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Optimal shape design as a material distribution problem

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Abstract. Shape optimization in a general setting requires the determination of the optimal spatial material distribution for given loads and boundary conditions. Every point in space is thus a material point or a void and the optimization problem is a discrete variable one. This paper describes various ways of removing this discrete nature of the problem by the introduction of a density function that is a continuous design variable. Domains of high density then define the shape of the mechanical element. For intermediate densities, material parameters given by an artificial material law can be used. Alternatively, the density can arise naturally through the introduction of periodically distributed, microscopic voids, so that effective material parameters for intermediate density values can be computed through homogenization. Several examples in two-dimensional elasticity illustrate that these methods allow a determination of the topology of a mechanical element, as required for a boundary variations shape optimization technique.

1 Introduction

Shape optimization in its most general setting should consist of a determination for every point in space whether there is material in that point or not. Alternatively, for a FEM discretization, every element is a potential void or structural member. In this setting, the topology of the structure is not fixed *a priori*, as in boundary variation techniques, and the general formulation should allow for the prediction of the layout of a structure. Shape design problems formulated this way are inherently discrete optimization problems and there are various ways of solving them without the use of discrete optimization algorithms. One way is to use continuous approximations based on heuristics and this is satisfactory in some cases. The most satisfactory approximations are obtained by introducing composites such as layered structures or porous, periodic media. This means that the shape can be described by a density of material that can take on all values between zero and one and for which intermediate values make physical sense. Also, the problem is now a standard sizing problem, which among other things implies that a fixed FEM discretization can be used through-out an iterative optimization procedure.

The standard approach to shape optimization is to introduce boundary variations for a given topology (lay-out) of the structure. This methodology for shape optimization has attracted a great deal of attention and the literature on the subject is quite extensive; we refer to the surveys by Ding (1986) and by Haftka and Gandhi (1986). The boundary variation method can be implemented in a number of ways, for example by employing certain mesh moving schemes to define the shape of a given structure. In this case, the design variables are the coordinates of nodal points of a finite element model of the structure. A different approach to representing boundaries in shape optimization is to introduce the boundary segment idea which describes the design boundary by a set of simple segments such as straight lines, circular arcs, elliptic arcs and splines. The optimum is then sought within this restricted definition of the boundary.

The boundary variations techniques are not straightforward to implement and normally require some method for FEM-remeshing which should be used for the structure at hand several times during an iterative optimization scheme. Also, the definition of the allowable boundary variations needs to be carefully considered, in order to obtain acceptable designs. However, the techniques have now reached a level of maturity that makes it viable to implement the boundary variations methods in CAE (Computer Aided Engineering) systems for production use.

The boundary variation techniques are limited in scope in the sense that the methods only allow for the prediction of the optimum shape of the boundaries of *a given initial topology*. A new method that can yield the optimal topology as well as the optimal shape, even in a rough form, of a structure would be a useful extension of the present methodology. Such a method should be seen as a preprocessor for the boundary variations techniques in cases where it is obvious that much can be gained by changing the topology as well as the shape of boundaries.

The formulation of shape design problems as pointwise material/no material problems was proposed by Kohn and Strang (1986a, 1986b) and the practical possibilities of this approach were first studied in a recent paper by Bendsøe and Kikuchi (1988). It turns out that, in general, existence of solutions cannot be expected unless the problem is turned into a material distribution one, using composite materials. For a periodic medium with known microstructure, homogenization theory can be applied to compute a relation between a material density and the effective material properties and in this way the shape design problem appears as a problem of finding the optimal density-distribution of material in a *fixed* domain.

We thus take an approach where a structural element is understood in a broad sense as being defined only by the loads it is supposed to carry; its volume (cost), and design requirements such as stress and strain limitations. The only restrictions on the allowable shapes is that the resulting structure should connect to the given surface tractions. The initial design in the iterative design optimization procedure is a rough block of space in which we fill material in an optimal way (or we have a rough block of material and remove material).

In the present paper, we compare the use of composites consisting of material with voids of square and rectangular shape with the use of a layered medium, where a very weak material plays the role of voids. Also, results that can be obtained from an artificial power-law for the dependence of rigidity on density are presented, with a linear law representing the design of variable thickness sheets in plane stress. The methods allow for a determination of the topology of a mechanical element and give useful information on the form of the boundaries of the optimal shape. For moderately low volume fractions the lay-out of truss-like structures is predicted, but for very low volume fractions it is recommended that the traditional lay-out theory be employed, as described by Rozvany (1984).

2 General problem formulation

In the following, the general formulation for optimal shape design of linearly elastic structures is presented. The set-up is analogous to the well-known formulation for sizing problems (cf. Olhoff and Taylor 1983).

Consider a mechanical element as a body occupying a domain Ω^m which is part of larger reference domain Ω in R^2. Refering to the reference domain Ω, we can define the optimal shape design problem as the problem of the optimal choice of elasticity tensor $E_{ijkl}(x)$, which is a variable over the domain and which takes the form

$$E_{ijkl}(x) = X(x)\overline{E}_{ijkl} \,. \tag{1}$$

Here $\overline{E}_{ijkl}$ is the constant rigidity tensor for the material employed for the construction of the mechanical element, and $X(x)$ is an indicator function for the part Ω^m of Ω that is occupied by the material:

$$X(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega^m \\ 0 & \text{if} \quad x \in \Omega \setminus \Omega^m \end{cases} . \tag{2}$$

Note that by defining the admissible tensor in this way for each point x in space (or rather, in Ω) one has the discrete choice of material or no material. That is, we have formulated a distributed parameter optimization problem with a *discrete valued* parameter function. A direct approach to such an optimization problem by discretization using finite elements thus requires the use of discrete optimization algorithms. However, such an approach would be unstable with respect to choice of elements and discretization mesh, as the distributed problem, in general, does *not* have a solution, unless composite materials are introduced (see Kohn and Strang 1986a, 1986b). The use of composites moves the on-off nature of the problem from the macroscopic scale to a microscopic scale.

In the following examples, various cases of the minimization of compliance for fixed volume are illustrated, with linear plane stress as the physical model.

Introducing the energy bilinear form

$$a(u,v) = \int_\Omega E_{ijkl}\epsilon_{kl}(u)\epsilon_{ij}(v)\,\mathrm{d}x \,, \tag{3}$$

with linearized strains $\epsilon(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and the load linear form $L(v) = \int_\Omega f{\cdot}v\mathrm{d}x + \int_{\Gamma_T} t{\cdot}v\mathrm{d}s$, the minimum compliance problem takes the form

$$\min L(u) \tag{4a}$$

$$\text{subject to} : a(u,v) = L(v), \quad \text{all } v \in U \,, \tag{4b}$$

$$\text{volume constraint} \,. \tag{4c}$$

Here the equilibrium equation is written in its weak, variational form, with U denoting the space of kinematically admissible displacement fields, $\mathbf{f}$ are the body forces and $\mathbf{t}$ the boundary tractions. For the choice of the elasticity tensors given by (1), the volume is

$$\mathrm{Vol} = \int_\Omega X(x)\,\mathrm{d}x = \text{measure}\,(\Omega^m) \,. \tag{5}$$

The minimum compliance problem can conveniently and efficiently be solved by using the so-called optimality criteria method, where the optimality condition for the problem is solved directly through an iterative scheme (cf. Cheng and Olhoff 1982; Bendsøe 1986; Rozvany 1989). For the compliance problem, the optimality conditions reduce to

$$\frac{\partial}{\partial D} E_{ijkl}(D)\epsilon_{kl}(u)\epsilon_{ij}(u) = \wedge \frac{\partial}{\partial D}(\mathrm{vol}), \qquad (6)$$

for each real valued design variable D. Here $\wedge$ is a positive constant, namely the Lagrange multiplier for the volume constraint and 'vol' denotes the pointwise expression for the volume, as expressed in terms of the design variables.

In the examples that follow, the analysis problem was solved through FEM and the design variables where discretized as element-wise constant. The iterative update scheme for the design variables was based on (6), with the value of $\wedge$ adjusted in an inner iteration loop.

3 The direct approach

For the sake of comparison, a direct approach to the solution of the shape optimization described above was tried on a number of examples. The first step of this approach is to choose a suitable reference domain Ω which allows for the definition of surface tractions and other boundary conditions. Then for a fixed FEM discretization of this domain, the elements that are voids are determined. This distribution of voids can be computed by employing a 0-1 discrete optimization method or, as was done here, by a suitable differentiable approximation of this on-off character of the problem. By introducing an artificial density function $\mu(x), x \in \Omega, 0 \leq \mu(x) \leq 1$ and with $p \gg 1$ letting

$$E_{ijkl}(x) = [\mu(x)]^p \overline{E}_{ijkl}, \quad \mathrm{Vol} = \int_\Omega \mu(x)\,\mathrm{d}x, \qquad (7)$$

we obtain an artificial material where intermediate values, $0 < \mu < 1$, give very little stiffness at an unreasonable cost (volume is linear in μ). This scheme works very effectively and results in μ-values 1 or 0 in most elements (see Fig. 1). However, the scheme is very dependent on the mesh and it is impossible to give any physical meaning to intermediate values of μ .

Note that if we set $p = 1$ in (4), we have the case of optimal design of variable thickness sheets as described by Rossow and Taylor (1973). This does not result in a material-void type structure, so it is unsatisfactory for generating two-dimensional shapes. However, as for generating three-dimensional shapes, the problem makes sense and it is computationally well-behaved (solutions exist, cf. Bendsøe 1983). Figure 2 shows an example for this case.

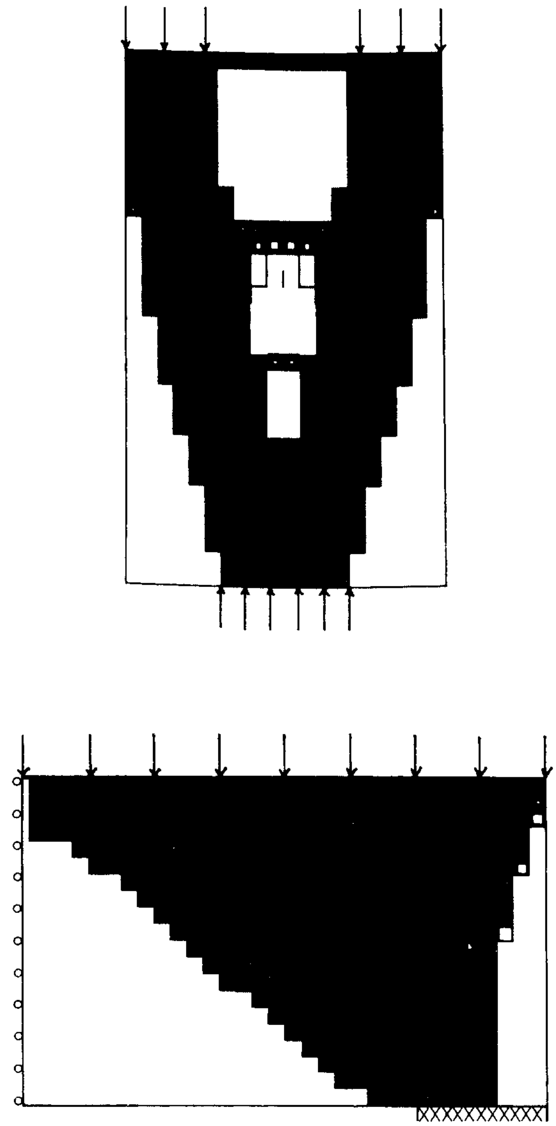

Fig. 1. Direct approach applied with μ^p, $p = 4$. The top picture is a support, the bottom one is half of a 'bridge', with symmetry around the left hand side. The volume constraint in both cases corresponds to 64% of the full area. The black areas indicate material, with intermediate values of μ shown as white, square holes with an area $(1-\mu)\times$ element size

4 Periodic media and homogenization

In the problem statement of Section 2, the optimal shape is, figuratively speaking, defined by the macroscopic distribution of voids. That is, at each point in space there is material or there is no material (void). Introducing a material density μ by constructing a composite material consisting of an infinite number of infinitely small holes periodically distributed through the base material, we can

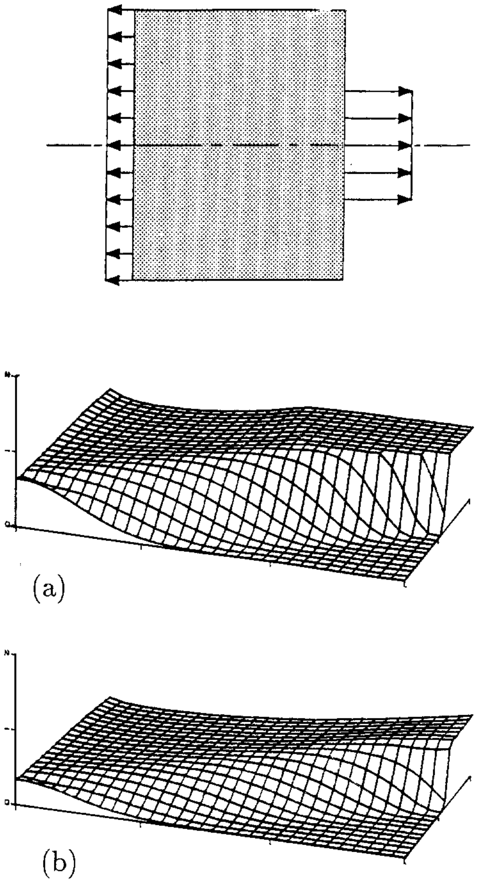

Fig. 2. Variable thickness sheets. Left hand picture shows the loadings and the design area, right hand side two optimal designs (lower half-part), corresponding to volumes of (a) 64%, (b) 36%

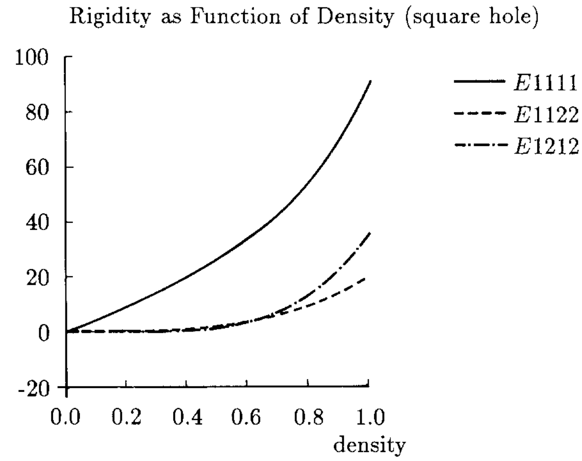

Fig. 3. E_{ijkl} as a function of the density μ for a periodic microstructure of square cells with square holes

transform the problem to the form of a sizing problem. The on-off nature of the problem is avoided through the introduction of μ, with $\mu = 0$ corresponding to a void, $\mu = 1$ to material and $0 < \mu < 1$ to the porous composite with voids at a microlevel. We now have a relationship

$$E_{ijkl}(x) = \tilde{E}_{ijkl}[\mu(x), \Theta(x)], \quad \mathrm{Vol} = \int_\Omega \mu(x)\,\mathrm{d}x, \tag{8}$$

where the effective material parameters $\tilde{E}_{ijkl}$ for the composite can be obtained analytically or numerically through the formulas of homogenization (see below). The composite material will, in general, be orthotropic so the angle Θ of rotation of the directions of orthotropy enters as a design variable, via well-known transformation formulas for frame rotations. Also, the density μ can in itself be a function of a number of design variables which describe the geometry of the holes at the microlevel and it is these variables that should be optimized. Figure 3 shows the density-rigidity relation for a composite with square holes, and as for the direct approach we have obtained a situation where intermediate density values give less than proportional rigidity. However, for composites, intermediate densities make physical sense.

Materials with microstructure play an important role in optimal structural design and have been introduced for a number of problems as a basis for regularizing ill-posed optimization problems, for example in plate design (Cheng and Olhoff 1982; Bendsøe 1986; Rozvany *et al.* 1987), or in the design of torsion bars (Lurie *et al.* 1982; Goodman *et al.* 1986). In this paper, composites are viewed as a practical tool for solving the discrete valued material distribution problem of shape design, but as noted by Kohn and Strang (1986a, 1986b), the optimal design problem as formulated in Section 2 also requires regularization through, for example, the introduction of materials with microstructure. The use of composites in the present setting may thus also achieve regularization, at least partly, even though this is not the primal goal of the approach.

For the sake of completeness of the presentation, the formulas of homogenization will be briefly recalled. For details, the reader is refered to Bensousson *et al.* (1978), Sanchez-Palencia (1980) and Bourgat (1977). Suppose that a periodic microstructure is assumed in the neighbourhood of an arbitrary point $\mathbf{x}$ of a given linearly elastic structure (see Fig. 4.). The periodicity is represented by a parameter ϵ which is very small and the elasticity tensor E^ϵ_{ijkl} is given in the form

$$E^\epsilon_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x}, \mathbf{x}/\epsilon), \tag{9}$$

where $\mathbf{y} \to E_{ijkl}(\mathbf{x},\mathbf{y})$ is Y-periodic, with cell Y of periodicity, $Y = [Y_{1R}, Y_{1L}] \times [Y_{2R}, Y_{2L}]$. Here $\mathbf{x}$ is the macroscopic variation of material parameters, while $\mathbf{x}/\epsilon$ gives the microscopic, periodic variations. Now, suppose that the

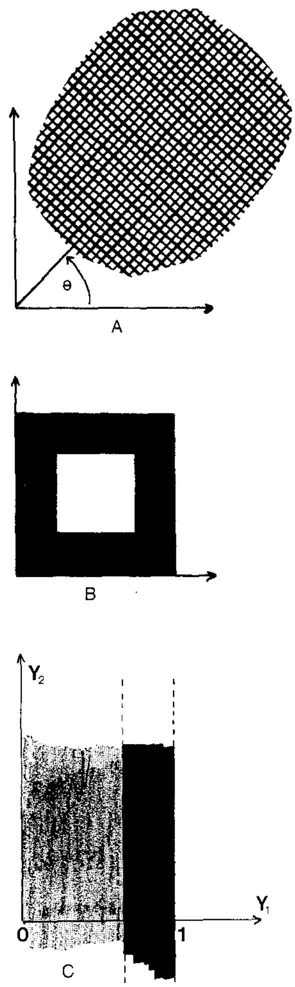

Fig. 4. A: a periodic microstructure with square holes. Θ is the angle of rotation. B: a square cell with a square hole. C: a layered structure

structure is subjected to a macroscopic body force and a macroscopic surface traction. The resulting displacement field $\mathbf{v}^\epsilon(\mathbf{x})$ can then be expanded as

$$\mathbf{v}^\epsilon(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \epsilon \mathbf{v}_1(\mathbf{x}, \mathbf{x}/\epsilon) + \ldots \tag{10}$$

where the leading term $\mathbf{v}_0(\mathbf{x})$ is a macroscopic deformation field that is independent of the microscopic variable $\mathbf{y}$. It turns out that this effective displacement field is the macroscopic deformation field that arises due to the applied forces when the rigidity of the structure is assumed to be given by the effective rigidity tensor

$$E^H_{ijkl}(\mathbf{x}) = \frac{1}{|Y|} \int_Y \left[E_{ijkl}(\mathbf{x},\mathbf{y}) - E_{ijpq}(\mathbf{x},\mathbf{y}) \frac{\partial \aleph^{kl}_p}{\partial y_q} \right] \mathrm{d}\mathbf{y}\,. \tag{11}$$

Here $\aleph^{kl}$ is a microscopic displacement field that is given as the Y-periodic solution of the cell-problem

$$\int_Y E_{ijpq}(\mathbf{x},\mathbf{y}) \frac{\partial \aleph^{kl}_p}{\partial y_q} \frac{\partial v_i}{\partial y_j} \,\mathrm{d}\mathbf{y} =$$

$$= \int_Y E_{ijkl}(\mathbf{x},\mathbf{y}) \frac{\partial v_i}{\partial y_j} \,\mathrm{d}\mathbf{y} \quad \text{for all } \mathbf{v}\,, \tag{12}$$

where $\mathbf{v}$ denotes Y-periodic virtual displacement fields. From (11) and (12) we see that the effective moduli E^H_{ijkl} for plane problems can be computed by solving three analysis problems for the unit cell Y. For most geometries this has to be done numerically, using FEM or, as can be advantageous, BEM or spectral methods. Equations (11) and (12) hold for mixtures of linearly, elastic materials and for materials with voids, where the boundary of the void does not intersect the boundary of the unit cell. Figure 3 shows the variation of the effective moduli for a material with square voids imbedded in square cells, as illustrated in Fig. 4.

For layered materials, the effective moduli can be computed analytically, using (11) and (12). Alternatively, the effective moduli can be derived by a smear-out technique that finds the effective moduli from a relationship between the direct averages of the strain and stress tensors in the unit cell, obtained through the use of interface conditions; for details see e.g. Olhoff *et al.* (1981), Bendsøe (1986).

Now, consider a layered material, as illustrated in Fig. 4, with layers directed along the y_2-direction and repeated periodically along the y_1-axis. The unit cell is $[0,1] \times \mathbb{R}$, and it is clear that the unit cell fields $\aleph^{kl}$ are independent of the variable y_2. Also note that in (11), the term involving the cell deformation field $\aleph^{kl}$ is of the form $E_{ijpq}(\mathbf{x},\mathbf{y}) \frac{\partial \aleph^{kl}_p}{\partial y_q}$, so an explicit expression for $\aleph^{kl}$ is not needed.

Taking first $k = 1$ and $l = 1$, using test functions $\mathbf{v} = [\varphi(y_1), 0]$ in (12), we obtain

$$E_{1111} \frac{\partial \aleph^{11}_1}{\partial y_1} = E_{1111} + c_1\,, \tag{13}$$

where c_1 is a constant. Periodicity of $\aleph^{11}$ implies that

$$\frac{1}{|Y|} \int_Y \frac{E_{1111} + c_1}{E_{1111}} \,\mathrm{d}\mathbf{y} = 0\,. \tag{14}$$

With the notation

$$M(f) = \frac{1}{|Y|} \int_Y f(\mathbf{y}) \,\mathrm{d}\mathbf{y}\,, \tag{15}$$

for the average over Y of a function f, we have shown that

$$E_{1111}\frac{\partial \aleph_1^{11}}{\partial y_1} = E_{1111} + c_1\,, \tag{16a}$$

with

$$c_1 = -[M(1/E_{1111})]^{-1}\,. \tag{16b}$$

Similarly with $k = 2$, $l = 2$ we obtain

$$E_{1111}\frac{\partial \aleph_1^{22}}{\partial y_1} = E_{1122} + c_2\,, \tag{17a}$$

$$c_2 = -M\left(\frac{E_{1122}}{E_{1111}}\right)\cdot\left[M\left(\frac{1}{E_{1111}}\right)\right]^{-1}. \tag{17b}$$

Finally for $k = 1$, $l = 2$, we use test functions $\mathbf{v} = [0, \varphi(y_1)]$ to obtain

$$E_{1212}\frac{\partial \aleph_2^{12}}{\partial y_1} = E_{1212} + c_3\,, \tag{18a}$$

$$c_3 = -\left[M\left(\frac{1}{E_{1212}}\right)\right]^{-1}. \tag{18b}$$

Assuming that the direction of the layering coalesces with the directions of orthotropy of the materials involved, the only non-zero elements of the tensor E are E_{1111}, E_{2222}, E_{1212}, E_{1122} $(= E_{2211})$. In this case, the information given in (16) through (18) is sufficient for calculating the effective moduli from (11). We get

$$E^H_{1111} = \left[M\left(\frac{1}{E_{1111}}\right)\right]^{-1},$$

$$E^H_{2222} = M(E_{2222}) - M\left(\frac{E^2_{2211}}{E_{1111}}\right) +$$

$$+\left[M\left(\frac{E_{2211}}{E_{1111}}\right)\right]^2\cdot\left[M\left(\frac{1}{E_{1111}}\right)\right]^{-1},$$

$$E^H_{1122} = M\left(\frac{E_{1122}}{E_{1111}}\right)\cdot\left[M\left(\frac{1}{E_{1111}}\right)\right]^{-1}, \tag{19}$$

$$E^H_{1212} = \left[M\left(\frac{1}{E_{1212}}\right)\right]^{-1}.$$

For a layering of two materials with the same Poisson ratio ν, with different Young's moduli E^+ and E^- and with layer thicknesses γ and $(1-\gamma)$, respectively, we get

$$E^H_{1111} = I_1\,, \qquad E^H_{2222} = (1-\nu^2)I_2 + \nu^2 I_1\,,$$

$$E^H_{1212} = \frac{1-\nu}{2}I_1\,, \quad E^H_{1122} = \nu I_1\,,$$

$$I_1 = E^- E^+/[\gamma E^- + (1-\gamma)E^+]\,,$$

$$I_2 = \gamma E^+ + (1-\gamma)E^-\,. \tag{20}$$

We note that for layered materials, voids should be represented by a very weak material, even if layers at multiple scales are introduced (see below). On the other hand, layered materials have analytical expressions for the effective moduli, which is a distinct advantage for optimization. For other types of microvoids, the effective moduli must be computed numerically for a number of dimensions of the voids in the unit cell and for other values of densities the effective moduli can be interpolated using, for example, Legendre polynomials or splines. Note that this only needs to be carried out for different values of Poisson's ratio, as Young's modulus enters as a scaling factor.

5 Material density approach by material with voids

The purpose of introducing composites with microvoids into the general formulation for shape design is to avoid the discrete valued nature of this formulation. The results of an optimization should preferably lead to a distribution of material where the density is 0 or 1 almost everywhere, corresponding to a design with only macroscopic holes. Thus the important quantity for shape design is the density of material while the underlying geometric quantities defining this density are of less interest. Employing microvoids which are square holes in square unit cells, we obtain that the density μ is described by just one geometric variable, namely the length of the sides of the square. Also the density μ can take on all values between 0 and 1, a feature that is not satisfied for e.g. circular holes. However, for square holes the effective material is orthotropic, with directions of orthotropy given by the angle of rotation of the unit cell. This means that this angle of rotation should also be considered as a design variable. To recapitulate, with a composite with square microvoids, the optimal shape that minimizes compliance for a given volume can be found by solving the problem given by (4), with two distributed design variables, $\mu(x)$ (a sizing variable), $\Theta(x)$ (a rotation angle), in the fixed domain Ω and with

$$E_{ijkl}(x) = \tilde{E}_{ijkl}[\mu(x), \Theta(x)]\,, \quad \text{Vol} = \int_\Omega \mu(x)\,\mathrm{d}x\,.$$

The reference domain Ω is a suitable chosen domain that allows for the proper assignment of boundary conditions and boundary tractions. This domain can be chosen simply connected or not, depending on design requirement.

The use of the fixed reference domain and the formulation of shape optimization as a sizing problem means that the same FEM mesh can be used throughout the iterative optimization scheme. For the optimization of the angle Θ of cell rotations, an iterative Newton type algo-

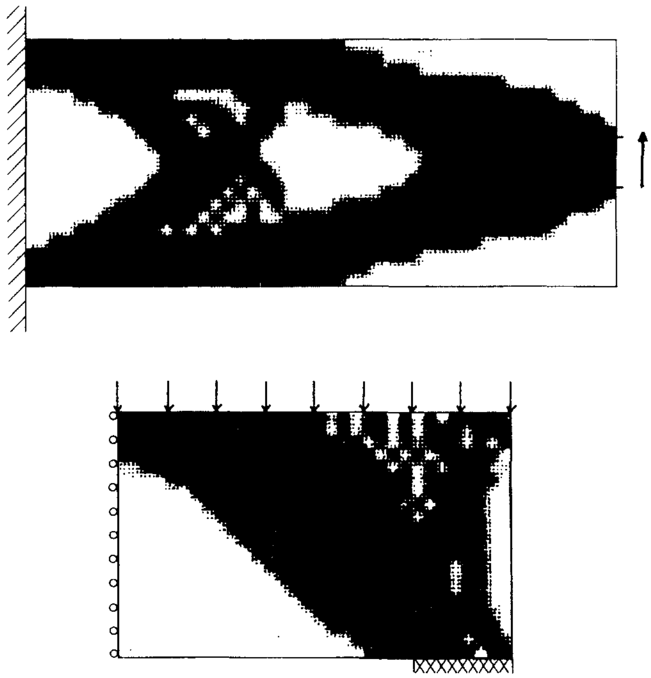

Fig. 5. Material density approach using square holes in square cells. The top picture is a clamped 'beam', the bottom picture is the right half-part of a bridge. The volume constraint is 64%. The picture shows, macroscopically, the size of holes in each element of the discretization. In reality these holes are at the microlevel

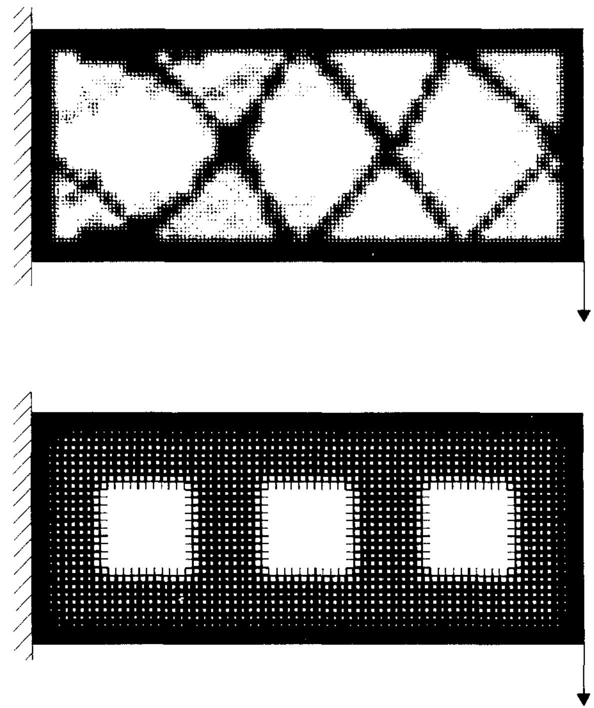

Fig. 6. Material density approach using square holes in square cells. Optimal design of a beam, clamped at one end and loaded at a corner at the other end (one-half of a simply supported beam). It is required that the rim of the beam is fixed (with $\mu = 1$) and there should be 3 holes inside the beam (with $\mu \simeq 0$). With these constraints, as indicated in the bottom picture, the optimal design is the truss-like structure shown at the top. Volume constraint is 45%

rithm can be employed or recent results on optimal rotation of orthotropic materials can be used (Pedersen 1989a, 1989b). For materials with square microvoids, the cells should be rotated along the directions of principal strain (which at the optimum coalesce with the directions of principal stress). Unfortunately, the optimization of the angle of cell rotation is prone to somewhat erratic behavior near the optimum, but this does not influence the overall shapes that are obtained. However, examples show that the angle of rotation should not be ignored in the optimization problem (Bendsøe and Kikuchi 1988).

The method has been tested on a large number of examples, a few being illustrated in Figs. 5 and 6. Note that for comparatively small volume fractions, the method predicts the lay-out of truss-like structures. For very low volume fractions, very fine discretization meshes are required (for coarse meshes the structures break up), so for these cases it is perhaps better to use traditional lay-out methods (Rozvany 1984). The method turns out to be stable with respect to the discretization of the domain. It is very fast with respect to computer time, it predicts topology as well as boundary form and it has a physical interpretation of intermediate density values.

The use of square holes at the microscopic level is but one, albeit the simplest, choice of composite that can be employed. More complicated microstructures invariably lead to more design variables with no apparent benefit. Several experiments show that similar shapes and compliance values are obtained, independently of the microstructure. The important feature is that a microstructure is introduced. Figure 7 shows examples computed by use of rectangular microvoids in square cells, a case where the density of material is given by two design variables.

6 Material density approach by layered material

Recent studies on bounds on the effective material properties of composite mixtures made of two materials have shown that for plane elasticity the strongest material can be obtained by a layered medium, with layering at two different microscales (cf. Avellaneda 1987; Kohn 1988).

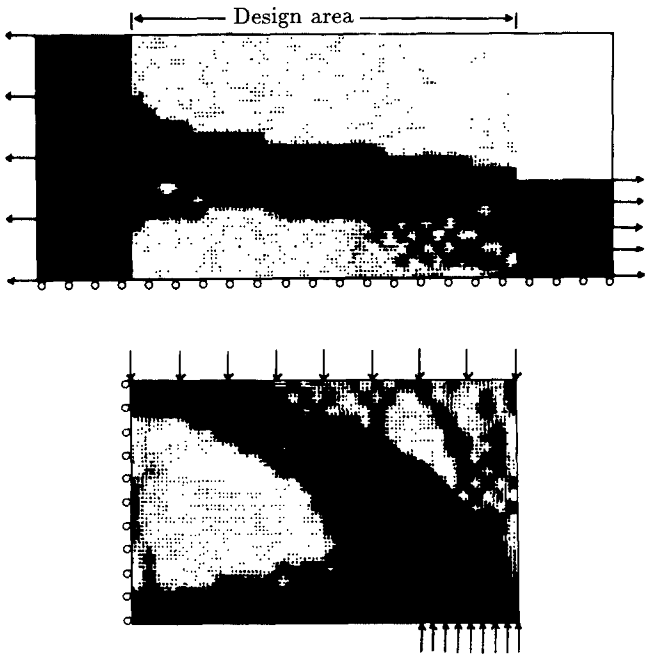

Fig. 7. Material density approach using rectangular holes in square cells. The picture at the top is the upper part of a fillet, with only part of the domain free to be designed; the volume constraint is 36% of this area. The bottom picture shows the right hand part of a beam loaded on the top and on a part of the bottom; volume constraint is 64%

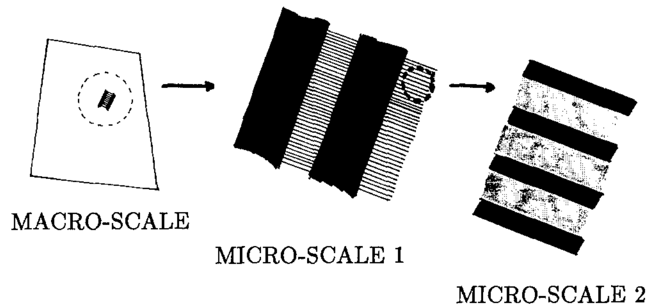

Fig. 8. The build-up of a second rank layered material, by successive layering

This means that the existence of solutions is assured for minimum compliance shape minimization problems. Voids should be exchanged by a very weak material and the composites to be used are those constructed by layering. As shown in Section 4, effective material properties for layered materials can be obtained analytically and for a so-called second rank layering two densities γ and δ of layers are needed to define the material properties and the total density of material. First, a (first order) layering of the strong and the weak material is constructed, the thicknesses of the strong and weak layers being γ and $(1-\gamma)$, respectively, in the unit cell, $[0,1] \times I\!R$ (see Fig. 8). This resulting composite material is then used as one of two components in a new layered material, with layers δ thick of the isotropic, strong material and with layers $(1-\delta)$ thick of the composite just constructed; the layers of this composite material are placed perpendicular to the direction of the new layering. The effective properties of the resulting material are computed by recursive use of the formulas in Section 4; the moduli are computed as the material is constructed, bottom-up. This computation is most conveniently carried out by the use of a computer language for symbolic computations, as derivatives of the moduli with respect to γ and δ are needed for optimization.

The layered materials can be used for shape design in a manner analogous to the use of composites with voids. The density of material now depends on the two design variables γ and δ, and in the optimization problem defined in (4), we have in this case

$$E_{ijkl}(x) = \tilde{E}_{ijkl}[\gamma(x), \delta(x), \Theta(x)]\,,$$

$$\mathrm{Vol} = \int_{\Omega} \{\delta(x) + [1-\delta(x)]\gamma(x)\}\,\mathrm{d}x\,,$$

where Θ denotes the angle of rotation of the layers of the composite. Figure 9 shows some example shapes obtained by use of layered composites. The resulting designs are very similar to those obtained by using square holes, so these simpler composites should be used when speed of computations is an issue. On the other hand, the analytical expressions for the effective moduli for layered materials simplifies the change of material coefficients for the base materials, especially where non-isotropic base materials are considered. This feature is important when considering shape design with composite materials.

7 Conclusions

The optimal topology of a mechanical element can be predicted in a number of ways, by introducing an artificial density or by introducing a density of a composite with voids. Weighing cost and complexity against generality it seems that the most satisfactory method is to employ the porous material approach, using simple square voids at the microscale.

The methods for predicting the topology of a mechanical element can be used as a preprocessor for a boundary variations technique for optimal shape design. The interfacing can be done semi-automatically through the use of graphics facilities (Bendsøe and Rodrigues 1989), but re-

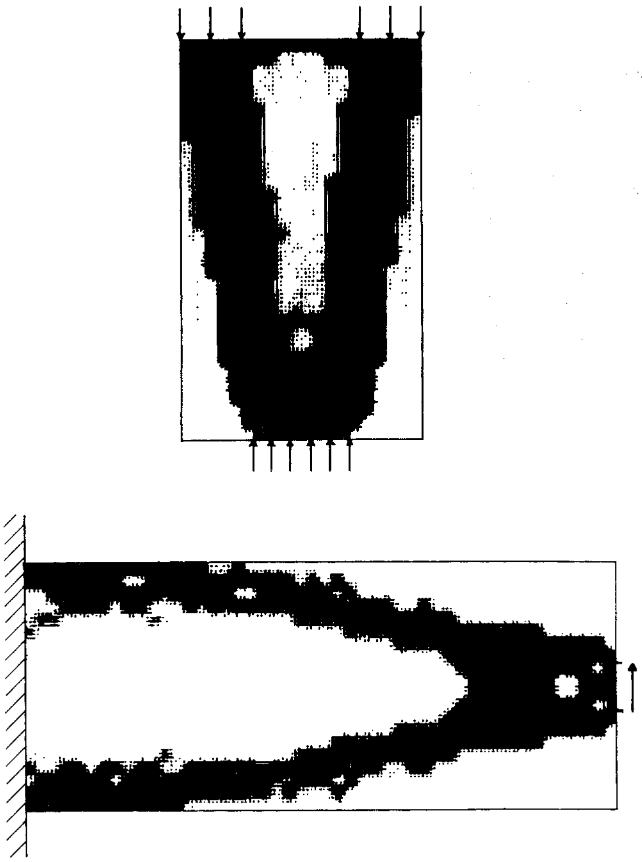

Fig. 9. Material density approach using layered materials. The top picture is a support with volume constraint 64%. The bottom picture is a clamped beam loaded at the right end; volume constraint is 36%

search should be carried out in order to develop methods for automatic interfacing. Notions from image processing and pattern recognition seem to be suitable for such automatic interfacing, and the possibility of choosing the final form of the structure from a discrete set of 'production-friendly' shapes should be investigated.

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