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An integral treatment for non-Darcy free convection over a vertical flat plate and cone embedded in a fluid-saturated porous medium

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Abstract. An integral treatment was proposed for analysis of non-Darcy free convection over a vertical flat plate and cone within a fluid-saturated porous medium. A flexible one-parameter family of third order polynomials was employed to cope with vast changes in the velocity and temperature profiles encountered in the Darcy flow limit through to the Forchheimer flow limit. Zero curvature requirement for the temperature profile at the wall was exploited as an auxiliary relation to determine the shape parameter. Comparison of the approximate results with the exact solution reveals a high performance of the present integral procedure for heat transfer rate prediction.

Eine Integral-Methode für freie „Non-Darcy"-Konvektion entlang einer vertikalen flachen Platte und einem Kegel in einem gesättigten porösen Medium

Zusammenfassung. Eine Integral-Methode wurde zur Analyse von freier „Non-Darcy"-Konvektion entlang einer vertikalen flachen Platte und einem Kegel in einem gesättigten porösen Medium herangezogen. Eine flexible einparametrige Schar von Polynomen dritter Ordnung wurde verwendet, um große Änderungen in den Geschwindigkeits- und Temperaturprofilen im „Darcy"- bis hin zum „Forchheimer"-Bereich erfassen zu können. Die Forderung, daß das Temperaturprofil an der Wand keine Krümmung aufweisen darf, wurde als Hilfsbeziehung benutzt, um den Formfaktor zu bestimmen. Ein Vergleich der angenäherten Ergebnisse mit der exakten Lösung offenbart, daß die dargestellte Integral-Methode sich sehr gut eignet, um Wärmeübertragungswerte berechnen zu können.

Nomenclature

A, B, D	shape factors
C	empirical constant associated with the Forchheimer term
g	tangential component of the acceleration due to gravity
Gr	modified Grashof number
i	1 for a flat plate and 3 for a cone
K	permeability
r	1 for a flat plate and x for a cone
Ra_x	local Rayleigh number
T	temperature
u, v	Darcian velocity components
x, y	boundary layer coordinates
α	effective thermal diffusivity
δ	boundary layer thickness
ν	kinematic viscosity of the fluid
Λ	shape parameter

Subscripts

e	external
w	wall

1 Introduction

Heat and fluid flow within porous media has attracted considerable attention because of numerous possible applications in both geophysics and engineering problems [1]. The Darcy flow model which assumes proportionality between the velocity and pressure gradient has been widely used to analyse free convection over heated bodies embedded in porous media e.g. [2, 3]. The Darcy flow model, however, breaks down, when the Rayleigh number becomes high. Plumb and Huenefeld [4] obtained a similarity solution for the non-Darcy free convection over a vertical flat plate, using the Ergun model, [5, 6] which accounts for the porous inertia by a velocity squared term, and supported the experimental observation made by Fand, Steinberger and Cheng [7].

Although numerical integration results have been furnished for a limited number of cases, a lengthy shooting process required for such numerical integrations, still invites a simpler and yet sufficiently accurate solution procedure for the heat transfer problems associated with porous media. Cheng [8] applied the Karman-Pohlhausen integral method to analyse various Darcy flow problems. His integral method was modified by the authors [9, 10] to achieve a better accuracy in integration results. The integral relation was also used by Bejan and Poulikakos [11] and Kaviany and Mittal [12] to attack non-Darcy flow problems. The integral treatment based on an exponential velocity profile, reported by Bejan and Poulikakos, however, overestimates the Nusselt number by 10 to 15%. Also as indicated by Kaviany and Mittal [12], one of the difficulties lies in the fact that the non-Darcy effect due to the porous inertia reflects on the velocity and temperature profiles so drastically that an exponential profile cannot cope with changes in the velocity and temperature fields within the boundary layer.

In the present paper, we shall propose a highly accurate integral method for analysing the non-Darcy free convection. A flexible one-parameter family of third order polynomials is introduced to describe vast changes in the velocity and temperature profiles, encountered in the Darcy flow limit through the Forchheimer limit. An effort is made to match the condition for the temperature profile curvature at the wall. It will be shown that this integral treatment leads to a substantial improvement in the accuracy of the heat transfer results. The closed form expression for the local Nusselt number, derived in this study, is found quite useful for speedy and accurate estimates of heat transfer rates.

2 Governing equations and boundary conditions

Figure 1 takes a vertical flat plate and cone under consideration. The surface wall temperature T_w exceeds the ambient temperature T_e. As a result, the buoyancy force drives the fluids upwards along the heated wall surface.

Using the usual boundary layer coordinates (x, y), the governing equations for the non-Darcy free convection, namely, the continuity equation, the Ergun model equation, and the energy conservation equation are given by

$$\frac{\partial r u}{\partial x}+\frac{\partial r v}{\partial y}=0, \tag{1}$$

$$u+\frac{C\sqrt{K}}{\nu}u^2=\frac{\beta K}{\nu}(T-T_e)\,g, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where

$$r=\begin{cases}1: & \text{a flat plate}\\ x: & \text{a cone}\end{cases}. \tag{4}$$

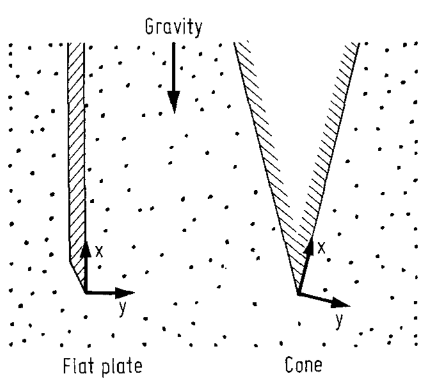

Fig. 1. Physical model and boundary layer coordinates

In the foregoing equations, u and v are the Darcian velocity components in the x and y directions, and T is the local temperature. The tangential component of the acceleration due to gravity is indicated by g. K is the permeability; ν the kinematic viscosity; α the equivalent thermal diffusivity of the porous medium; β the coefficient of thermal expansion; and C the empirical constant associated with the porous inertia term which becomes quite significant as the Rayleigh number increases. The appropriate boundary conditions are

$$y=0:\quad v=0,\quad T=T_w, \tag{5a, b}$$

$$y\to\infty:\quad u=0,\quad T=T_e, \tag{5c, d}$$

where the subscripts w and e refer to the wall and ambient.

3 Integral procedure

Let us integrate the energy Eq. (3) across the boundary layer $0\leqq y\leqq\delta$, exploiting the continuity Eq. (1) and the boundary conditions given by Eqs. (5a) and (5b):

$$\frac{d}{dx}\int_0^\delta r\,u(T-T_e)\,dy=-r\alpha\left.\frac{\partial T}{\partial y}\right|_{y=0}. \tag{6}$$

Substituting Eq. (2) into Eq. (6), we have

$$\frac{d}{dx}\int_0^\delta r\,u\left(u+\frac{C\sqrt{K}}{\nu}u^2\right)dy = -r\alpha\frac{\partial}{\partial y}\left.\left(u+\frac{C\sqrt{K}}{\nu}u^2\right)\right|_{y=0}. \tag{7}$$

The Ergun model equation, which is a quadratic equation in terms of u, can be solved for u as

$$u=\frac{\nu}{2C\sqrt{K}}\left[\left(1+4\,Gr\left(\frac{T-T_e}{T_w-T_e}\right)\right)^{1/2}-1\right], \tag{8a}$$

where

$$Gr=CK^{3/2}\,g\,\beta(T_w-T_e)/\nu^2 \tag{8b}$$

is a modified Grashof number representing the relative significance of the porous inertia (Forchheimer term). Eq. (8a) along with Eq. (5b) provides the slip velocity $u_w=u|_{y=0}$:

$$u_w=\frac{\nu}{2C\sqrt{K}}[(1+4\,Gr)^{1/2}-1]. \tag{9}$$

Upon integrating Eq. (7), and utilizing the foregoing relation, we obtain the following explicit expression for the boundary layer thickness δ:

$$(\delta/x)^2 Ra_x=\frac{8\,Gr(1+4\,Gr)^{1/2}D}{i\,[(1+4\,Gr)^{1/2}-1]\,[2A+[(1+4\,Gr)^{1/2}-1]\,B]}, \tag{10}$$

where

$$i=\begin{cases}1: & \text{a flat plate}\\ 3: & \text{a cone}\end{cases}, \tag{11a}$$

$$A = \int_0^\delta (u/u_w)^2 \, dy/\delta \,, \tag{11b}$$

$$B = \int_0^\delta (u/u_w)^3 \, dy/\delta \,, \tag{11c}$$

$$C = -\frac{\delta}{u_w} \frac{\partial u}{\partial y}\bigg|_{y=0} \tag{11d}$$

and

$$Ra_x = K g \beta (T_w - T_e) \, x/\nu \alpha \tag{11e}$$

is the local Rayleigh number. The integer i for the axisymmetric case ($i = 3$) accounts for the three-dimensional (thinning) effect on the boundary layer growth. The shape factors A, B and D can be expressed in terms of functions of a certain shape parameter, as we specify the velocity profile:

$$u/u_w = 1 - \frac{6-\Lambda}{4}\left(\frac{y}{\delta}\right) - \frac{\Lambda}{2}\left(\frac{y}{\delta}\right)^2 + \frac{2+\Lambda}{4}\left(\frac{y}{\delta}\right)^3 . \tag{12}$$

The foregoing equation with Eq. (8 a) automatically satisfies the required boundary conditions given by Eqs. (5 b) and (5 c). The shape parameter Λ is set to the negative of the second derivative at the wall, namely,

$$\Lambda = -\frac{\delta^2}{u_w} \frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} . \tag{13}$$

Substitution of Eq. (12) into (11 b), (11 c) and (11 d) yields

$$A(\Lambda) = \frac{396 + 32\Lambda + \Lambda^2}{1680} , \tag{14a}$$

$$B(\Lambda) = \frac{9288 + 892\Lambda + 46\Lambda^2 + \Lambda^3}{53760} \tag{14b}$$

and

$$D(\Lambda) = \frac{6 - \Lambda}{4} . \tag{14c}$$

For determination of the shape parameter Λ, an additional relation is required. Through the previous studies on the Darcy flows [3, 9, 10], we have found it quite effective to use the compatible condition associated with the temperature profile curvature at the wall, which is implicit in the energy equation, namely,

$$u_w \frac{dT_w}{dx} = \alpha \frac{\partial^2 T}{\partial y^2}\bigg|_{y=0} . \tag{15a}$$

Since the wall is assumed to be isothermal, the foregoing relation simply suggests zero curvature of the temperature profile at the wall:

$$\frac{\partial^2 T}{\partial y^2}\bigg|_{y=0} = \frac{\partial}{\partial y^2}\left(u + \frac{C\sqrt{K}}{\nu} u^2\right)\bigg|_{y=0} = 0 . \tag{15b}$$

Substituting the velocity profile given by Eq. (12) into the foregoing equation, we obtain a quadratic equation for the parameter Λ, which can easily be solved for Λ as

$$\Lambda = \frac{2\,[7(1+4\,Gr)^{1/2} - 3] - 2\,[[7(1+4\,Gr)^{1/2} - 3]^2 - 9\,[(1+4\,Gr)^{1/2} - 1]^2]^{1/2}}{(1+4\,Gr)^{1/2} - 1} \tag{16}$$

Once the shape parameter Λ is determined for given Gr using the foregoing equation, the local Nusselt number of our primary concern may be evaluated from

$$Nu_x = -\frac{\partial T}{\partial y}\bigg|_{y=0} \bigg/ \left(\frac{T_w - T_e}{x}\right) = -\frac{\nu x}{K g \beta (T_w - T_e)} \frac{\partial}{\partial y}\left(u + \frac{C\sqrt{K}}{\nu} u^2\right)\bigg|_{y=0} = \frac{u_w}{K g \beta (T_w - T_e)}\left(\frac{x}{\delta}\right)\left(1 + 2\frac{C\sqrt{K}\,u_w}{\nu}\right) D$$

$$= (i\,Ra_x)^{1/2}\left[\frac{(1+4\,Gr)^{1/2}[(1+4\,Gr)^{1/2} - 1]^3\,[2A + [(1+4\,Gr)^{1/2} - 1]\,B]\,D}{32\,Gr^3}\right]^{1/2} , \tag{17}$$

where Eqs. (9) and (10) were used to obtain the final expression for Nu_x. It is particularly interesting to note that the vertical flat plate solution can readily be translated for the cone solution, simply by substituting $3\,Ra_x$ in place of Ra_x.

4 Results and discussion

The shape parameter Λ versus Gr curve was generated from Eq. (16), and plotted in Fig. 2, the limiting values for the Darcy flow and Forchheimer flow limits may easily be extracted from the equation, namely,

$$\Lambda = \begin{cases} \frac{9}{2}\,Gr & \text{for} \quad Gr \ll 1 \quad (18a) \\ 14 - 4\sqrt{10} = 1.351 & \text{for} \quad Gr \gg 1 \quad (18b) \end{cases}$$

Upon substituting the foregoing limiting values into Eqs. (14) and (17), we obtain the asymptotic expressions for the local Nusselt number as follows:

$$Nu_x/(i\,Ra_x)^{1/2} = \begin{cases} 0.4205 & \text{for} \quad Gr \ll 1 \quad (18c) \\ 0.4782/Gr^{1/4} & \text{for} \quad Gr \gg 1 \quad (18d) \end{cases}$$

Thus, in the Forchheimer flow limit ($Gr \gg 1$), Nu_x varies in proportion to

$$(Ra_x^2/Gr)^{1/4} = (\sqrt{K}\, g\, \beta (T_w - T_e)\, x^2/C\,\alpha^2)^{1/4}$$

instead of $Ra_x^{1/2}$. The values 0.4205 and 0.4782 are in good agreement with the exact values, namely, 0.444 and 0.494, respectively. Bejan and Poulikakos [11] used an exponential-

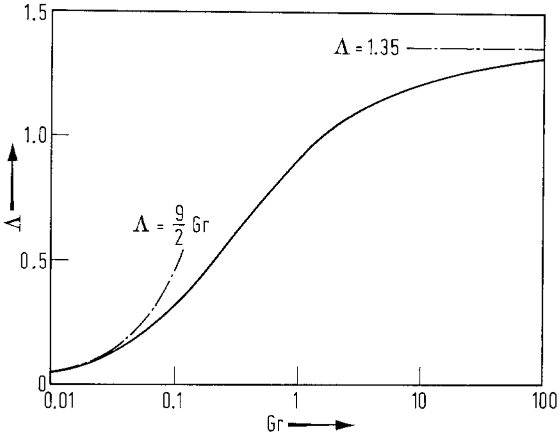

Fig. 2. Shape parameter

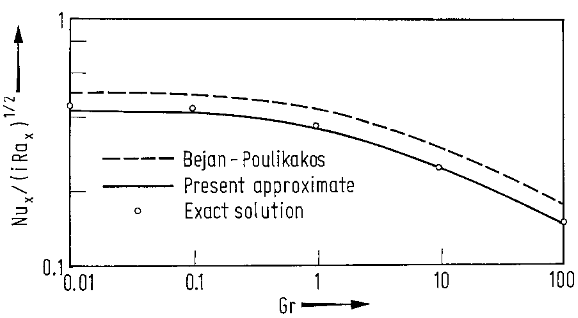

Fig. 3. Heat transfer results

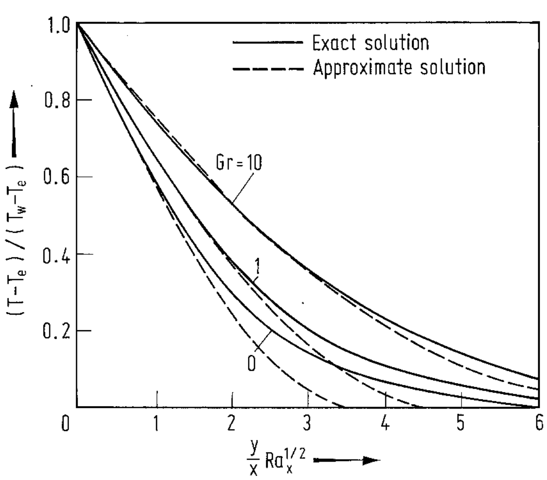

Fig. 4. Temperature profiles

decay velocity profile and temperature profile shapes based on the Oseen's linearized solution, and derived the following integral solution for the isothermal flat plate:

$$Nu_x/Ra_x^{1/2} = \left[\frac{[(1+4\,Gr)^{1/2}-1]\,[16\,Gr^2-[(1+4\,Gr)^{1/2}-1]^2]}{96\,Gr^3}\right]^{1/2}. \tag{19}$$

Table 1. $Nu_x/Ra_x^{1/2}$

Gr	Present approximation	Exact number	Bejan-Poulikakos' approximation
0	0.4205	0.4439	0.5000
10^{-2}	0.4191	0.4423	0.4992
10^{-1}	0.4085	0.4297	0.4912
1	0.3528	0.3662	0.4317
10	0.2435	0.2513	0.2973
10^2	0.1466	0.1519	0.1779

The present heat transfer formula Eq. (17) and the Bejan-Poulikakos' formula Eq. (19) are compared against the exact solution [14] in Fig. 3. The present formula provides a close approximation to the exact solution, while the Bejan-Poulikakos' formula overestimates the heat transfer rates by 10 to 15%. The values based on both formulas are tabulated in Table 1 for a direct comparison with the exact solution. Excellent agreement observed between the present approximate solution, and the exact solution may be attributed to our effort to match the compatibility condition at the wall, namely, zero curvature of the temperature profile at the wall.

Although we cannot expect an approximate method of this kind to predict accurate temperature profiles, it would still be worthwhile to check the assumed temperature profiles, so that the accuracy of the integral treatment can be examined further. Equation (2) along with (9) provides the dimensionless temperature distribution, namely,

$$\frac{T-T_e}{T_w-T_e} = \frac{(1+4\,Gr)^{1/2}-1}{4\,Gr}\left(\frac{u}{u_w}\right)\left[2+[(1+4\,Gr)^{1/2}-1]\left(\frac{u}{u_w}\right)\right], \tag{20}$$

where the function (u/u_w) is given by Eq. (12). In Fig. 4, the temperature profiles for $Gr = 0$, 1 and 10, obtained from Eq. (20), are presented with those generated from the exact solution. The abscissa variable is chosen to be

$$\frac{y}{x}Ra_x^{1/2} = \left[\frac{\delta}{x}Ra_x^{1/2}\right]\left(\frac{y}{\delta}\right), \tag{21}$$

where the value in the bracket is evaluated from Eq. (10). The Assumed profiles near the wall appear to be in close agreement with those of the exact solution. This fact guarantees us a high performance of the present integral procedure for predicting heat transfer rates.

5 Concluding remarks

In this article, we proposed a consistent integral treatment for analysing free convection over a vertical flat plate and cone embedded in a fluid saturated porous medium. A one-parameter family of third order polynomials was used to describe the velocity and temperature profiles. Obviously, an effort to match the compatibility condition for the temperature profile at the wall was successful for improving the accuracy of the heat transfer results. The present heat transfer results fall within about 5% of the exact results.

Although more elaborate numerical integration schemes are now available, integral treatments such a presented here should keep being exploited for speedy and yet sufficiently accurate estimations of heat transfer rates.

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