x', u_0', v_0' is associative. If $y \in E$, then

$$(u_0 \cdot y v_0)' = (y \cdot u_0 v_0)' = y' \cdot u_0' v_0' = u_0' \cdot y' v_0' = u_0' \cdot (y v_0)'. \tag{8}$$

Here we have again used Lemma 4 and Eq. (7). Let u and v be arbitrary elements of $G(a,b)$ such that $u = x u_0$ and $v = y v_0$, where $x, y \in E$, and $u_0, v_0 \in \bar{G}(a,b)$. Then, substituting $y v_0$ for v_0 in (7), which is possibly by virtue of (8), we get

$$(uv)' = x' \cdot (u_0 \cdot y v_0)' = x' u_0' \cdot (y v_0)' = (x u_0)'(y v_0)' = u'v'.$$

The lemma is proved.

The following lemma is now easily obtained.

LEMMA 6. Let G be a semisimple analytic Moufang loop of form (6), G' be a connected analytic Moufang loop, and φ be a local homomorphism of G into G'. Then φ can be uniquely extended to a homomorphism $\tilde{\varphi}$ of the loop G into G'.

Indeed, the above-constructed mapping $\tilde{\varphi}$ extends the local homomorphism φ. If $a, b \in G$, then $a, b \in G(a,b)$ and, by Lemma 5, we have $(ab)\tilde{\varphi} = a\tilde{\varphi} \cdot b\tilde{\varphi}$. The uniqueness of the homomorphism $\tilde{\varphi}$ is obvious. The lemma is proved.

We now formulate the main theorem.

THEOREM 1. Let G and G' be connected analytic Moufang loops, G be simply connected, and φ be a local homomorphism of the loop G into G'. Then φ can be uniquely extended to a homomorphism $\tilde{\varphi}$ of the loop G into G' into the large. If φ is a local isomorphism and the loop G' is simply connected, then $\hat{\varphi}$ is an isomorphism of the loop G onto G'.

Proof. We start the proof with the last statement. In this case, without loss of generality we can consider the simply connected analytic Moufang loop, constructed in [5], as G. Then $G = P \cdot N$, where P is a semisimple subloop of the form (6), N is a simply connected solvable normal divisor of the loop G. Let φ_P and φ_N denote the restrictions of the local homomorphism φ to P and N, respectively. By Lemma 6, φ_P can be extended to a homomorphism $\tilde{\varphi}_P$ of the loop P into G'. In its turn, φ_N can also be extended to a homomorphism $\hat{\varphi}_N$ of the loop N into G' by virtue of [5]. If u is an arbitrary element of G and u = pa, where $p \in P, a \in N$, then we set

$$u\tilde{\varphi} = p\tilde{\varphi}_P \cdot a\tilde{\varphi}_N. \tag{9}$$

It is easily seen that $\tilde{\varphi}$ is a properly defined mapping of the loop G into G' that extends the local homomorphism φ. We prove that $\tilde{\varphi}$ is a homomorphism of the loop G into G'. For this let us consider an infinitesimally generated element $x \in P$ and an arbitrary $a \in N$. As in [5], we can show that

$$aT_x\tilde{\varphi} = a\tilde{\varphi}T_{x\tilde{\varphi}}. \tag{10}$$

If p and s are arbitrary elements of $p = \langle p_1 \dots p_n \rangle$, and $s = \langle s_1 \dots s_m \rangle$, where $p_1, \dots, p_n$, $s_1, \dots, s_m$ are infinitesimally generated elements of the loop P, and $a, b \in N$, then by [5]

$$pa \cdot sb = ps \cdot (aW_1 \cdot bW_2)W, \tag{11}$$

where W, W_1, and W_2 are T-words $p_1,\dots,p_n, s_1,\dots,s_m$. An analogous equality holds G' also for arbitrary a', b' and $p' = \langle p'_1 \dots p'_n \rangle, s' = \langle s'_1 \dots s'_m \rangle$ if p'_i and s'_j are infinitesimally generated elements of the loop G'. Using (9)-(11), we get

$$(pa \cdot sb)\tilde{\varphi} = (ps)\tilde{\varphi} \cdot (aW_1 \cdot bW_2)W\tilde{\varphi} = (p\tilde{\varphi} \cdot s\tilde{\varphi}) \cdot (a\tilde{\varphi}W'_1 \cdot b\tilde{\varphi}W'_2)W' = (p\tilde{\varphi} \cdot a\tilde{\varphi})(s\tilde{\varphi} \cdot b\tilde{\varphi}) = (pa)\tilde{\varphi} \cdot (sb)\tilde{\varphi},$$

where W', W'_1, and W'_2 are T-words of the loop G' that are obtained from W, W_1, and W_2 by replacing the elements $p_1,\dots,p_n,s_1,\dots,s_m$ by $p_1\tilde{\varphi},\dots,p_n\tilde{\varphi},s_1\tilde{\varphi},\dots,s_m\tilde{\varphi}$, respectively. Here the homomorphicity of the mapping $\tilde{\varphi}$ on P and N has also been used.

Thus, $\tilde{\varphi}$ is a homomorphism of G into G' that generates the local homomorphism φ. Since φ is a local isomorphism and G' is generated by each of its neighborhoods of the identity, it follows that $\tilde{\varphi}$ is a covering of G onto G'. Since G' is simply connected, it now follows that $\tilde{\varphi}$ is an isomorphism of the loop G onto the loop G', which was required to be proved.

Let us consider the general case of a local homomorphism φ of a simply connected loop G into a connected analytic Moufang loop G'. By virtue of what we have proved above, we can consider the simply connected analytic Moufang loop, constructed in [5], as G, and the extending homomorphism $\tilde{\varphi}$ can be defined by Eq. (9).

The theorem is proved.

2. We formulate some consequences of Theorem 1.

THEOREM 2. Let G be a connected analytic Moufang loop, A be the tangent Mal'tsev algebra of the loop G, and A_0 be a Lie subalgebra of A. Then the loop G_0, corresponding to the subalgebra A_0, is a Lie group. In particular, if A is a Lie algebra, then G is a Lie group.

Proof. Let us consider the simply connected Lie group $\tilde{G}_0$ with the tangent algebra A_0. The local homomorphism $\varphi: \tilde{G}_0 \to G$, induced by the inclusion $A_0 \subset A$, can be extended to a homomorphism in the large by Theorem 1. The image of the Lie group $\tilde{G}_0$ under this homomorphism is the subloop G_0; whence the statement of the theorem follows.

Remark. It should be observed that the subgroup G_0, corresponding to the subalgebra A_0, is understood in the sense of the definition given at the beginning of this article, i.e., as a group equipped with the intrinsic topology. But if G_0 is equipped with the subspace topology of the space G, then G_0 may turn out to a nonclosed subspace and may not be a Lie group.

The following theorem gives a classification of the connected analytic Moufang loops that are locally isomorphic to a given loop.

THEOREM 3. Let $\mathfrak{A}$ be the class of all connected analytic Moufang loops that are locally isomorphic to a given loop. Then the class $\mathfrak{A}$ contains a unique (up to isomorphism) simply connected loop $\tilde{G}$. An arbitrary loop G from the class $\mathfrak{A}$ is a homomorphic image of the loop $\tilde{G}$ such that the kernel of the covering homomorphism $\varphi: \tilde{G} \to G$ is a discrete central normal subgroup of the loop $\widetilde{G}$.

This theorem follows easily from [3, 5] and Theorem 1.

In conclusion, we formulate two theorems that are analogs of theorems of Pontryagin [7] and Mal'tsev [4] and characterize normal subloops of simply connected analytic Moufang loops.

THEOREM 4. Let G be a simply connected analytic Moufang loop and N' be a local normal subloop of it. Then a certain neighborhood of the identity of the loop N' can be embedded as a neighborhood of the identity in a normal subgroup N of the loop G in the large.

THEOREM 5. Every connected normal subloop N of a simply connected analytic Moufang loop G is simply connected.

With regard for [5] and Theorem 1, the proof of Theorem 4 is carried out verbatim as in [7]. To prove Theorem 5 we can use the plan of arguments from [4], applying, where necessary, the results of [5] and Theorem 1.

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