STABILITY AND UNIQUENESS OF THE SOLUTION OF THE INVERSE KINEMATIC PROBLEM OF SEISMOLOGY IN HIGHER DIMENSIONS

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The following inverse kinematic problem of seismology is considered. In the compact domain M of dimension $\nu \geqslant 2$ with the metric $ds^2 = g_{ij}dx^i dx^j$, we consider the problem of constructing a new metric $du = n\,ds$ according to the known formula $\tau(\xi,\eta) = \int_{K_{\xi,\eta}} n\,ds$ where $\xi,\eta \in \partial M$ and $K_{\xi,\eta}$ is the geodesic in the metric du, connecting the points ξ, η. One proves uniqueness and one obtains a stability estimate

$$\int_M (n_2-n_1)(n_2^{\nu-1}-n_1^{\nu-1})\,dx^1\wedge\ldots\wedge dx^\nu \leqslant \int_{\partial M\times\partial M}\Omega^{\tau_1,\tau_2},$$

where the refraction indices n_1, n_2 are the solutions of the inverse kinematic problem, constructed relative to the functions τ_1, τ_2, respectively, $g=\det g_{ij}$; Ω^{τ_1,τ_2} is the differential form on $\partial M\times\partial M$

$$\Omega^{\tau_1,\tau_2} = -\frac{\Gamma(\frac{\nu}{2})(-1)^{\frac{(\nu-1)(\nu-2)}{2}}}{2\pi^{\nu/2}(\nu-1)!}\sum_{\alpha+\beta=\nu-2} D_\eta\tau\wedge D_\xi\tau\wedge(D_\eta D_\xi\tau_1)^\alpha\wedge(D_\tau D_\xi\tau_2)^\beta$$

where $\tau=\tau_2-\tau_1$, $D_\xi = d\xi^i\frac{\partial}{\partial\xi^i}$, $D_\eta = d\eta^i\frac{\partial}{\partial\eta^i}$.

One of the fundamental problems of geophysics is the investigation of the inner construction of the Earth, including the investigation of the distribution of the velocities of the elastic waves inside the terrestrial sphere.

The problem of the determination of the velocity from the known running times of waves between points on the surface is called the inverse kinematic problem of seismology. A wide literature has been devoted to its mathematical aspects. One has proved the uniqueness of the solution in the class of refraction indices close to a constant for sufficiently small convex domains in the plane [1]. One has obtained a stability estimate and one has proved uniqueness in the case of a bounded domain in the plane [2, 3].

In the present note we give a stability estimate and we prove the uniqueness of the solution of the inverse kinematic problem of seismology for the case of domains of dimension greater than two.*

We consider a compact domain M of dimension $\nu \geqslant 2$ with a smooth boundary ∂M. Assume that in M there is given the metric $\partial s^2 = g_{ij}\,dx^i dx^j$. In geophysics, M is a three-dimen-

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sional ball, being the model of the Earth, and $g_{ij} = \delta_{ij}$. Assume that a function τ, defined on $\partial M \times \partial M$, is known, its physical meaning being the running time of waves between points on the surface.

The question is, can one introduce a new metric $du = n\,ds$, such that the function τ is representable in the form

$$\tau(\xi, \eta) = \int_{\mathcal{K}_{\xi,\eta}} n\,ds$$

where $\xi, \eta \in \partial M$, and $\mathcal{K}_{\xi,\eta}$ is the geodesic in the metric du, joining the points ξ and η? The formulated problem (the determination of the refraction index n from the hodograph of τ) is called the inverse kinematic problem.

It is directly related to the boundary value problem

$$\begin{cases} \nabla_\xi (\nabla_x u)^2 = 0 & x \in M \setminus \partial M, \xi \in \partial M \\ u|_{\partial M \times \partial M} = \tau \end{cases} \tag{1}$$

where ∇_x, ∇_ξ are the gradients in the metrics of M and ∂M, respectively.

Clearly, the inverse kinematic problem is nonlinear since the family of geodesics is not known in advance.

We describe the class of refraction indices in which a stability estimate is obtained. Let $S^{\nu-1}(x)$ be a sphere of sufficiently small radius with center at an arbitrary interior point x of the domain M and let $\{\mathcal{K}_{\xi,x}\}$ be the family of geodesics of the metric nds, joining x with the points ξ of the boundary ∂M. Then, through each point on the sphere $S^{\nu-1}(x)$ there passes a unique geodesic of the family $\{\mathcal{K}_{\xi,x}\}$. We consider the mapping K_x:

$$K_x : S^{\nu-1}(x) \longrightarrow \partial M, \tag{2}$$

which associates to the point $\theta \in S^{\nu-1}(x)$ the point ξ of the boundary ∂M, lying on the same geodesic from the family $\{\mathcal{K}_{\xi,x}\}$ as the point θ.

We denote by N the class of refraction indices (real positive functions in the domain M) such that for any point $x \in M \setminus \partial M$ the mapping (2) is an orientation preserving C^1-diffeomorphism.

We elucidate the physical sense of the last condition. It is satisfied if the domain and the refraction index are such that a source, placed at an arbitrary interior point, illuminates the entire boundary ∂M, which can be smoothly "contracted" along the geodesics into a small sphere around the source.

<u>THEOREM.</u> If the solution of the inverse kinematic problem exists in the class N, then it is unique and we have the stability estimate

$$\int_M (n_2 - n_1)(n_2^{\nu-1} - n_1^{\nu-1}) \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^\nu \leqslant \int_{\partial M \times \partial M} \Omega^{\tau_1, \tau_2}, \tag{3}$$

where the refraction indices n_1, n_2 are the solutions of the inverse kinematic problem from the class N, constructed from the functions τ_1, τ_2, respectively; $g = \det g_{ij}$. Here Ω^{τ_1,τ_2} is a differential form which can be written in the form

$$\Omega^{\tau_1,\tau_2}(\xi,\eta) = -\frac{\Gamma(\frac{\nu}{2})(-1)^{\frac{(\nu-1)(\nu-2)}{2}}}{2\pi^{\nu/2}(\nu-1)!} \sum_{\alpha+\beta=\nu-2} D_\eta \tau \wedge D_\xi \tau \wedge (D_\eta D_\xi \tau_1)^\alpha \wedge (D_\eta D_\xi \tau_2)^\beta,$$

where $\tau = \tau_2 - \tau_1$, $D_\eta = d\eta^i \frac{\partial}{\partial \eta^i}$, $D_\xi = d\xi^i \frac{\partial}{\partial \xi^i}$,

$$D_\eta D_\xi = d\eta^i \wedge d\xi^j \frac{\partial^2}{\partial \eta^i \partial \xi^j}, \quad \xi, \eta \in \partial M, \quad i,j = 1, \ldots \nu-1.$$

By $\wedge$ we have denoted the operation of the exterior multiplication of differential forms.

We consider the functions $u_\sigma(\xi, x)$ $\sigma = 1,2$.

$$u_\sigma(\xi, x) = \int_{K^\sigma_{\xi,x}} n\, ds,$$

which are the solutions of the problem (1), with the boundary condition $u_\sigma(\xi,\eta) = \tau_\sigma(\xi,\eta)$ where $\eta \in \partial M$. With the functions u_σ one can associate the differential forms "solid angle" [4] $\omega_0^\sigma(\theta, x)$, $\theta \in S^{\nu-1}(x)$, such that

$$\int_{S^{\nu-1}(x)} \omega_0^\sigma(\theta, x) = \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})} \qquad \sigma = 1,2$$

Here $S^{\nu-1}(x)$ is a sphere of sufficiently small radius with center at the point $x \in M \setminus \partial M$.

A fundamental role in the proof of the theorem is played by

LEMMA. If the conditions of the theorem hold, then we have the identity

$$\sqrt{g}\,(n_1^\nu \omega_0^1 + n_2^\nu \omega_0^2 - g^{ij} \frac{\partial u_1}{\partial x^i} \frac{\partial u_2}{\partial x^j} (n_1^{\nu-2} \omega_0^1 + n_2^{\nu-2} \omega_0^2)) \wedge dx^1 \wedge \ldots \wedge dx^\nu = D\Omega^{u_1,u_2}, \tag{4}$$

where Ω^{u_1,u_2} is a differential form representable in the form

$$\Omega^{u_1,u_2}(\xi, x) = -\frac{(-1)^{\frac{(\nu-1)(\nu-2)}{2}}}{(\nu-1)!} \sum_{\alpha+\beta=\nu-2} D_x w \wedge D_\xi w \wedge (D_x D_\xi u_1)^\alpha \wedge (D_x D_\xi u_2)^\beta,$$

where

$$w = u_2 - u_1, \quad D_x = dx^i \frac{\partial}{\partial x^i}, \quad D_\xi = d\xi^k \frac{\partial}{\partial \xi^k},$$

$$D_x D_\xi = dx^i \wedge d\xi^k \frac{\partial^2}{\partial x^i \partial \xi^k}, \quad D = D_x + D_\xi,$$

$$x \in M \setminus \partial M; \quad \xi \in \partial M \quad i = 1, \ldots \nu; \quad k = 1, \ldots, \nu-1.$$

In order to obtain the estimate (3) one has to integrate the identity (4) over $\partial M \times M$ and to make use of Stokes' formula. The uniqueness of the solution of the inverse kinematic problem follows from the estimate (3) if one notes that from the fact that $\tau_1 = \tau_2$ there follows the equality $n_1 = n_2$.

Remark 1. From the theorem of the present paper follow the results of [1-3].

Remark 2. The estimate (3) for the case $\nu = 3$, which presents interest in geophysics, has the form

$$\int_M (n_1 - n_2)^2 (n_1 + n_2) \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^\nu \leq \int_{\partial M \times \partial M} \Omega^{\tau_1,\tau_2},$$

where

$$\Omega^{\tau_1,\tau_2}=\frac{1}{8\pi}\det\begin{vmatrix} 0 & \dfrac{\partial(\tau_2-\tau_1)}{\partial\xi^1} & \dfrac{\partial(\tau_2-\tau_1)}{\partial\xi^2} \\ \dfrac{\partial(\tau_2-\tau_1)}{\partial\eta^1} & \dfrac{\partial^2(\tau_1+\tau_2)}{\partial\eta^1\partial\xi^1} & \dfrac{\partial^2(\tau_1+\tau_2)}{\partial\eta^1\partial\xi^2} \\ \dfrac{\partial(\tau_2-\tau_1)}{\partial\eta^2} & \dfrac{\partial^2(\tau_1+\tau_2)}{\partial\eta^2\partial\xi^1} & \dfrac{\partial^2(\tau_1+\tau_2)}{\partial\eta^2\partial\xi^2} \end{vmatrix} d\xi^1\wedge d\xi^2\wedge d\eta^1\wedge d\eta^2.$$

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LITERATURE CITED

1. V. G. Romanov, "On the uniqueness of the definition of an isotropic Riemannian metric inside a domain in terms of the distances between points of the boundary," Dokl. Akad. Nauk SSSR, 218, No. 2, 295-297 (1974).
2. R. G. Mukhometov, "The problem of the recovery of a two-dimensional Riemann metric and integral geometry," Dokl. Akad. Nauk SSSR, 232, No. 1, 32-35 (1977).
3. R. G. Mukhometov, "The inverse kinematic problem of seismology on the plane," in: Mathematical Problems of Geophysics (collection of papers of the Computational Center, Siberian Branch, Academy of Sciences of the USSR), No. 6, Part 2 (1975), pp. 243-254.
4. L. Shvarts (Schwartz), Analysis [Russian translation], Vol. 2, Mir, Moscow (1972).

FINITE-DIMENSIONAL OSCILLATORY MODELS IN THE GENERAL THEORY OF RELATIVITY AND IN GAS DYNAMICS

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The methods of the qualitative theory of differential equations are applied for the investigation of homogeneous cosmological models and to models of stellar explosion in the classical gas dynamics. The strong degeneracy of the singular points of a dynamical system are investigated by their successive solving with the aid of special changes of coordinates. The trajectories of the system are approximated by sequences of separatrices of nondegenerate singular points. The limit cycles are investigated.

1. Qualitative Theory of Homogeneous Cosmology Models

As it is known, in the general theory of relativity, one investigates the four-dimensional space-time manifold M^4 with the Einstein metric g_{ij} $(i,j=0,1,2,3)$, satisfying the Einstein equations

$$R_{ij}-\frac{1}{2}g_{ij}R=\frac{8\pi k}{c^4}T_{ij}. \tag{1.1}$$

The homogeneous cosmology models are determined by the fact that the manifold M^4 admits a group G of motions (for the sake of definiteness, acting on the right-hand side) with space-

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