PRETABULAR EXTENSIONS OF LEWIS $S4$

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In this article further results related to the lattice M of all normal modal logics containing Lewis $S4$ are found. The study of the M lattice was begun in [1], where it was established that there exists a close relation between M and a lattice $\mathcal{L}$ of superintuitionistic logics. So-called tabular logics were investigated in one of the sections of this article, and the finite approximability of pretabular logics was also established. We will use previous results [1, 2] to describe all pretabular modal logics which are extensions of $S4$. It has been proved that there are, in all, five such logics, including $S5$, whose pretabular property has been previously established [8]. Pretabular logics will be denoted in this article by PM1-PM5. All these logics are finitely axiomatizable, so that there exists an effective criterion for the tabular property of modal logics. The natural extensions of each of PM1-PM5 form an infinitely decreasing chain.

Because of the dual automorphism between M and the lattice of manifolds of topological Boolean algebras, the results obtained are carried over to this lattice of manifolds. In particular, there exist precisely five pretabular manifolds of topological Boolean algebras.

In this work, notation and results from a previous article [1], familiarity with which is presupposed, are used. The methods of proof are similar to those of [2], in which all pretabular extensions of intuitionistic logic were described.

1. DEFINITION OF PM1-PM5 LOGICS

We recall [1] that a logic is tabular if it can be represented as a set of formulas true in a finite algebra. A logic is called pretabular if it is tabular and if all its natural extensions are tabular.

It has been previously proved [2] that there exist exactly three pretabular superintuitionistic logics, namely, LC, $\mathcal{L}_2$, and $\mathcal{L}_3$. A mapping σ has been constructed ([1], Sec. 3 (7)) from the lattice $\mathcal{L}$ of superintuitionistic logics into M. Let us now introduce the notation

$$\mathrm{PM1} = \sigma(LC), \quad \mathrm{PM2} = \sigma(\mathcal{L}_2), \quad \mathrm{PM3} = \sigma(\mathcal{L}_3).$$

We define two families of finite topological Boolean algebras (TBA).

For $n \geqslant 1$ we denote by

$$\mathcal{U}_n = \langle \mathcal{U}_n, \&, \vee, \rightarrow, \sim, \Box, 1 \rangle$$

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the algebra, such that $\langle \mathcal{U}_n, \&, \vee, \rightarrow, \sim, 1\rangle$ is a finite Boolean algebra with n atomic formulas $a_1, \ldots, a_n$.

$$\Box x = \begin{cases} 1, & \text{if } x=1, \\ a_n, & \text{if } a_n \leqslant x < 1, \\ 0, & \text{if } a_n \nleqslant x. \end{cases}$$

We denote by

$$V_n = \langle V_n, \&, \vee, \rightarrow, \sim, \Box, 1\rangle$$

the TBA, such that $\langle V_n, \&, \vee, \rightarrow, \sim, 1\rangle$ is a finite Boolean algebra with n atomic formulas.

$$\Box x = \begin{cases} 1, & \text{if } x=1, \\ 0, & \text{if } x \neq 1. \end{cases}$$

We denote by PM4 the set of formulas that are true in all TBA $\mathcal{U}_n$ $(n=1,2,\ldots)$ and let PM5 be the set of formulas true in all TBA V_n $(n=1,2,\ldots)$. We note that PM5 coincides with Lewis $S5$ (cf., for example, [8]).

We will also obtain an algebraic characterization of PM1, PM2, and PM3. We recall [2] that the superintuitionistic logic LC can be defined as the set of formulas of intuitionistic logic true in all pseudo-Boolean algebras (PBA) L_n $(n=2,3,\ldots)$, where L_n is a linearly ordered PBA containing n elements. The superintuitionistic logic $\mathcal{L}_2$ coincides with the set of formulas true in all PBA B_n $(n=0,1,2,\ldots)$, where B_0 is a two-element Boolean algebra and $B_{n+1} = B_0 + B_0^n + B_0$. $\mathcal{L}_3$ coincides with the set of formulas true in all PBA C_n $(n=0,1,2,\ldots)$, where $C_n = \overline{B_0^n} + B_0$.

It was previously proved ([1], Sec. 3) that using any PBA $\mathfrak{A}$ it is possible to construct a TBA $S(\mathfrak{A})$ which would be the least TBA containing $\mathfrak{A}$ as a sublattice. We find that PM1 $= \sigma(LC)$ is the set of formulas true in all TBA $S(L_n)$ $(n=2,3,\ldots)$, by using Theorem 6b and the corollary of Theorem 5 from [1]. Similarly, PM2 $= \sigma(\mathcal{L}_2)$ is the set of formulas true in the TBA $S(B_n)$ $(n=0,1,2,\ldots)$, and PM3 $= \sigma(\mathcal{L}_3)$ is the set of formulas true in all TBA $S(C_n)$ $(n=0,1,2,\ldots)$.

2. REPRESENTING QUASIORDERED SETS

In this section we will study the relation between topological Boolean algebras and quasiordered sets.

Suppose Q is a nonempty set, and let R be a quasiordering, i.e., a reflexive and transitive ordering on Q. It is known that Q is a topological space if the interiority operator $\Box$ is defined by the equality

$$\Box X = \{x \mid \forall y\,(xRy \Rightarrow y \in X)\}$$

for $X \subseteq Q$. We may then define on the set of all subsets $P(Q)$ of the space Q the TBA

$$\mathcal{T}(Q) = \langle P(Q), \&, \vee, \rightarrow, \sim, \Box, 1\rangle,$$

where $\&, \vee$ and $\sim$ denote set-theoretic intersection, union, and complementation, respectively, $X \rightarrow Y = \sim X \vee Y$, $1 = Q$.

We now note lemmas similar to those from Sec. 2 of [2].

LEMMA 1. Suppose Q is quasiordered by the relation R, and let Q' be an open subset of Q. Then there exists a homomorphism h from $\mathcal{T}(Q)$ onto $\mathcal{T}(Q')$

The desired homomorphism is given by $h(X) = X \cap Q'$ for $X \subseteq Q$ (cf., [3], Chap. 3.1).

LEMMA 2. Suppose the mapping θ of a set Q with quasiordering R onto the set Q' quasiordered by the relation R' satisfies the conditions

(1) $xRy \Rightarrow \theta(x)R'\theta(y)$,

(2) $\theta(x)R'\theta(y) \Rightarrow \exists y'(xRy'$ and $\theta(y) = \theta(y'))$.

Then the algebra $\mathcal{T}(Q')$ is isomorphically embedded in $\mathcal{T}(Q)$.

It can be easily verified that $h(Y) = \theta^{-1}(Y)$ for $Y \subseteq Q'$ is an isomorphism of into $\mathcal{T}(Q')$ and $\mathcal{T}(Q)$.

We now correlate to every TBA $\mathcal{L} = \langle \mathcal{B}; \&, \vee, \rightarrow, \sim, \Box, 1 \rangle$ its representing set, i.e., the space $Q_{\mathcal{L}}$ of all simple filters of the algebra $\mathcal{L}$ quasiordered by the relation

$$\Phi_1 R \Phi_2 \leftrightharpoons (\forall x \in \mathcal{B})(\Box x \in \Phi_1 \Rightarrow \Box x \in \Phi_2),$$

where $\Phi_1, \Phi_2 \in Q_{\mathcal{L}}$. The following assertion is well known.

LEMMA 3. The mapping $\varphi : \mathcal{L} \rightarrow \mathcal{T}(Q_{\mathcal{L}})$, determined by the condition $\varphi(x) = \{\Phi / \Phi \in Q_{\mathcal{L}}$ $\& x \in \Phi\}$, is an isomorphism from the algebra $\mathcal{L}$ into the algebra $\mathcal{T}(Q_{\mathcal{L}})$. In particular, if $\mathcal{L}$ is a finite TBA, φ is an isomorphism of $\mathcal{L}$ into $\mathcal{T}(Q_{\mathcal{L}})$.

Remark. If Q is a finite quasiordered set, $Q_{\mathcal{T}(Q)}$ is isomorphic to Q.

The relation between operations over TBA and operations over the representing sets is characterized by the next lemmas.

LEMMA 4. a) Suppose $\mathcal{L}$ is a TBA, and let h be a homomorphism from $\mathcal{L}$ onto the TBA $\mathcal{L}_1$. Then $Q_{\mathcal{L}_1}$ is isomorphic to an open subset of $Q_{\mathcal{L}}$.

b) Suppose $\mathcal{L}$ is a capital TBA, $\Phi_0 \in Q_{\mathcal{L}}$, and let $Q = \{\Phi / \Phi \in Q_{\mathcal{L}} \;\&\; \Phi_0 R \Phi\}$. Then there exists a TBA $\mathcal{L}_1$ such that the set $Q_{\mathcal{L}_1}$ is isomorphic to Q, and $\mathcal{L}_1$ is the homomorphism of the TBA $\mathcal{L}$.

Proof. a) Suppose $\mathcal{L} = \langle \mathcal{B}, \&, \vee, \rightarrow, \sim, \Box, 1 \rangle$, and let $\mathcal{L}_1 = \langle \mathcal{B}_1, \&, \vee, \rightarrow, \sim, \Box, 1 \rangle$. Since $\mathcal{L}$ and $\mathcal{L}_1$ are pseudo-Boolean algebras in $\&, \vee, \rightarrow, \sim, 1$, the mapping $\alpha(\Phi) = h^{-1}(\Phi)$ for $\Phi \in Q_{\mathcal{L}_1}$ constructed in Lemma 4 [2] is a one-to-one correspondence between the sets $Q_{\mathcal{L}_1}$ and $Q' = \{\Phi' \in Q_{\mathcal{L}} \mid \Phi' \supseteq h^{-1}(1)\}$.

It is easy to verify that for any $\Phi_1, \Phi_2 \in Q_{\mathcal{L}_1}$:

$$\Phi_1 R \Phi_2 \Leftrightarrow h^{-1}(\Phi_1) R h^{-1}(\Phi_2).$$

It remains for us to note that Q' is an open subset of $Q_{\mathcal{L}}$. In fact, we have from $\Phi_1 R \Phi_2$ and $\Phi_1 \supseteq h^{-1}(1)$ for $\Phi_1, \Phi_2 \in Q_{\mathcal{L}}$,

$$x \in h^{-1}(1) \Rightarrow h(x) = 1 \Rightarrow h(\Box x) = \Box h(x) = 1 \Rightarrow \Box x \in \Phi_1 \Rightarrow \Box x \in \Phi_2 \Rightarrow x \in \Phi_2,$$

i.e., $\Phi_2 \supseteq h^{-1}(1)$, QED.

b) We define on $\mathcal{L}$ the relation

$$x \approx_{\Phi_0} y \leftrightharpoons \Box(x \leftrightarrow y) \in \Phi_0 .$$

It is easy to verify that $\approx_{\Phi_0}$ is a congruence on $\mathcal{L}$. We note that

$$x \approx_{\Phi_0} 1 \Longleftrightarrow \Box x \in \Phi_0 .$$

Now suppose that h is a homomorphism from $\mathcal{L}$ onto $\mathcal{L}_1 = \mathcal{L}/\approx_{\Phi_0}$. By a) of Lemma 4 the mapping $\alpha : Q_{\mathcal{L}_1} \to Q_{\mathcal{L}}$, where $\alpha(\Phi) = h^{-1}(\Phi)$ for $\Phi \in Q_{\mathcal{L}_1}$, is an isomorphism between $Q_{\mathcal{L}_1}$ and the set

$$Q' = \{\Phi' \in Q_{\mathcal{L}} \mid \Phi' \supseteq h^{-1}(1)\}.$$

It remains for us to prove that $Q' = Q$. Suppose $\Phi \in Q_{\mathcal{L}}$. Then

$$\Phi \in Q' \Leftrightarrow \Phi \supseteq h^{-1}(1) \Leftrightarrow \forall x\,[h(x) = 1 \Rightarrow x \in \Phi] \Leftrightarrow$$

$$\Leftrightarrow \forall x\,[\Box x \in \Phi_0 \Rightarrow x \in \Phi] \Leftrightarrow \forall x\,[\Box x \in \Phi_0 \Rightarrow \Box x \in \Phi] \Leftrightarrow \Phi_0 R \Phi \Leftrightarrow \Phi \in Q.$$

The lemma is proved.

LEMMA 5. Suppose $\mathcal{L} = \langle \mathcal{B}, \&, \vee \rangle$ is a distributive lattice, and let $\mathcal{L}_1 = \langle \mathcal{B}_1, \&, \vee \rangle$ be a sublattice of $\mathcal{L}$. Then any simple filter Φ_1 on $\mathcal{L}_1$ can be extended to a simple filter Φ_0 on $\mathcal{L}$, such that $\Phi_0 \cap \mathcal{B}_1 = \Phi_1$

Proof. Suppose Φ_1 is a simple filter on $\mathcal{L}_1$. We set $\overline{\Phi}_1 = \mathcal{B}_1 \setminus \Phi_1$ and examine the set: $\Sigma = \{\Phi / \Phi$ is a filter on $\mathcal{L}$, $\Phi_1 \subseteq \Phi$ and $\Phi \cap \overline{\Phi}_1 = \emptyset\}$.

The set Σ is nonempty, since it contains the filter $\Phi = \{x \mid x \in \mathcal{L}$ and $(\exists y \in \Phi_1)(y \leqslant x)\}$. Further, the union of any chain of filters in Σ again belongs to Σ. By Zorn's lemma, Σ contains a maximal element Φ_0. It is easy to verify that Φ_0 is a simple filter on $\mathcal{L}$. Since $\Phi_0 \in \Sigma$, we have $\Phi_0 \cap \mathcal{B}_1 = \Phi_1$.

The lemma is proved.

LEMMA 6. Suppose $\mathcal{L}$ and $\mathcal{L}_1$ are TBA, and let $\mathcal{L}_1$ be a subalgebra of $\mathcal{L}$. Then there exists a mapping θ from $Q_{\mathcal{L}}$ onto $Q_{\mathcal{L}_1}$ that satisfies conditions (1) and (2) of Lemma 2.

Proof. By Lemma 5, the mapping $\theta(\Phi) = \Phi \cap \mathcal{L}_1$, where $\Phi \in Q_{\mathcal{L}}$, is a mapping from $Q_{\mathcal{L}}$ onto $Q_{\mathcal{L}_1}$. Evidently, for any $\Phi_1, \Phi_2 \in Q_{\mathcal{L}}$ $\Phi_1 R \Phi_2$ implies $\theta(\Phi_1) R \theta(\Phi_2)$.

Now suppose $\theta(\Phi_1) R \theta(\Phi_2)$, where $\Phi_1, \Phi_2 \in Q_{\mathcal{L}}$. This means that $\Phi_1 \cap \mathcal{L}_1 \cap G(\mathcal{L}_1) \subseteq \Phi_2 \cap \mathcal{L}_1 \cap G(\mathcal{L}_1)$, where $G(\mathcal{L}_1)$ is the set of all open elements of the algebra $\mathcal{L}_1$, i.e., $G(\mathcal{L}_1) = \{x \mid x \in \mathcal{L}_1$ and $x = \Box x\}$. Hence,

$$\Phi_1 \cap G(\mathcal{L}_1) \subseteq \Phi_2 \cap G(\mathcal{L}_1). \tag{*}$$

We set $\widetilde{\Phi}_2 = \mathcal{L}_1 \setminus \Phi_2$ and consider the set

$$\Sigma = \{\Phi / \Phi \text{ is a filter on } \mathcal{L} \text{ and } (\Phi_1 \cap G(\mathcal{L})) \cup (\Phi_2 \cap \mathcal{L}_1) \subseteq \Phi \text{ and } \Phi \cap \widetilde{\Phi}_2 = \emptyset\}.$$

The set Σ is nonempty, since it contains the element $\Phi_0 = \{x \mid x \in \mathcal{L}$ and

$$(\exists y \in \Phi_1 \cap G(\mathcal{L}))(\exists z \in \Phi_2 \cap \mathcal{L}_1)(y \,\&\, z \leqslant x)\}.$$

In fact, let us assume that $\Phi_0 \cap \widetilde{\Phi}_2 \neq \emptyset$, $x \in \Phi_0 \cap \widetilde{\Phi}_2$. Then y and z can be found such that

$$y \in \Phi_1 \cap G(\mathcal{L}),\ z \in \Phi_2 \cap \mathcal{L}_1 \text{ and } y \,\&\, z \leqslant x.$$

Hence, $y \leqslant z \to x$ and $y = \square y \leqslant \square(z \to x)$, so that $\square(z \to x) \in \Phi_1$. Moreover, $z \in \mathcal{L}_1$ and $x \in \tilde{\Phi}_2 \subseteq \mathcal{L}_1$, so that $\square(z \to x) \in G(\mathcal{L}_1)$. Consequently, by (*) we have $\square(z \to x) \in \Phi_2$, that is, $(z \to x) \in \Phi_2$. But we also have $z \in \Phi_2$, so that $x \in \Phi_2$ contradicts $x \in \tilde{\Phi}_2$. Thus, $\Phi_0 \cap \tilde{\Phi}_2 = \emptyset$. Evidently, Φ_0 is a filter on $\mathcal{L}$ and

$$(\Phi_1 \cap G(\mathcal{L})) \cup (\Phi_2 \cap \mathcal{L}_1) \subseteq \Phi_0 .$$

Further, the union of any chain of elements in Σ again occurs in Σ, so that, by Zorn's lemma, Σ has a maximal element in Φ_2'. It is verified in a standard fashion that Φ_2' is a simple filter, i.e., $\Phi_2' \in Q_{\mathcal{L}}$.

It follows from the definition of Σ that $\Phi_2' \cap \mathcal{L}_1 = \Phi_2 \cap \mathcal{L}_1$. In fact, $\Phi_2 \cap \mathcal{L}_1 \subseteq \Phi_2'$. On the other hand, if $x \in \Phi_2' \cap \mathcal{L}_1$ we have $x \in \mathcal{L}_1$ and $x \notin \tilde{\Phi}_2$, i.e., $x \in \Phi_2$. Thus, $\theta(\Phi_2') = \theta(\Phi_2)$. Moreover, $\Phi_1 \cap G(\mathcal{L}) \subseteq \Phi_2' \cap G(\mathcal{L})$, i.e., $\Phi_1 R \Phi_2'$. Thus, we have

$$\theta(\Phi_1) R \theta(\Phi_2) \Longrightarrow \exists \Phi_2' (\Phi_1 R \Phi_2' \text{ and } \theta(\Phi_2) = \theta(\Phi_2')).$$

The lemma is proved.

We recall [2] that as the representing set of a pseudo-Boolean algebra $\mathfrak{A}$ is understood as the space $S_{\mathfrak{A}}$ of simple filters on $\mathfrak{A}$ partially ordered by inclusion. The PBA $G(\mathcal{L})$, consisting of all open elements of $\mathcal{L}$, corresponds to any TBA $\mathcal{L}$. The relation between $Q_{\mathcal{L}}$ and $S_{G(\mathcal{L})}$ is established by the following assertion.

<u>LEMMA 7.</u> Suppose $\mathcal{L}$ is a TBA, and let $G(\mathcal{L})$ be the PBA of all open elements of $\mathcal{L}$. Then $S_{G(\mathcal{L})}$ is isomorphic to the quotient set of $Q_{\mathcal{L}}$ by means of the equivalence

$$\Phi_1 =_R \Phi_2 \Leftrightarrow \Phi_1 R \Phi_2 \quad \text{and } \Phi_2 R \Phi_1 .$$

<u>Proof.</u> Suppose $\Phi \in Q_{\mathcal{L}}$. We set $\psi(\Phi) = \Phi \cap G(\mathcal{L})$. Then, evidently, $\psi(\Phi)$ is a simple filter on $G(\mathcal{L})$. Further, by the definition of the relation R on $Q_{\mathcal{L}}$, we have

$$\Phi_1 R \Phi_2 \Longleftrightarrow \Phi_1 \cap G(\mathcal{L}) \subseteq \Phi_2 \cap G(\mathcal{L}).$$

The following assertions are therefore true.

$$\Phi_1 R \Phi_2 \Longleftrightarrow \psi(\Phi_1) \subseteq \psi(\Phi_2),$$

$$\Phi_1 =_R \Phi_2 \Longleftrightarrow \psi(\Phi_1) = \psi(\Phi_2).$$

By Lemma 5, ψ is a mapping of $Q_{\mathcal{L}}$ onto $S_{G(\mathcal{L})}$.

The lemma is proved.

It is possible to construct a TBA $\mathcal{L} = S(\mathfrak{A})$ by means of any PBA $\mathfrak{A}$ (cf., [1], Sec. 3 (4)). We have the following assertion.

<u>LEMMA 8.</u> Suppose $\mathfrak{A}$ is a PBA, and let $S(\mathfrak{A})$ be the TBA corresponding to it. Then the set $Q_{S(\mathfrak{A})}$ is partially ordered by the relation R and isomorphic to the set $S_{\mathfrak{A}}$.

<u>Proof.</u> The relation R is a quasiordering on $Q_{S(\mathfrak{A})}$. We will prove that R is antisymmetric. Suppose $\Phi_1, \Phi_2 \in Q_{S(\mathfrak{A})}$ and let $\Phi_1 \neq \Phi_2$. Then there exists an $x \in \Phi_1$ such that $x \notin \Phi_2$. The TBA $S(\mathfrak{A})$ is generated by the set $G(S(\mathfrak{A}))$, by Lemma 3.2 from [1], and by definition of the algebra $S(\mathfrak{A})$. Therefore, x can be represented in the form

$$x = (\sim a_1 \vee b_1) \& \ldots \& (\sim a_n \vee b_n),$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n \in G(S(\mathfrak{A}))$. We have $(\sim a_j \vee b_j) \notin \Phi_2$ for given j $(1 \leq j \leq n)$. Then $\sim a_j \notin \Phi_2$, and $b_j \notin \Phi_2$, i.e., $a_j \in \Phi_2$, and $b_j \notin \Phi_2$. On the other hand, $(\sim a_j \vee b_j) \in \Phi_1$, that is, $a_j \notin \Phi_1$ or $b_j \in \Phi_1$. Hence,

$$\Phi_1 \cap G(S(\mathfrak{A})) \neq \Phi_2 \cap G(S(\mathfrak{A})),$$

that is, we cannot have simultaneously $\Phi_1 R \Phi_2$ and $\Phi_2 R \Phi_1$. Thus, R is a partial order on $Q_{S(\mathfrak{A})}$, and the relation $=_R$ defined in Lemma 7 coincides with equality. By Lemma 7, $S_{G(S(\mathfrak{A}))}$ is isomorphic to $Q_{S(\mathfrak{A})}$. Therefore, the assertion follows from the isomorphism between $\mathfrak{A}$ and $G(S(\mathfrak{A}))$ (cf., [1], Sec. 3 (5)).

We will use Lemma 8 to describe the representing sets of the TBA $S(L_n)$, $S(B_n)$ and $S(C_n)$(cf. Sec. 1). By Lemma 8, $Q_{S(L_n)}$, $Q_{S(B_n)}$, and $Q_{S(C_n)}$ are partially ordered and isomorphic to S_{L_n}, S_{B_n}, and S_{C_n}, respectively. Thus, the set $Q_{S(L_n)}$ for $n=2,3,\ldots$ is a linearly ordered set containing $(n-1)$ elements. The set $Q_{S(B_n)}$ for $n = 0,1,2,\ldots$ consists of $(n+1)$ elements, including a greatest and a least element, the remaining elements being pairwise incomparable.

In conclusion, we note that for $n \geq 1$ the representing set Q_{U_n} of the algebra U_n constructed in Sec. 1 is isomorphic to Y_n, where Y_n is the set $\{1,\ldots,n\}$ with the quasiorder R : $uRv \Leftrightarrow u \leq n-1$ or $v = n$.

The algebras Q_{V_n} that represent the set V_n are isomorphic to X_n, where X_n is the set $\{1,\ldots,n\}$ with a quasiorder R such that $(\forall uv)uRv$.

3. SPECIAL PROPERTIES OF TBA THAT CAN BE DETERMINED FROM THEIR REPRESENTING SETS

We recall that a TBA is called completely bound if it satisfies the condition $\Box x \vee \Box y = 1 \Rightarrow \Box x = 1$ or $\Box y = 1$.

The PBA is called completely bound if we have in it $x \vee y = 1 \Rightarrow x = 1$ or $y = 1$.

LEMMA 9. The following conditions are equivalent for any TBA $\mathfrak{B}$:

(a) $\mathfrak{B}$ is a completely bound TBA;

(b) $G(\mathfrak{B})$ is a completely bound TBA;

(c) the representing set $Q_{\mathfrak{B}}$ satisfies the condition $\exists u \forall v\, uRv$.

Proof. Equivalence between (a) and (b) is evident.

Suppose we have (b). Then there exists in $S_{G(\mathfrak{B})}$ a least element since $\Phi_0 = \{1\}$ is a simple filter in $G(\mathfrak{B})$. By Lemma 7, there exists a homomorphism ψ from $Q_{\mathfrak{B}}$ onto $S_{G(\mathfrak{B})}$ such that for $\Phi, \Phi_1 \in Q_{\mathfrak{B}}$ we have

$$\psi(\Phi) \subseteq \psi(\Phi_1) \Longleftrightarrow \Phi R \Phi_1.$$

Let us take any $\Phi \in Q_{\mathfrak{B}}$ such that $\psi(\Phi) = \Phi_0$. Then for any $\Phi_1 \in Q_{\mathfrak{B}}$ we have $\Phi R \Phi_1$, i.e., condition (c).

On the other hand, if (c), by Lemma 7, $S_{G(\mathfrak{B})}$ has a least element Φ_0. Since

Φ_0 is contained in any simple filter in $G(\mathcal{L})$, we have $\Phi_0 = \{1\}$, that is, $G(\mathcal{L})$ is completely bound. Thus, conditions (b) and (c) are equivalent, and the lemma is proved.

ASSERTION 1. For any TBA $\mathcal{L}$ the following conditions are equivalent:

a) $\Box(x \leftrightarrow \Diamond x) \vee \Box(x \leftrightarrow \Box x) = 1$ for all $x \in \mathcal{L}$;

b) the representing set $Q_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the condition

$$(\forall u v w)(uRv \text{ and } vRw \Rightarrow u = v \text{ or } v = w).$$

Proof. We assume that (b) does not hold. Then there exist simple filters Φ_1, Φ_2, and Φ_3 in $\mathcal{L}$ such that $\Phi_1 R \Phi_2$, $\Phi_2 R \Phi_3$, and $x \in \Phi_2 \setminus \Phi_1$, $y \in \Phi_2 \setminus \Phi_3$ for given x and y in $\mathcal{L}$. Suppose that $z = x \& y$. We have $z \in \Phi_2$, $z \notin \Phi_1$, and $z \notin \Phi_3$. We obtain $\Phi_1 R \Phi_2$ and $\Phi_2 R \Phi_3$ from conditions $\Diamond z = \sim\Box\sim z \in \Phi_1$ and $\Box z \notin \Phi_2$. Therefore, $(z \leftrightarrow \Diamond z) \notin \Phi_1$ and $\Box(z \leftrightarrow \Box z) \notin \Phi_2$. Consequently, $\Box(z \leftrightarrow \Diamond z) \notin \Phi_1$, and $\Box(z \leftrightarrow \Box z) \notin \Phi_1$, that is, $\Box(z \leftrightarrow \Diamond z) \vee \Box(z \leftrightarrow \Box z) \notin \Phi_1$ and $\Box(z \leftrightarrow \Diamond z) \vee \Box(z \leftrightarrow \Box z) \neq 1$.

Conversely, suppose condition (b). We take an arbitrary subset $X \subseteq Q_{\mathcal{L}}$, where $u \in Q_{\mathcal{L}}$. We assume that $u \notin \Box(X \leftrightarrow \Box X)$. Then there exists a $v \in Q_{\mathcal{L}}$, such that uRv and $v \notin (X \leftrightarrow \Box X)$. Hence, $v \in X$ and $v \notin \Box X$. There exists a $w \in Q_{\mathcal{L}}$ such that vRw and $w \notin X$, consequently, $w \neq v$. Condition (b) implies $u = v$ and $u \in X$. We take an arbitrary u' such that uRu'. Then we have from $u' \in \Diamond X$ that there exists a w' such that $u'Rw'$ and $w' \in X$. We obtain $Q_{\mathcal{L}}$ or $u' = u$ from the condition on $u' = w'$ in both cases $u' \in X$. That is, $u' \in (X \leftrightarrow \Diamond X)$. Hence, $u \in \Box(X \leftrightarrow \Diamond X)$. Thus, $u \in \Box(X \leftrightarrow \Box X) \vee \Box(X \leftrightarrow \Diamond X)$ for any $u \in Q_{\mathcal{L}}$ and $X \subseteq Q_{\mathcal{L}}$. Since $\mathcal{L}$ is isomorphically embedded in the TBA $\mathcal{T}(Q_{\mathcal{L}})$ of all subsets of $Q_{\mathcal{L}}$, we have $\Box(x \leftrightarrow \Diamond x) \vee \Box(x \leftrightarrow \Box x) = 1$ for all $x \in \mathcal{L}$.

The assertion is proved.

ASSERTION 2. For any TBA $\mathcal{L}$ the following conditions are equivalent:

(a) $\Box\Diamond x \leq \Diamond\Box x$ for all $x \in \mathcal{L}$;

(b) the representative set $Q_{\mathcal{L}}$ satisfies the condition

$$\forall u \exists v \; (uRv \text{ and } \forall w (vRw \Rightarrow v = w)).$$

Proof. Suppose the TBA $\mathcal{L}$ satisfies condition (a) and let Φ_0 be a simple filter in $\mathcal{L}$. Then $F_0 = \Phi_0 \cap G(\mathcal{L})$ is a filter on the PBA $G(\mathcal{L})$. It can be expanded to a maximal filter F_1 in the PBA $G(\mathcal{L})$. We note that F_1 satisfies the condition (cf. [3], 1, 13.10)

$$(\forall z \in G(\mathcal{L}))(z \in F_1 \text{ or } \neg z \in F_1),$$

or, equivalently,

$$(\forall z \in \mathcal{L})(\Box z \in F_1 \text{ or } \Box\sim\Box z \in F_1). \qquad (*)$$

By Lemma 5, there exists a simple filter Φ_1 on $\mathcal{L}$ such that $\Phi_1 \cap G(\mathcal{L}) = F_1$. Now suppose Φ is an arbitrary simple filter on $\mathcal{L}$ such that $\Phi_1 R \Phi$. Then

$$F_1 = \Phi_1 \cap G(\mathcal{L}) \subseteq \Phi \cap G(\mathcal{L}) \in S_{G(\mathcal{L})}.$$

Therefore, $\Phi \cap G(\mathcal{L}) = F_1$.

Let us assume that $\Phi \neq \Phi_1$. Then $x \in \Phi$ and $\sim x \in \Phi_1$ for some $x \in \mathcal{L}$. Hence,

$\Box x \notin F_1$ and $\Box \sim x \notin F_1$, since $F_1 \subseteq \Phi \cap \Phi_1$. That is, by condition (*), $\Box \sim \Box x \in F_1$ and $\Box \sim \Box \sim x \in F_1$. However, by condition (a), $\Box \sim \Box \sim x \,\&\, \Box \sim \Box x = \Box \Diamond x \,\&\, \sim \Diamond \Box x = 0$ which contradicts $F_1 \in S_{G(\mathcal{B})}$. That is, $\Phi = \Phi_1$, i.e., condition (b).

Conversely, suppose (b) and let $X \subseteq Q_{\mathcal{B}}$. We assume that $u \in \Box \Diamond X$. Let us take a $v \in Q_{\mathcal{B}}$, such that uRv and $\forall w\,(vRw \Rightarrow v = w)$. Then $v \in \Diamond X$, i.e., there exists a w such that vRw and $w \in X$. We find $w = v$ and $v \in X$. From the properties of v it follows that $v \in \Box X$ and $u \in \Diamond \Box X$. Thus, $\Box \Diamond X \subseteq \Diamond \Box X$. By Lemma 3 we have $\Box \Diamond x \leq \Diamond \Box x$ for $x \in \mathcal{B}$. The assertion is proved.

4. CLASSIFICATION OF QUASIORDERED SETS AND PM1-PM5 LOGICS

Suppose the set Q is quasiordered by the relation R. We call the quotient set $Q/_{=_R}$ with respect to the equivalence $u =_R v \iff uRv$ the skeleton $\sigma(Q)$ of the set Q, and let vRu be partially ordered by the relation $[u] \leq [v] \iff uRv$, where $[u] = \{v / u =_R v\}$ is the residue class. The residue class $[u]$ is called exterior if $\forall v\,(uRv \Rightarrow vRu)$, i.e., $[u]$ is a maximal element in $\sigma(Q)$. A residue class is called interior if it is not exterior. We use the following notation:

$$\mu_1(Q) = \sup\{\overline{\overline{[u]}} / [u] \text{ is an exterior class in } Q\},$$
$$\mu_2(Q) = \sup\{\overline{\overline{[u]}} / [u] \text{ is an interior class in } Q\}.$$

(Here $[\bar{\bar{u}}]$ is the cardinality of $[u]$.) We denote by $u \in Q$ for $\nu(u)$ the number of classes in $[v]$ such that $[u] \leq [v]$. We set

$$\nu(Q) = \sup\{\nu(u) \mid u \in Q\}.$$

For example, if Q is partially ordered, $\mu_1(Q) = 1$ and $\mu_2(Q) \leq 1$. If a partially ordered set Q has a least element, then $\nu(Q) = \bar{\bar{Q}}$.

For the set X_n and Y_n (where $n \geq 1$) constructed at the end of Sec. 2, we have

$$\mu_1(X_n) = n, \quad \mu_2(X_n) = 0, \quad \nu(X_n) = 1;$$
$$\mu_1(Y_n) = 1, \quad \mu_2(Y_n) = n-1,$$
$$\nu(Y_n) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n > 1. \end{cases}$$

We recall that if $\mathcal{B}$ is a TBA, $M\mathcal{B}$ denotes the set of all formulas for modal logic true in $\mathcal{B}$.

<u>THEOREM 1.</u> Suppose $\{\mathcal{B}_i\}_{i \in I}$ is a family of TBA, and let there exist a natural number K such that $\nu(Q_{\mathcal{B}_i}) \leq K$, $\mu_1(Q_{\mathcal{B}_i}) \leq K$, and $\mu_2(Q_{\mathcal{B}_i}) \leq K$ for every $i \in I$. Then the logic $M = \bigwedge_{i \in I} M\mathcal{B}_i$ is tabular.

<u>Proof.</u> Suppose $\{\mathcal{B}_j\}_{j \in J}$ is a class of all subdirectly nonfactorable TBA which are homomorphisms of algebras of the family $\{\mathcal{B}_i\}_{i \in I}$. Then by the Birkhoff theorem [4] every algebra $\mathcal{B}_i$ $(i \in I)$ can be represented as a subdirect product of given algebra $\mathcal{B}_j$. Therefore, $M = \bigwedge_{j \in J} M\mathcal{B}_j$.

By Lemma 4a, $\nu(Q_{\mathcal{B}_j}) \leq K$, $\mu_1(Q_{\mathcal{B}_j}) \leq K$, and $\mu_2(Q_{\mathcal{B}_j}) \leq K$ for every $j \in J$. Moreover, by Lemma 9, for every $j \in J$ the set $Q_{\mathcal{B}_j}$ is such that since any subdirect nonfactorable TBA is completely bound [4]. Therefore, $Q_{\mathcal{L}_j}$ contains less than $\nu(Q_{\mathcal{L}_j}) \cdot max\{\mu_1(Q_{\mathcal{L}_j}), \mu_2(Q_{\mathcal{L}_j})\} \leq \kappa^2$ elements. Consequently, $\mathcal{L}_j$ is finite, and its cardinality is less than 2^{κ^2}. Therefore, there exists in the family $\{\mathcal{L}_j\}_{j\in J}$ only a finite number of pairwise nonfactorable TBA and $M = M\mathcal{L}$, where $\mathcal{L}$ is the direct product of all such $\mathcal{L}_j$.

The theorem is proved.

We now need the following simple lemma from the theory of Boolean algebras.

LEMMA 10. Suppose $\mathcal{L}$ is a Boolean algebra, and let $Q_{\mathcal{L}}$ be a set of simple filters on $\mathcal{L}$ that contains at least n elements. Then $\mathcal{L}$ contains a finite Boolean algebra with n atomic formulas as its subalgebras.

LEMMA 11. Suppose $\mathcal{L}$ is a TBA, and let $\mu_1(Q_{\mathcal{L}}) \geq n$. Then all formulas in $M\mathcal{L}$ are true in the TBA U_n.

Proof. Suppose $\Phi_0 \in Q_{\mathcal{L}}$ is such that the residue class $[\Phi_0]$ is exterior and $\overline{[\Phi_0]} \geq n$. By Lemma 4b there exists a TBA $\mathcal{L}_1$ such that $Q_{\mathcal{L}_1}$ is isomorphic to $[\Phi_0]$ and $\mathcal{L}_1$ is a homomorphism of the TBA $\mathcal{L}$. The set $Q_{\mathcal{L}_1}$ will then satisfy $(\forall \Phi_1 \Phi_2)$. $\Phi_1 R \Phi_2$ Therefore, for every $x \in \mathcal{L}_1$ we have

$$\Box x > 0 \Rightarrow (\exists \Phi \in Q_{\mathcal{L}_1})(\Box x \in \Phi) \Rightarrow (\forall \Phi \in Q_{\mathcal{L}_1})(\Box x \in \Phi) \Rightarrow \Box x = 1 \Longleftrightarrow x = 1.$$

Since $\overline{Q_{\mathcal{L}_1}} \geq n$, $\mathcal{L}_1$ contains, by Lemma 10, a finite Boolean subalgebra $\mathcal{L}_2$ with n atomic formulas. By definition of the operation $\Box$ on $\mathcal{L}_1$ we have that the TBA U_n is isomorphic to the subalgebra $\mathcal{L}_1$. Therefore $M\mathcal{L} \subseteq M\mathcal{L}_1 \subseteq MU_n$.

The lemma is proved.

LEMMA 12. Suppose $\mathcal{L}$ is a finite TBA, and let $\mu_2(Q_2) \geq n > 0$. Then all formulas in $M\mathcal{L}$ are true in the TBA U_{n+1}

Proof. By virtue of Lemma 3, $\mathcal{L}$ is isomorphic to $\mathcal{T}(Q_{\mathcal{L}})$. We will find a $\Phi_1 \in Q_{\mathcal{L}}$ such that Φ_1 is an exterior class and $\overline{[\Phi_1]} \geq n$. Suppose $[\Phi_1] = \{\Phi_1, \ldots, \Phi_m\}$, $m \geq n$. We select $Q' = \{\Phi \in Q_{\mathcal{L}} \mid \Phi_1 R \Phi\}$. By Lemma 1, $\mathcal{T}(Q')$ is a homomorphism of the algebra $\mathcal{L}$. Let us now construct a mapping θ from Q' onto Y_{n+1} in the following manner: $\theta(\Phi_i) = i$ for $i = 1, \ldots, n$; $\theta(\Phi_i) = n$ for $n \leq i \leq m$; $\theta(\Phi) = n+1$ if $\Phi \notin [\Phi_1]$.

It is easy to verify that θ satisfies conditions (1) and (2) of Lemma 2, so that $U_{n+1} \cong \mathcal{T}(Y_{n+1})$ is isomorphically embedded in $\mathcal{T}(Q')$. Consequently, $M\mathcal{L} \subseteq MU_{n+1}$, QED.

THEOREM 2. Suppose M is a modal logic which is not tabular. Then M is contained in at least one of PM1-PM5.

Proof. Suppose M is not tabular. We will consider the superintuitionistic logic $\rho(M)$ (cf. [1], Sec. 3 (6)).

Two cases are possible.

1. $\rho(M)$ is not tabular. Then according to the fundamental theorem [2], $\rho(M)$

is contained in one of LC, $\mathcal{L}_2$, and $\mathcal{L}_3$. We now use relations from [1], Sec. 3.

(P7) $M \subseteq \sigma\rho(M)$,

(P3) $L_1 \subseteq L_2 \Rightarrow \sigma(L_1) \subseteq \sigma(L_2)$.

We find $M \subseteq \sigma(LC) = PM1$, or $M \subseteq \sigma(\mathcal{L}_2) = PM2$, or $M \subseteq \sigma(\mathcal{L}_3) = PM3$.

2. $\rho(M)$ is tabular. According to Lemma 4.9 [1] M is finitely approximable, i.e., $M = \bigwedge_{i \in I} M\mathcal{L}_i$, where $\mathcal{L}_i$ are finite TBA. We may assume (cf. [1], Lemma 4.3) that all $\mathcal{L}_i$ are pairwise nonfactorable and, consequently, completely bound. By virtue of Lemma 3.5 and Theorem 6a from [1],

$$\rho(M) = \bigwedge_{i \in I} \rho(M\mathcal{L}_i) = \bigwedge_{i \in I} LG(\mathcal{L}_i), \text{where } LG(\mathcal{L}_i)$$

is the superintuitionistic logic consisting of all formulas true in the PBA $G(\mathcal{L}_i)$. Since $\rho(M)$ is tabular, there exists a $K \geqslant 2$ such that the formula $A(K) \leftrightharpoons \bigvee_{1 \leqslant i < j \leqslant K} (p_i \equiv p_j)$ occurs in $\rho(M)$ (cf. for example, [7]), that is, is true in all $G(\mathcal{L}_i)$. Since, according to Lemma 9, the $G(\mathcal{L}_i)$ are completely bound, $\overline{\overline{G(\mathcal{L}_i)}} < K$ for all $i \in I$. That is, $\overline{\overline{S}}_{G(\mathcal{L}_i)} < K$. By virtue of Lemma 7 we find that $\nu(Q_{\mathcal{L}_i}) \leqslant \overline{\overline{S}}_{G(\mathcal{L}_i)} < K$. Let us recall that M is not tabular and now use Theorem 1. We conclude that $\sup_{i \in I} \{\mu_1(Q_{\mathcal{L}_i})\} = \infty$ or $\sup_{i \in I} \{\mu_2(Q_{\mathcal{L}_i})\} = \infty$. If $\sup_{i \in I} \{\mu_1(Q_{\mathcal{L}_i})\} = \infty$, for any n we can find an $i \in I$ such that $\mu_1(Q_{\mathcal{L}_i}) \geqslant n$ and, by Lemma 11, $M\mathcal{L}_i \subseteq M\mathfrak{V}_n$. Therefore,

$$M = \bigwedge_{i \in I} M\mathcal{L}_i \subseteq \bigwedge_{n=1}^{\infty} M\mathfrak{V}_n = PM5.$$

If

$$\sup_{i \in I} \{\mu_2(Q_{\mathcal{L}_i})\} = \infty,$$

then by Lemma 12

$$M = \bigwedge_{i \in I} M\mathcal{L}_i \subseteq \bigwedge_{n=1}^{\infty} M\mathfrak{U}_{n+1} = \bigwedge_{n=1}^{\infty} M\mathfrak{U}_n = PM4.$$

The theorem is proved.

Remark. Whenever $\rho(M)$ is not tubular we may use Theorems 2 and 3 from [2] and Lemma 7 in order to determine which of PM1-PM3 contain a given logic M.

5. AXIOMATIZATION

In this section, we will prove that all the PM1-PM4 logics are finitely axiomatizable. The PM5 logic coincides with $S5$ and, consequently, is also finitely axiomatizable.

We recall [1] that if Γ is a set of formulas, $[\Gamma]$ denotes the set of formulas derivable from Γ and the axioms of $S4$ by means of rules of substitution, modus ponens, and $\frac{\alpha}{\Box\alpha}$.

We have as a direct corollary from Lemma 4.2 of [1] the following assertion.

COMPLETENESS LEMMA. Suppose Γ is a set of formulas, and let α be a formula. Then $\alpha \in [\Gamma]$ if and only if α is true in any completely bound TBA in which all formulas of Γ are true. If $[\Gamma]$ is finitely approximable, $\alpha \in [\Gamma]$ if and only if α is true in any completely bound finite TBA in which all formulas of Γ are true.

We denote by T as was done in [1], Sec. 2, the translation of the formulas of intuitionistic logic into formulas of modal logic.

PROPOSITION 1. Suppose $\{\mathfrak{B}_i\}_{i\in I}$ is a family of all finite completely bound nondegenerate TBA in which the formulas E and $T(Z)$ are true, where

$$E = \Box p \vee \Diamond[(\Box(q \vee \Box p)\,\&\sim\Box p) \vee (\Box(\sim q \vee \Box p)\,\&\sim \Box p)],$$

$$Z = (p \supset q) \vee (q \supset p).$$

Then

a) for every $i \in I$ an $n \geqslant 2$ can be found such that $\mathfrak{B}_i$ is isomorphic to $S(L_n)$, where L_n is a linearly ordered n-element PBA;

b) if $\overline{\overline{\mathfrak{B}}}_i \leqslant \overline{\overline{\mathfrak{B}}}_j$, $\mathfrak{B}_i$ is isomorphically embedded in $\mathfrak{B}_j$.

Proof. a) Suppose $\mathfrak{B}$ is a finite completely bound nondegenerate TBA. Then $Q_{\mathfrak{B}}$ is nonempty and, by virtue of Lemma 9, contains an element u_0 such that $u_0 R v$ for any $v \in Q_{\mathfrak{B}}$. It is easy to verify that if $\mathfrak{B} \models T(Z)$, the set $Q_{\mathfrak{B}}$ is such that $(\forall uv)(uRv \vee vRu)$.

Further, $\mathfrak{B}$ is isomorphic to $\mathcal{T}(Q_{\mathfrak{B}})$ according to Lemma 3. If E is true in $\mathfrak{B}$, by Theorem 1 of [6], $Q_{\mathfrak{B}}$ is partially ordered. Therefore, R is a linear ordering on $Q_{\mathfrak{B}}$. Let us assume that $Q_{\mathfrak{B}}$ contains n elements. Then $Q_{\mathfrak{B}}$ is isomorphic to a representing partially ordered set $S_{L_{n+1}}$ of the linearly ordered PBA L_{n+1}. We note that by virtue of Lemma 8 the representing set $Q_{S(L_{n+1})}$ of the TBA $S(L_{n+1})$ is also isomorphic to $S_{L_{n+1}}$. Therefore, $Q_{\mathfrak{B}}$ and $Q_{S(L_{n+1})}$ are isomorphic, that is, by Lemma 3, $\mathfrak{B}$ is isomorphic to $S(L_{n+1})$.

b) In view of a), it is sufficient to prove that for any $n \geqslant 2$ the algebra $S(L_n)$ is isomorphically embedded in $S(L_{n+1})$. However, since $Q_{S(L_n)}$ and $Q_{S(L_{n+1})}$ are linearly ordered and contain $(n-1)$ and n elements, respectively, it is easy to construct a mapping θ from $Q_{S(L_{n+1})}$ onto $Q_{S(L_n)}$ that satisfies conditions (1) and (2) of Lemma 2. Then $S(L_n)$ which is isomorphic to $\mathcal{T}(Q_{S(L_n)})$ is embedded in $S(L_{n+1})$, QED.

THEOREM 3. PM1 coincides with the set $[E, T(Z)]$.

Proof. We have $\mathrm{PM1} = \bigwedge_{n=2}^{\infty} MS(L_n)$ (Sec. 1).

Since the formula $Z = (p \supset q) \vee (q \supset p)$ is true in all PBA L_n, the formula $T(Z) = \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$ is true in all TBA $S(L_n)$, by virtue of relation (8) from [1], Sec. 3. Therefore, $T(Z) \in \mathrm{PM1}$. Since the sets $Q_{S(L_n)}$ are finite and linearly ordered, E, according to Theorem 1 of Ref. [6], is true in $\mathcal{T}(Q_{S(L_n)}) \cong S(L_n)$. Therefore, $[E, T(Z)] \subseteq \mathrm{PM1}$.

We now note that the $[E, T(Z)]$ logic is an extension of $S4.3 = [T(Z)]$. It was previously proved [5] that all extensions of $S4.3$ are finitely approximable. Suppose now the formula α does not occur in $[E, T(Z)]$. Then using the Completeness Lemma, a finite completely bound TBA $\mathfrak{B}$ can be found such that E and $T(Z)$ are true in $\mathfrak{B}$ and such that α is not true in $\mathfrak{B}$. In accordance with Proposition 1a,

α is contradicted in $S(L_n)$ for given n. Consequently, it does not occur in PM1. The theorem is proved.

<u>THEOREM 4.</u> **PM2** coincides with the set $[T(A_2), \alpha_2]$, where

$$A_2 = (\neg p \vee \neg\neg p) \,\&\, [(\neg\neg p \,\&\, ((q \supset p) \supset (r \supset q)) \,\&\, ((r \supset q) \supset r)) \supset r],$$

$$\alpha_2 = \Box(p \leftrightarrow (\Diamond p \,\&\, \sim\Diamond(\Diamond p \,\&\, \sim p))) \vee \Box(p \leftrightarrow (\Box p \vee \Diamond(\Diamond \sim p \,\&\, p))).$$

<u>Proof.</u> We will first prove that $PM2 \subseteq [T(A_2), \alpha_2]$. For this purpose it is sufficient to prove that all formulas in **PM2** are true in any completely bound TBA in which $T(A_2)$ and α_2 are true. We recall (cf. [1], No. 3 (7)), that $\mathbf{PM2} = \sigma(\mathscr{L}_2) = \{\alpha \mid \forall \mathfrak{A}\ (\mathfrak{A} \text{ is a TBA and } L\mathfrak{A} \supseteq \mathscr{L}_2 \Rightarrow S(\mathfrak{A}) \models \alpha)\}$.

Suppose the TBA $\mathscr{L}$ is completely bound and that the formula α_2 is true in it. Then the algebra $\mathscr{L}$ is generated by the set $\mathfrak{A} = G(\mathscr{L})$. In fact, suppose $x \in \mathscr{L}$. Then since $\mathscr{L}$ is completely bound, we have $\Box(x \leftrightarrow (\Diamond x \,\&\, \sim\Diamond(\Diamond x \,\&\, \sim x))) = 1$ or $\Box(x \leftrightarrow (\Box x \vee \Diamond(\Diamond \sim x \,\&\, x))) = 1$. In the first case, $x = \sim\Box\sim x \,\&\, \Box\sim(\Diamond x \,\&\, \sim x)$ and $\Box\sim x, \Box\sim(\Diamond x \,\&\, \sim x) \in G(\mathscr{L})$. In the second case, $x = \Box x \vee \sim\Box\sim(\Diamond\sim x \,\&\, x)$ and $\Box x, \Box\sim(\Diamond\sim x \,\&\, x) \in G(\mathscr{L})$. By Lemma 3.4 from [1], the TBA $\mathscr{L}$ and $S(\mathfrak{A})$ are isomorphic.

Now suppose the formula $T(A_2)$ is true in $\mathscr{L}$. Then A_2 is true in $\mathfrak{A} = G(\mathscr{L})$ so that all the formulas of the pretabular superintuitionistic logic $\mathscr{L}_2$ are true in $\mathfrak{A}$ (cf. [2]) and therefore, in $S(\mathfrak{A})$ also, that is, all formulas of $\sigma(\mathscr{L}_2)$ – **PM2** are true in $\mathscr{L}$.

It therefore follows that $\mathbf{PM2} \subseteq [T(A_2), \alpha_2]$.

We now prove that the formulas $T(A_2)$ and α_2 belong to **PM2**. It is sufficient to prove that these formulas are true in all algebras $S(B_n)$, since $\mathbf{PM2} = \bigwedge_{n=0}^{\infty} MS(B_n)$ (cf. Sec. 1). Since A_2 is true in all B_n, $T(A_2)$ is true in all $S(B_n)$ (cf. [1], Sec. 3 (8)). Let us prove that α_2 is true in any TBA $\mathscr{L}$ such that the representing set $Q_{\mathscr{L}}$ is partially ordered, has a least element 0, and a greatest element ∞, the remaining elements being pairwise noncomparable. According to Lemma 8, all $Q_{S(B_n)}$ satisfy these conditions. Thus, suppose $Q_{\mathscr{L}}$ satisfies these conditions, and let X be an arbitrary subset of $Q_{\mathscr{L}}$. We consider two cases.

1. $\infty \notin X$.

If $X = \emptyset$, $\Diamond X = \emptyset$, and $\Diamond X \,\&\, \sim\Diamond(\Diamond X \,\&\, \sim X) = X$.

If $X \neq \emptyset$ and $0 \notin X$, $\Diamond X = X \cup \{0\}$, $\Diamond X \,\&\, \sim X = \{0\}$ and $\Diamond(\Diamond X \,\&\, \sim X) = \{0\}$, $\Diamond X \,\&\, \sim\Diamond(\Diamond X \,\&\, \sim X) = X$.

If $0 \in X$, we have $\Diamond X = X$, $\Diamond X \,\&\, \sim X = \emptyset$, and $\Diamond X \,\&\, \sim\Diamond(\Diamond X \,\&\, \sim X) = X \,\&\, \sim\emptyset = X$. Therefore, $\Box(X \leftrightarrow \Diamond X \,\&\, \sim\Diamond(\Diamond X \,\&\, \sim X)) = 1$, and α_2 is true for $p = X$.

2. $\infty \in X$.

Then $\infty \notin \sim X$; we may substitute $\sim X$ for X in the equality obtained in case 1. We will have

$$\Box(\sim X \leftrightarrow \Diamond\sim X \,\&\, \sim\Diamond(\Diamond\sim X \,\&\, X)) = 1,$$

i.e., $\Box(X \leftrightarrow \Box X \vee \Diamond(\Diamond \sim X \& X)) = 1$, and α_2 is true for $p = X$. We have found that α_2 is true in the TBA $\mathcal{T}(Q_{\mathfrak{B}})$, that is, in $\mathfrak{B}$. Consequently, α_2 is true in all $S(B_n)$, that is, it occurs in PM2.

The theorem is proved.

PROPOSITION 2. Suppose $\{\mathfrak{B}_i\}_{i \in I}$ is a family of all finite completely bound nondegenerate TBA in which the formulas $T(A_2)$ and α_2 are true. Then

(a) for any $i \in I$ an $n \geqslant 0$ can be found such that the algebra $\mathfrak{B}_i$ is isomorphic to $S(B_n)$, where B_0 is a two-element PBA and

$$B_{n+1} = B_0 + B_0^n + B_0;$$

(b) if $\bar{\mathfrak{Z}}_i \leqslant \bar{\mathfrak{Z}}_j$, $\mathfrak{B}_i$ is isomorphically embedded in $\mathfrak{B}_j$.

Proof. Suppose a nondegenerate TBA $\mathfrak{B}$ is finite and completely bound, and let $T(A_2)$ and α_2 be true in it. We will establish, as in the first part of the proof of Theorem 2, that $\mathfrak{B}$ is isomorphic to $S(\mathfrak{A})$, where $\mathfrak{A} = G(\mathfrak{B})$, and that A_2 is true in $G(\mathfrak{B})$. The formula A_2 is the conjunction of the formulas $\neg p \vee \neg\neg p$ and

$$D_8 = (\neg\neg p \& ((q \supset p) \supset (r \supset q)) \& ((r \supset q) \supset r)) \supset r.$$

The set $S_{G(\mathfrak{B})}$ is nonempty and, by Propositions 1 and 2 of [2], satisfies the condition

$$(\forall u_1 u_2)[\exists v\,(v \leqslant u_1 \text{ and } v \leqslant u_2) \Rightarrow \exists w\,(u_1 \leqslant w \text{ and } u_2 \leqslant w)],$$

$$(\forall u_1 u_2 u_3 u_4) \neg (u_1 < u_2 < u_3 < u_4).$$

Since $G(\mathfrak{B})$ is completely bound, $S_{G(\mathfrak{B})}$ contains a least element 0. Then $S_{G(\mathfrak{B})}$ also contains, under these conditions, a greatest element ∞, all the remaining elements being pairwise noncomparable. The set $S_{G(\mathfrak{B})}$ is finite, so that the TBA $\mathfrak{B}$ is finite. Suppose $S_{G(\mathfrak{B})}$ contains n elements. Then the PBA $G(\mathfrak{B})$ is isomorphic to B_0 for $n = 1$ and isomorphic to $B_0 + B_0^{n-2} + B_0$ for $n > 1$. Assertion (a) is proved.

Assertion (b) follows from (a), since, according to Lemma 8, $Q_{S(B_n)}$ is isomorphic to S_{B_n}, and it is easy to construct a mapping from $S_{B_{n+1}}$ onto S_{B_n} that satisfies conditions (1) and (2) of Lemma 2.

The proposition is proved.

THEOREM 5. PM3 coincides with the set $[\alpha_3]$, where

$$\alpha_3 = \Box(p \leftrightarrow \Box p) \vee \Box(p \leftrightarrow \Diamond p).$$

Proof. We will first prove that all formulas of PM3 are deducible from α_3, using the Completeness Lemma. Suppose the TBA $\mathfrak{B}$ is completely bound, and let $\mathfrak{B}$ be true in α_3. Then $\mathfrak{B}$ is generated by the set $\mathfrak{A} = G(\mathfrak{B})$. In fact, let us take the element x from $\mathfrak{B}$. Then $\Box(x \leftrightarrow \Box x) \vee \Box(x \leftrightarrow \Diamond x) = 1$ in $\mathfrak{B}$. Since $\mathfrak{B}$ is completely bound, $\Box(x \leftrightarrow \Box x) = 1$ or $\Box(x \leftrightarrow \Diamond x) = 1$. In the first case, $x = \Box x \in G(\mathfrak{B})$, while in the second case, $x = \sim\Box\sim x$. By Lemma 3.4 of [1], the TBA $\mathfrak{B}$ is isomorphic to $S(\mathfrak{A}) = S(G(\mathfrak{B}))$.

Moreover, by virtue of Assertion 1, $Q_{\mathfrak{B}}$ satisfies the condition that uRv and

$vRw \Rightarrow u=v \vee v=w$. Therefore, the length of the R chains is less than 2, and $Q_{\mathfrak{B}}$ is partially ordered. By Lemma 8, $S_{G(\mathfrak{B})}$ is isomorphic to $Q_{\mathfrak{B}}$. According to Proposition 1 of [2], the formula

$$D_2 = (\neg\neg p \,\&\, ((q \supset p) \supset q)) \supset q$$

is true in $\mathfrak{A}=G(\mathfrak{B})$ so that all formulas of the pretabular superintuitionistic logic $\mathcal{L}_3$ are true in $\mathfrak{A}$ (cf. [2]). Let us take an arbitrary formula α of PM3 $= \sigma(\mathcal{L}_3)$. By definition of $\sigma(\mathcal{L}_3)$ we find that α is true in $S(\mathfrak{A})$, that is, in $\mathfrak{B}$. Consequently, PM3 $\subseteq [\alpha_3]$ according to the Completeness Lemma.

Let us prove that $\alpha_3 \in$ PM3. We have PM3 $= \bigwedge_{n=0}^{\infty} MS(C_n)$. By Lemma 8, the representing set $Q_{S(C_n)}$ is isomorphic to S_{C_n}, that is, it satisfies the condition

$$uRv \text{ and } vRw \Rightarrow u=v \quad \vee \quad v=w.$$

By Assertion 1, $S(C_n)$ is true in α_3.

The theorem is proved.

PROPOSITION 3. Suppose $\{\mathfrak{B}_i\}_{i\in I}$ is a family of all finite completely bound nondegenerate TBA in which α_3 is true. Then

(a) for any $i \in I$ an $n>0$ can be found such that $\mathfrak{B}_i$ is isomorphic to $S(C_n)$, where $C_n = B_0^n + B_0$;

(b) if $\overline{\mathfrak{B}}_i \leq \overline{\mathfrak{B}}_j$, $\mathfrak{B}_i$ is isomorphically embedded in $\mathfrak{B}_j$.

Proof. Suppose a finite nondegenerate TBA $\mathfrak{B}$ is completely bound and that α_3 is true in it. It was noted in the first part of the proof of Theorem 5 that $\mathfrak{B}$ is isomorphic to $S(\mathfrak{A})$, where $\mathfrak{A} = G(\mathfrak{B})$, and that the length of chains in the set $S_{G(\mathfrak{B})}$ is less than two. Moreover, $S_{G(\mathfrak{B})}$ has a least element and is finite. Therefore, the PBA $G(\mathfrak{B})$ is isomorphic to $B_0^{n-1} + B_0 = C_{n-1}$, where n is the number of elements in $S_{G(\mathfrak{B})}$. Assertion (a) is proved.

By Lemma 8, $Q_{S(C_n)}$ is isomorphic to S_{C_n}. It is easy to construct a mapping from $S_{C_{n+1}}$ into S_{C_n}, that satisfies conditions (1) and (2) of Lemma 2. Therefore, assertion (b) follows from (a). The proposition is proved.

PROPOSITION 4. Suppose $\{\mathfrak{B}_i\}_{i\in I}$ is a family of all completely bound nondegenerate TBA in which the formulas $T(Z)$, $T(D_2)$, and α_4 are true, where

$$Z = (p \supset q) \vee (q \supset p),$$
$$D_2 = (\neg\neg p \,\&\, ((q \supset p) \supset q)) \supset q,$$
$$\alpha_4 = \Box\Diamond p \rightarrow \Diamond\Box p.$$

Then for any $i,j \in I$ (a) $\mathfrak{B}_i$ is locally finite; (b) if $\mathfrak{B}_i$ and $\mathfrak{B}_j$ are finite and $\overline{\mathfrak{B}}_i < \overline{\mathfrak{B}}_j$, $\mathfrak{B}_i$ is isomorphically embedded in $\mathfrak{B}_j$; (c) if $\mathfrak{B}_i$ is finite, $\mathfrak{B}_i$ is isomorphic to $\mathcal{U}_n$ for some n.

Proof. Suppose the nondegenerate TBA $\mathfrak{B}$ is completely bound and that $T(Z)$ and $T(D_2)$ are true in it. Then the formulas Z and D_2 are true in the PBA $G(\mathfrak{B})$. Since $G(\mathfrak{B})$ is completely bound, it is linearly ordered. The representing set $S_{G(\mathfrak{B})}$ is also linearly ordered. By virtue of Proposition 1 of [2], the length of chains in $S_{G(\mathcal{B})}$ is less than 2. Therefore, $G(\mathcal{B})$ is isomorphic to the linearly ordered PBA $L_2 = B_0$ or to $L_3 = \{0, a_1, 1\}$.

We therefore find, in particular, the following assertion. Suppose A is any subset of $\mathcal{B}$, and let $\mathcal{B}_1$ be a subalgebra of $\mathcal{B}$ generated by A. Then $\mathcal{B}_1$ is contained in the set A_1 obtained by the closure $A \cup G(\mathcal{B})$ relative to the Boolean operations. Therefore, if A is finite, then $\mathcal{B}_1$ is also finite, and assertion (a) is proved.

We now assume that the formula α_4 is also true in $\mathcal{B}$. Then, by virtue of Assertion 2, the set $Q_{\mathcal{B}}$ satisfies the condition

$$\forall u \exists v \ (u R v \text{ and } \forall w \ (v R w \Longrightarrow v = w)).$$

It follows from the conditions of $S_{G(\mathcal{B})}$ and Lemma 7 that if the TBA $\mathcal{B}$ is finite, then $Q_{\mathcal{B}}$ is isomorphic to Y_n, where $n = \overline{\overline{Q}}_{\mathcal{B}}$, that is, $\mathcal{B}$ is isomorphic to U_n, and assertion (c) is proved.

Assertion (b) follows from (a) and Lemma 2. The required mapping of Y_{n+1} onto Y_n is constructed in the following fashion:

$$\theta(i) = \min(i, n-1) \text{ for } 1 \leq i \leq n; \quad \theta(n+1) = n.$$

THEOREM 6. PM4 coincides with the set $[T(Z),\ T(D_2), \alpha_4]$.

Proof. We first prove that all formulas of PM4 are deducible from $\{T(Z), T(D_2), \alpha_4\}$.

Let us assume that there exists a formula α in PM4, such that α does not occur in $[T(Z), T(D_2), \alpha_4]$. According to the Completeness Lemma, there exists a completely bound TBA $\mathcal{B}$ such that α is not true in $\mathcal{B}$, while the formulas $T(Z)$, $T(D_2)$, and α_4 are true in $\mathcal{B}$. Suppose $p_1, \ldots, p_n$ are all the variables of α. There exists a limit $v: \{p_1, \ldots, p_n\} \to \mathcal{B}$ such that $\alpha(v(p_1), \ldots, v(p_n)) \neq 1$ in $\mathcal{B}$. We take the subalgebra $\mathcal{B}_1$ of $\mathcal{B}$ generated by the set $\{v(p_1), \ldots, v(p_n)\}$. Evidently, $\mathcal{B}_1$ is also completely bound; by Proposition 4a, $\mathcal{B}_1$ is finite. We have from Proposition 4c that $\mathcal{B}_1$ is isomorphic to U_n for given n. By definition of PM4, all formulas of PM4 are true in $\mathcal{B}_1$. However, $v(p_1), \ldots, v(p_n) \in \mathcal{B}_1$, and $\alpha(v(p_1), \ldots, v(p_n)) \neq 1$. We have obtained a contradiction.

It remains for us to prove that the formulas $T(Z), T(D_2)$, and α_4 occur in PM4. We consider the algebra U_n for given $n \geq 1$. The algebra $G(U_n)$ is isomorphic to the two-element Boolean algebra L_2 if $n = 1$, and to the three-element PBA L_3 if $n > 1$. Evidently, the formula $Z = (p \supset q) \vee (q \supset p)$ is true in L_2 and in L_3. The formula D_2 is true in L_2 and L_3 by Proposition 1 of [2]. Therefore, $T(Z)$ and $T(D_2)$ are true in U_n. The formula α_4 is true in U_n by virtue of Assertion 2, since Q_{U_n} is isomorphic to Y_n. Thus, all three formulas are true in all U_n, that is, occur in PM4 $= \bigwedge_{n=1}^{\infty} MU_n$.

6. PRETABULAR LOGICS AND THEIR EXTENSIONS

FUNDAMENTAL THEOREM. There exist exactly five pretabular modal logics containing $S4$:

1) PM1 with the axioms $T(Z)$ and E, where

$$Z = (p \supset q) \vee (q \supset p);$$

$$E = \Box p \vee \Diamond [(\Box(q \vee \Box p) \& \sim \Box p) \vee (\Box(\sim q \vee \Box p) \& \sim \Box p)];$$

2) PM2 with the axioms α_2 and $T(A_2)$, where

$$\alpha_2 = \Box[p \leftrightarrow (\Diamond p \& \sim \Diamond(\Diamond p \& \sim p))] \vee \Box[p \leftrightarrow (\Box p \vee \Diamond(\Diamond \sim p \& p))],$$

$$A_2 = (\neg p \vee \neg\neg p) \& [(\neg\neg p \& ((q \supset p) \supset (r \supset q)) \& ((r \supset q) \supset r)) \supset r];$$

3) PM3 with the axiom

$$\alpha_3 = \Box(p \leftrightarrow \Box p) \vee \Box(p \leftrightarrow \Diamond p);$$

4) PM4 with the axioms $T(Z), T(D_2)$, and α_4, where

$$D_2 = (\neg\neg p \& ((q \supset p) \supset q)) \supset q,$$

$$\alpha_4 = \Box\Diamond p \rightarrow \Diamond\Box p;$$

5) PM5, which is equivalent to $S5$, with the axiom

$$\alpha_5 = \Diamond p \rightarrow \Box\Diamond p.$$

Proof. We first note that none of these logics is contained in any of the other logics. In fact, all formulas of $S(L_5)$ are true in PM1 except for $T(A_2)$, α_3, $T(D_2)$, and α_5, according to Proposition 1 of [2], the relationship $G(S(L_5)) \cong L_5$, and Assertion 1. In $S(B_3)$ all formulas of PM2 are true except for $T(Z)$, α_3, and α_5. In $S(C_2)$ $T(Z)$, $T(A_2)$, and α_5 are not true, but $MS(C_2) \supset$ PM3. In U_3 the formulas E, α_2, α_3, and α_5 are not true, but $MU_3 \supset$ PM4. Finally, PM5 $\subset MV_2$, but E, α_2, α_3, and α_4 are not true in V_2.

Now suppose M is any of PM1-PM5. Then M does not contain the formula $\alpha(n) \rightleftharpoons \bigvee_{1 \leq i < j \leq n+1} \Box(p_i \leftrightarrow p_j)$ for any n. By Lemma 4.5 of [1], M is not tabular. M is contained in some pretabular logic M_0 according to Lemma 4.6 of [1], and in accordance with Theorem 2, M_0 is contained in one of the PM1-PM5. Since no two PM1-PM5 are comparable, we find that $M \subseteq M_0 \subseteq M$, i.e., $M = M_0$, and M is pretabular.

Suppose M_0 is any pretabular logic. According to Theorem 2, $M_0 \subseteq M$, where M is one of PM1-PM5. Since M is not tabular, we have $M_0 = M$.

The theorem is proved.

As a corollary, we obtain the following assertion.

TABULAR TEST. There exists an algorithm that allows us to determine whether any logic $[\alpha]$ is tabular for any formula α.

In fact, the logic $[\alpha]$ is tabular if and only if α does not occur in any of PM1-PM5 and all these logics are finitely axiomatizable and finitely approximable.

THEOREM 7. There exists a PM1 for any consistent logic M which is a natural extension of $n \geq 2$ such that $M = MS(L_n)$ and such that M can be axiomatized by means of the axioms of PM1 with the additional formula $\alpha(K)$ for arbitrary K $(2^{n-1} \leq K < 2^n)$, where

$$\alpha(\kappa) \leftrightharpoons \bigvee_{1 \leq i < j \leq \kappa+1} (p_i \leftrightarrow p_j).$$

Proof. We use the Fundamental Theorem and Proposition 1. Suppose M is consistent, and let $M \supset \mathrm{PM1}$, $M \neq \mathrm{PM1}$. Since PM1 is pretabular, M is tabular; there exists a nondegenerate finite TBA $\mathfrak{B}$ such that $M = M\mathfrak{B}$. According to Lemma 4.3 of [1], $M = \bigwedge_{i \in I} M\mathfrak{B}_i$, where all the $\mathfrak{B}_i$ are subdirectly nonfactorable and $\bar{\bar{\mathfrak{B}}}_i \leq \bar{\bar{\mathfrak{B}}}$. Then there exists $i_0 \in I$ such that $\bar{\bar{\mathfrak{B}}}_i \leq \bar{\bar{\mathfrak{B}}}_{i_0}$ for all $i \in I$. By virtue of Proposition 1b, all the $\mathfrak{B}_i$ are isomorphically embedded in $\mathfrak{B}_{i_0}$. Therefore, $M\mathfrak{B}_i \supseteq M\mathfrak{B}_{i_0}$ for all $i \in I$ and $M = M\mathfrak{B}_{i_0}$. We find by using Proposition 1a that $\mathfrak{B}_{i_0}$ is isomorphic to $S(L_n)$ for given n, that is, $M = MS(L_n)$.

Evidently, the formula $\alpha(\kappa)$ for $2^{n-1} \leq \kappa < 2^n$ is true in $MS(L_n)$. On the other hand, if $\alpha(\kappa)$ for given κ $(2^{n-1} \leq \kappa < 2^n)$ is true in the subdirectly nonfactorable algebra $\mathfrak{B}$, we have $\bar{\bar{\mathfrak{B}}} < 2^n$. If $M\mathfrak{B} \supseteq \mathrm{PM1}$, by Proposition 1, $\mathfrak{B}$ is isomorphic to $S(L_i)$ for some $i \leq n$, that is, $M\mathfrak{B} \supseteq MS(L_n)$. Therefore,

$$[\mathrm{PM1} \cup \{\alpha(\kappa)\}] = \bigwedge_{M\mathfrak{B} \supseteq \mathrm{PM1} \cup \{\alpha(\kappa)\}} M\mathfrak{B} = MS(L_n)$$

where $2^{n-1} \leq \kappa < 2^n$.

The theorem is proved.

THEOREM 8. There exists a PM2 for any consistent logic M which is a natural extension of $n \geq 0$ such that $M = MS(B_n)$ and $M = [\mathrm{PM2} \cup \{\alpha(\kappa)\}]$ for arbitrary κ $(2^{n+1} \leq \kappa < 2^{n+2})$.

THEOREM 9. There exists a PM3 for any consistent logic M which is a natural extension of $n \geq 0$, such that $M = MS(B_n)$ and $M = [\mathrm{PM3} \cup \{\alpha(\kappa)\}]$ for arbitrary κ $(2^{n+1} \leq \kappa < 2^{n+2})$.

THEOREM 10. Suppose a consistent logic M is a natural extension of PM4. Then there exists an $n \geq 1$ such that $M = MU_n$ and $M = [\mathrm{PM4} \cup \{\alpha(\kappa)\}]$ for arbitrary κ $(2^n \leq \kappa < 2^{n+1})$.

The proofs of Theorems 8, 9, and 10 are similar to the proof of Theorem 7. We must now use in place of Proposition 1, Propositions 2, 3, and 4, respectively.

A similar theorem for $\mathrm{PM5} = S5$ was proved in [8].

COROLLARY. A normal modal logic containing $S4$ is pretabular if and only if the set of all its natural extensions has a linear ordering of the type ω^*.

Proof. If the logic $M \in \mathfrak{M}$ is pretabular, it coincides with one of PM1-PM5. It immediately follows from Theorems 7-10 that the extensions of each of PM1-PM4 have ordering of type ω^*. A similar assertion for $\mathrm{PM5} = S5$ was proved in [8].

To prove the converse assertion, we recall that tabular logics have only a finite number of extensions (Theorem 7 of [1]) and that any nontabular logic is contained in a pretabular logic (Lemma 4.6 of [1]), so that the set of its extensions contains a subset of type $1+\omega^*$

Note Added in Proof. The assertion that there exist exactly five pretabular modal logics was also published in the note of V. Yu. Meskhi and L. L. Esakia, "Five 'critical' modal systems," in: Theory of Logical Inference (Summaries of Reports of the All-Union Symposium, Moscow, 1974), Part 1.

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