manuscripta math. 28, 269 - 291 (1979)

manuscripta mathematica

UNIQUENESS AND STABILITY OF HARMONIC MAPS AND THEIR JACOBI FIELDS

Willi Jäger and Helmut Kaul

Dedicated to Hans Lewy and Charles B. Morrey

Let M,N be Riemannian manifolds and let $f_1, f_2 : M \longrightarrow N$ be harmonic maps. Using a maximum principle, an estimate of the distances of these maps by the distances of their boundary values will be proved. Corresponding estimates will be stated for the norm of Jacobi fields along harmonic maps, and for the distances of solutions of the heat equation.

Let M be a connected C^2-Riemannian manifold with nonvoid boundary ∂M, and let N be a C^3-Riemannian manifold without boundary. We denote by $\overset{o}{M} := M - \partial M$ the interior of M. Suppose that $\kappa \geq 0$ is an upper bound of the sectional curvature of N. An open subset A of N satisfies the cut locus condition, if every pair of points in A can be joined by exactly one minimizing geodesic arc of N. We denote by

$$B_r(a) := \{ y \in N : \mathrm{dist}(a,y) < r \}$$

the open r-ball in N with center $a \in N$, and define

0025-2611/79/0028/0269/$04.60

$$q_\kappa : \mathbb{R} \longrightarrow \mathbb{R} \quad , \quad q_\kappa(t) := \begin{cases} (1-\cos\sqrt{\kappa}t)/\kappa & \kappa > 0 \\ t^2/2 & \kappa = 0 . \end{cases}$$

Our results are:

THEOREM A. _Let_ $f_1, f_2 : M \longrightarrow N$ _be continuous mappings which are harmonic in the interior_ $\overset{\circ}{M}$ _of_ M. _Suppose that there is a ball_ $B_r(a) \subset N$ _satisfying the cut locus condition and_

$$f_1(M) \, , f_2(M) \subset B_r(a) \quad , \quad r < \frac{\pi}{2\sqrt{\kappa}} .$$

Then the function

$$\theta = \theta(f_1, f_2) : M \longrightarrow \mathbb{R} ,$$

$$x \longmapsto \frac{q_\kappa(\operatorname{dist}(f_1(x), f_2(x)))}{\cos(\sqrt{\kappa}\operatorname{dist}(a, f_1(x))) \cdot \cos(\sqrt{\kappa}\operatorname{dist}(a, f_2(x)))}$$

satisfies the maximum principle

$$\sup_M \theta \leq \sup_{\partial M} \theta .$$

In particular, from $f_1|_{\partial M} = f_2|_{\partial M}$ _follows_

$$f_1 = f_2 .$$

THEOREM B. _Let_ $f : M \longrightarrow N$ _be a continuous map which is harmonic on_ $\overset{\circ}{M}$. _Assume that there is a ball_ $B_r(a) \subset N$ _satisfying the cut locus condition and_

$$f(M) \subset B_r(a) \quad , \quad r < \frac{\pi}{2\sqrt{\kappa}} .$$

Then for every continuous vector field X _along_ f _which is a Jacobi field on_ $\overset{\circ}{M}$, _the function_

$$\theta = \theta_f(X) : M \longrightarrow \mathbb{R} \quad , \quad x \longmapsto \frac{|X(x)|^2}{\cos^2(\sqrt{\kappa}\,\mathrm{dist}(a,f(x)))}$$

<u>satisfies the maximum principle</u>

$$\sup_M \theta \le \sup_{\partial M} \theta \ .$$

<u>In particular,</u> $X = 0$ <u>is the only Jacobi field along</u> f <u>with</u> $X|_{\partial M} = 0$.

For a real number $T > o$ we consider the product manifold

$$M_T := [0,T] \times M$$

and the boundary of M_T without the top,

$$\partial_T M := (\{0\}\times M) \cup ([0,T]\times\partial M) \ .$$

In the <u>heat equation</u>

$$(\Delta - \partial_t) f = o \quad \text{for mappings} \quad f : M_T^0 \longrightarrow N \ ,$$

∂_t denotes the derivative with respect to the first ("time") variable and the generalized Laplacian Δ, defined by (3), is acting on the ("space") variables of M as usual ([1],[3],[5]).

<u>THEOREM C</u>. <u>Let</u> $f_1, f_2 : M_T \longrightarrow N$ <u>be continuous maps and solutions of the heat equation in the interior of</u> M_T. <u>Suppose that there is a ball</u> $B_r(a) \subset N$ <u>satisfying the cut locus condition and</u>

$$f_1(M_T) \ , \ f_2(M_T) \subset B_r(a) \quad , \quad r < \frac{\pi}{2\sqrt{\kappa}} \ .$$

<u>Then we have</u>

$$\sup_{M_T} \theta \leq \sup_{\partial_T M} \theta ,$$

<u>where</u> $\theta = \theta(f_1,f_2) : M_T \longrightarrow \mathbb{R}$ <u>is the function in Theorem A with</u> M <u>replaced by</u> M_T.

REMARKS. (i) Theorem A improves a result of the authors ([7], Corollary 3) where for a similiar function θ the maximum principle has been proved under the stronger condition

$$r < 1.00291/\sqrt{\kappa} .$$

(ii) Geodesic arcs on the standard sphere, the endpoints of which are antipodal give rise to the conjecture, that the bound $\pi/(2\sqrt{\kappa})$ in Theorem A and Theorem B is the best possible.

(iii) Consider the Dirichlet problem for harmonic maps

$$f : M \longrightarrow B_r(a) \subset N ,$$

M is compact, $B_r(a)$ satisfies the cut locus condition, and $r < \pi/(2\sqrt{\kappa})$. Then we conclude from Theorem A and the result of Hildebrandt, Kaul and Widman in [6] for continuous boundary datas the existence, uniqueness and the continuous dependence of solutions on the boundary values.

(iv) Since Jacobi's differential equation is the linearized Laplace-Beltrami equation (see (6) and (7)), Theorem B may be considered as an infinitesimal version of Theorem A. The fact, that under the conditions in Theorem B $X = 0$ is the only Jacobi field along f with $X|_{\partial M} = 0$, expresses the nondegeneracy of the Hessian $\nabla^2 E_f$ of the energy

$$\phi \longmapsto E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, d\,\mathrm{Vol}_M .$$

This fact generalizes the part of the Morse-Schoenberg Theorem which says that any geodesic arc of length less than $\pi/\sqrt{\kappa}$ is free of conjugate points.

(v) Theorem B answers a question posed to us by Eells and

Lemaire. Theorem A and Theorem B establish assumptions of Theorem (3.1), Corollary (3.2) and (4.3) in [2], where Eells and Lemaire prove the smooth dependence of harmonic maps on their boundary values and the metrics of M and N.

(vi) The cut locus condition in Theorem B can be weakened to the assumption: every point of the ball $B_r(a)$ can be joined with the center a by exactly one minimizing geodesic arc.

The main idea in the proof of our results is to use the maximum principle for a properly chosen elliptic operator $\mathcal{L}$ on the manifold M and a $\mathcal{L}$-subharmonic function $\theta : M \longrightarrow \mathbb{R}$, which controls the distance of solutions $f_1, f_2 : M \longrightarrow N$ or the norm of the Jacobi fields. If the target manifold N has nonpositive sectional curvature, the problem is easy, since the geodesic distance of two harmonic maps is subharmonic with respect to the Laplace-Beltrami operator on the manifold M. However, if the sectional curvature of N gets positive, the situation becomes more delicate. The techniques successful in the harmonic case are easily carried over to the corresponding heat equation.

The authors want to thank Professor Eells and Professor Lemaire for very interesting and helpful discussions. W.Jäger is appreciating the support he got for his study as a guest of the Department of Mathematics of the University of Utah.

We start with some preparations. The following standard notations on Riemannian manifolds are used:

$\langle \, , \, \rangle$	scalar product
$\| \; \|$	the corresponding norm
dist	geodesic distance
d	differential
D	covariant differentiation

R	Riemannian curvature tensor
∇	gradient
div	divergence,

and with respect to a co-ordinate chart for $\overset{\circ}{M}$,

$g_{\alpha\beta}$	coefficients of the scalar product	
$g^{\alpha\beta}$	coefficients of the inverse matrix of $(g_{\alpha\beta})$	
$\Gamma_{\alpha\beta}{}^{\gamma}$	Christoffel symbols	
∂_α	partial differentiation	in direction of the α-th base vector field.
D_α	covariant differentiation	

Let S be a Riemannian manifold, let $f: \overset{\circ}{M} \longrightarrow S$ be a C^2-map and $X: \overset{\circ}{M} \longrightarrow TS$ a C^2-vector field along f (i.e. $X(x) \in T_{f(x)}S \quad \forall x \in \overset{\circ}{M}$). We define with respect to a co-ordinate chart for $\overset{\circ}{M}$:

(1) $$|df|^2 := \sum_{\alpha,\beta} g^{\alpha\beta} \langle \partial_\alpha f, \partial_\beta f \rangle$$

(2) $$|DX|^2 := \sum_{\alpha,\beta} g^{\alpha\beta} \langle D_\alpha X, D_\beta X \rangle$$

(3) $$\Delta f := \sum_{\alpha,\beta} g^{\alpha\beta} (D_\alpha \partial_\beta f - \sum_\gamma \Gamma_{\alpha\beta}{}^{\gamma} \partial_\gamma f)$$

(4) $$\Delta X := \sum_{\alpha,\beta} g^{\alpha\beta} (D_\alpha D_\beta X - \sum_\gamma \Gamma_{\alpha\beta}{}^{\gamma} D_\gamma X)$$

(5) $$R_f(X) := \sum_{\alpha,\beta} g^{\alpha\beta} R(X, \partial_\alpha f) \partial_\beta f \; .$$

All these quantities do not depend on the choice of the chart. $|df|$, $|DX|$ are real functions on $\overset{\circ}{M}$ and Δf , ΔX , $R_f(X)$ are vector fields along f . Δ is called the <u>(generalized) Laplace-Beltrami operator</u> and R_f is the <u>Riemann curvature tensor along</u> f .
The map f is <u>harmonic</u>, iff

(6) $$\Delta f = 0 \; ,$$

and X is called a <u>Jacobi field along</u> a harmonic map f iff

(7) $$\Delta X + R_f(X) = 0 \; .$$

Let

$$E(f) := \frac{1}{2} \int_M |df|^2 \, d\mathrm{Vol}_M$$

be the <u>energy</u> of f. One calculates that the gradient (the "first variation") of the energy is given by

$$\nabla E_f(X) = -{}_M\int \langle \Delta f, X\rangle \, d\,\mathrm{Vol}_M$$

for smooth vector fields X along f with $X|_{\partial M} = 0$, and the Hessian (the "second variation") is

$$\nabla^2 E_f(X,Y) = -{}_M\int \langle \Delta X + R_f(X), Y\rangle \, d\,\mathrm{Vol}_M$$

for smooth vector fields X , Y along a harmonic map f with $X|_{\partial M} = Y|_{\partial M} = 0$. Hence, the harmonic maps are the critical points of the energy functional, and a vector field X along a harmonic map f is a Jacobi field iff

$$\nabla^2 E_f(X,Y) = 0$$

for every vector field Y along f with $Y|_{\partial M} = 0$. For further details about the variational calculus of the energy we refer to [1],[3].

Let $h : S \longrightarrow \mathbb{R}$ be a C^2-map. The Hessian of h is the symmetric bilinear form

(8) $\quad \nabla^2 h : T_y S \times T_y S \longrightarrow \mathbb{R} \quad , \quad \nabla^2 h(u,v) := \langle D_u \nabla h, v\rangle$

$\forall\, y \in \overset{\circ}{S}$. From the chain rule we get for the composition map $h\circ f : \overset{\circ}{M} \longrightarrow \mathbb{R}$

(9) $\quad \langle \nabla(h\circ f)(x), u\rangle = \langle (\nabla h)(f(x)), df(u)\rangle \quad \forall\, u \in T_x M \ , \ x \in M,$

(1o) $\quad \Delta(h\circ f) = \sum_{\alpha\beta} g^{\alpha\beta} \nabla^2 h(\partial_\alpha f, \partial_\beta f) + \langle (\nabla h)\circ f, \Delta f\rangle$,

(11) $\quad |\nabla(h\circ f)|^2 = \sum_{\alpha\beta} g^{\alpha\beta} \langle (\nabla h)\circ f, \partial_\alpha f\rangle \langle (\nabla h)\circ f, \partial_\beta f\rangle$.

In the special case $S = \mathbb{R}$, the equations (9) and (1o) reduce to

(12) $\quad \nabla(h\circ f) = (h'\circ f)\nabla f$

(13) $\quad \Delta(h\circ f) = (h''\circ f)|\nabla f|^2 + (h'\circ f)\,\Delta f$.

Let

$$\overset{\circ}{M} \xrightarrow{\psi_1,\dots,\psi_k} [0,\kappa^{-1}[\xrightarrow{\omega} \mathbb{R}$$

be C^2-maps, and set

$$\Phi := \frac{2}{k}\sum_{i=1}^{k} \omega\circ\psi_i .$$

Let $\mathcal{L}$ be the formally self adjoint elliptic operator on M

(14) $\mathcal{L}(u) := \operatorname{div}(e^{-\Phi}\nabla u) \quad \forall\, u\in C^2(\overset{\circ}{M},\mathbb{R})$.

<u>Lemma 1</u>. <u>Suppose that</u> $\omega'^2 \le \omega''$ <u>holds. Then for every</u> $\psi \in C^2(\overset{\circ}{M},\mathbb{R}_+)$ <u>satisfying</u>

$\nabla\psi = 0$ <u>on the set</u> $\psi^{-1}\{0\}$ <u>of zeros of</u> ψ ,

<u>we have</u>

$$\mathcal{L}(e^{\Phi}\psi) \ge \begin{cases} \Delta\psi & \text{on } \psi^{-1}\{0\} \\ \Delta\psi - \dfrac{|\nabla\psi|^2}{2\psi} + \dfrac{2}{k}\psi\displaystyle\sum_{i=1}^{k}(\omega'\circ\psi_i)\Delta\psi_i & \text{on } \overset{\circ}{M}-\psi^{-1}\{0\} \end{cases}$$

<u>Proof</u>. We have

$$\mathcal{L}(e^{\Phi}\psi) = \operatorname{div}(e^{-\Phi}\nabla(e^{\Phi}\psi)) = \operatorname{div}(\nabla\psi + \psi\nabla\Phi)$$
$$= \Delta\psi + \psi\Delta\Phi + \langle\nabla\psi,\nabla\Phi\rangle .$$

On $\psi^{-1}\{0\}$ holds $\psi = 0$, $\nabla\psi = 0$, whence

$$\mathcal{L}(e^{\Phi}\psi) = \Delta\psi .$$

On $\overset{\circ}{M}-\psi^{-1}\{0\}$ it follows from (12) and (13) that

$$\mathcal{L}(e^{\Phi}\psi) = \Delta\psi + \frac{2}{k}\psi\sum_{i=1}^{k}(\omega''\circ\psi_i)|\nabla\psi_i|^2$$
$$+ \frac{2}{k}\psi\sum_{i=1}^{k}(\omega'\circ\psi_i)\Delta\psi_i$$
$$+ \frac{2}{k}\sum_{i=1}^{k}(\omega'\circ\psi_i)\langle\nabla\psi,\nabla\psi_i\rangle ,$$

and by application of Young's inequality

$$(\omega'\circ\psi_i)\langle\nabla\psi,\nabla\psi_i\rangle \geq -\psi\cdot(\omega'\circ\psi_i)^2|\nabla\psi_i|^2 - \frac{|\nabla\psi|^2}{4\psi}$$

$\forall\, i=1,\ldots,k$, hence, observing $\omega'' \geq \omega'^2$, the second estimate. □

Let $s_\kappa : \mathbb{R} \longrightarrow \mathbb{R}$ be the solution of

(15) $$s_\kappa'' + \kappa\cdot s_\kappa = 0 \quad , \quad s_\kappa(0) = 0 \quad , \quad s_\kappa'(0) = 1$$

and

(16) $$q_\kappa(t) := \int_0^t s_\kappa \quad ,$$

or explicitely

$$s_\kappa(t) = \begin{cases} (\sin\sqrt{\kappa}t)/\sqrt{\kappa} \\ t \end{cases} , \quad q_\kappa(t) = \begin{cases} (1-\cos\sqrt{\kappa}t)/\kappa & \kappa > 0 \\ t^2/2 & \kappa = 0. \end{cases}$$

<u>Lemma 2</u>. <u>If</u> X <u>is a Jacobi field along a geodesic arc</u> $\sigma : [0,\rho] \longrightarrow N$ <u>such that</u>

$$\langle X,\sigma'\rangle = 0 \quad , \quad |\sigma'| = 1 \; ,$$

<u>then under the assumption</u>

$$0 < \rho < \pi/\sqrt{\kappa}$$

<u>the estimate</u>

$$\langle X,X'\rangle\Big|_0^\rho \geq \frac{s_\kappa'(\rho)}{s_\kappa(\rho)}\left(|v_1|^2 + |v_2|^2\right) - \frac{2}{s_\kappa(\rho)}|v_1|\cdot|v_2|$$

<u>holds with</u>

$$v_1 := X(0) \quad , \quad v_2 := X(\rho)$$

(σ' <u>is the tangent vector of</u> σ <u>and the prime at vector fields indicates the covariant differentiation</u> $D/\partial t$).

<u>Proof</u>. This proof is based on Karcher's Jacobi field tech-

niques ([8],App.A). From the assumptions

$$|\sigma'| = 1 \quad , \quad \langle X,\sigma'\rangle = 0$$

there follows

$$\langle X,R(X,\sigma')\sigma'\rangle \leq \kappa\cdot(|X|^2|\sigma'|^2 - \langle X,\sigma'\rangle^2) \leq \kappa\cdot|X|^2 \ ,$$

since κ is supposed to be an upper bound for the sectional curvature of N . Let $s : \mathbb{R} \longrightarrow \mathbb{R}$ be the solution of the boundary value problem

$$s'' + \kappa s = 0 \ , \ s(0) = |X(0)| = |v_1| \ , \ s(\rho) = |X(\rho)| = |v_2| .$$

The function

$$g := s\cdot|X|' - s'\cdot|X| : [0,\rho] \longrightarrow \mathbb{R}$$

is differentiable on every intervall $]\alpha,\beta[\subset]0,\rho[$ without zeros of X . From Jacobi's equation

(17) $\quad X'' + R(X,\sigma')\sigma' = 0$

and the last inequality we get

$$g' = s\cdot|X|'' - s''\cdot|X| = s\cdot\left(\frac{\langle X,X'\rangle}{|X|}\right)' + \kappa\cdot s\cdot|X|$$

$$= s\cdot\frac{\langle X',X'\rangle + \langle X,X''\rangle}{|X|} - s\cdot\frac{\langle X,X'\rangle^2}{|X|^3} + \kappa\cdot s\cdot|X|$$

$$= s\cdot|X|^{-3}(|X|^2|X'|^2 - \langle X,X'\rangle^2)$$

$$- s\cdot|X|^{-1}\langle X,R(X,\sigma')\sigma'\rangle + \kappa\cdot s\cdot|X| \geq 0 \ ,$$

hence

(18) $\quad g(\alpha+0) \leq g(\beta-0)$.

The function s is explicitely given by

$$s(t) = \frac{1}{s_\kappa(\rho)}\cdot(|v_1|\cdot s_\kappa(\rho-t) + |v_2|\cdot s_\kappa(t)) \ ,$$

from which follows

$$s \geq 0 \quad \text{on} \quad [0,\rho]$$

and

$$s'(0) = \frac{-|v_1|s_\kappa'(\rho)+|v_2|}{s_\kappa(\rho)} \quad , \quad s'(\rho) = \frac{-|v_1|+|v_2|s_\kappa'(\rho)}{s_\kappa(\rho)} \ .$$

If $X = 0$, the assertion is trivial. In the case $X \neq 0$, X has only finitely many zeros in $]0,\rho[$,

$$0 < t_1 < \ldots < t_k < \rho \quad ,$$

since X is solution of a second order differential equation. From

$$|X|'(t_i \pm 0) = \pm \, |X'(t_i)| \ ,$$

it follows

$$g(t_i+0) - g(t_i-0) = 2s(t_i)|X'(t_i)| \geq 0 \ ,$$

hence by application of inequality (18) to the intervalls $]t_{i-1},t_i[$ $(t_o:=0 \ , \ t_{k+1}:=\rho)$, we obtain

$$g(0+0) \leq g(t_1-0) \leq g(t_1+0) \quad \ldots \leq g(t_k+0) \leq g(\rho-0) \ ,$$

therefore

$$\begin{aligned} 0 \leq g(\rho-0) - g(0+0) &= s(\rho)|X|'(\rho-0) - s'(\rho)|X(\rho)| \\ &\quad - s(0)|X|'(0+0) + s'(0)|X(0)| \\ &= |X(\rho)|\cdot|X|'(\rho-0) \\ &\quad - \frac{s_\kappa'(\rho)|v_2|-|v_1|}{s_\kappa(\rho)}|v_2| \\ &\quad - |X(0)|\cdot|X|'(0+0) \\ &\quad + \frac{|v_2|-s_\kappa'(\rho)|v_1|}{s_\kappa(\rho)}|v_1| \\ &= \langle X,X'\rangle(\rho) - \langle X,X'\rangle(0) \\ &\quad - \frac{s_\kappa'(\rho)}{s_\kappa(\rho)}\,(|v_1|^2 + |v_2|^2) \end{aligned}$$

$$+ \frac{2}{s_\kappa(\rho)} |v_1| \cdot |v_2| \quad . \square$$

On the product $N \times N$ we introduce the Riemannian metric

$$\langle u_1 \oplus u_2 , v_1 \oplus v_2 \rangle := \langle u_1 , v_1 \rangle + \langle u_2 , v_2 \rangle$$

for $u_i , v_i \in T_{y_i} N$, $y_i \in N$, $i=1,2$. One verifies that for smooth maps

$$f_1 , f_2 : \overset{\circ}{M} \longrightarrow N \quad \text{and} \quad f_1 \times f_2 : \overset{\circ}{M} \longrightarrow N \times N$$

the identity

$$(19) \qquad \Delta(f_1 \times f_2) = \Delta f_1 \oplus \Delta f_2$$

holds.
The distance function on N,

$$\mathrm{dist} : N \times N \longrightarrow \mathbb{R}$$

is of class C^2 on $U_1 \times U_2$ outside the diagonal, if U_1 and U_2 are open subsets of N satisfying the condition

(2o) $\begin{cases} \text{every pair } y_1 \in U_1 , \ y_2 \in U_2 \text{ can be joined by ex-} \\ \text{actly one minimizing geodesic arc of } N. \end{cases}$

If $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2-map such that $\phi'(0) = 0$ (for instance, $\phi = q_\kappa$ defined in (16)), the composition map

$$\phi \circ \mathrm{dist} : U_1 \times U_2 \longrightarrow \mathbb{R}$$

is of class C^2 ([4],§5.1).

Lemma 3. *Let* $U_1 , U_2 \subset N$ *be open sets satisfying the condition* (2o) *and*

$$\mathrm{dist}(y_1 , y_2) < \pi/\sqrt{\kappa} \quad \forall (y_1 , y_2) \in U_1 \times U_2 \ .$$

Then the Hessian of

$$Q_\kappa := q_\kappa \circ \mathrm{dist} : U_1 \times U_2 \longrightarrow \mathbb{R}$$

(q_κ *is defined by* (16)) *admits the following estimates*:

$$\nabla^2 Q_\kappa(v,v) \geqq \begin{cases} |v|^2 & y_1 = y_2 \\ \dfrac{\langle \nabla Q_\kappa(y), v\rangle^2}{2Q_\kappa(y)} - \kappa Q_\kappa(y)|v|^2 & y_1 \neq y_2 \end{cases}$$

$\forall\, v \in T_y(N\times N)$, $y=(y_1,y_2)\in U_1\times U_2$, *and*

$$\nabla^2 Q_\kappa(v,v) \geqq (1-\kappa Q_\kappa(y))|u|^2 \ ,$$

if v *has the special form* $0\oplus u$ *or* $u\oplus 0$.

Proof. Let be $y=(y_1,y_2)\in U_1\times U_2$, $\rho := \operatorname{dist}(y_1,y_2)$, and $v = v_1\oplus v_2 \in T_{y_1}N\oplus T_{y_2}N \cong T_y(N\times N)$. In the case $\rho>0$ ($\Longleftrightarrow$ $y_1\neq y_2$) we denote by

$$e_i(y)\in T_{y_i}N \qquad i=1,2$$

the outside directed unit tangent vectors of the unique minimal geodesic segment joining y_1 and y_2. Let

$$v_i^{tan} := \langle v_i, e_i(y)\rangle e_i(y) \quad , \quad v_i^{nor} := v_i - v_i^{tan}$$

be the tangential part and the normal part respectively, of v_i with respect to this geodesic segment for $i=1,2$. Then the gradient of the distance function is

$$\nabla \operatorname{dist}(y) = e_1(y)\oplus e_2(y) \quad \text{if} \quad \rho>0 \ , \tag{21}$$

and using the chain rule we get

$$\nabla Q_\kappa(y) = \begin{cases} 0 & \rho \neq 0 \\ s_\kappa(\rho)(e_1(y)\oplus e_2(y)) & \rho>0 \ , \end{cases} \tag{22}$$

$$\nabla^2 Q_\kappa(v,v) = \langle D_v \nabla Q_\kappa, v\rangle = \langle D_v((s_\kappa\circ \operatorname{dist})\nabla \operatorname{dist}), v\rangle \ ,$$

hence

$$\nabla^2 Q_\kappa(v,v) = \begin{cases} |v|^2 & \rho = 0 \\ s_\kappa'(\rho)\langle \nabla \operatorname{dist}(y), v\rangle^2 + s_\kappa(\rho)\nabla^2 \operatorname{dist}(v,v) \end{cases} \tag{23}$$

$\rho > 0.$

In case $\rho > 0$, the Hessian of the distance function can be expressed in terms of Jacobi fields: for sufficiently small $\varepsilon \in \mathbb{R}$, let

$$\sigma_\varepsilon : [0,\rho] \longrightarrow N$$

be the uniquely determined geodesic arc such that

$$\sigma_\varepsilon(0) = \exp_{y_1}(\varepsilon v_1^{nor}) \quad , \quad \sigma_\varepsilon(\rho) = \exp_{y_2}(\varepsilon v_2^{nor}) .$$

The family $(\varepsilon,t) \longmapsto \Phi(\varepsilon,t) := \sigma_\varepsilon(t)$ constitutes a variation of the geodesic arc $\sigma = \sigma_0$, and the variational vector field of this variation,

$$X(t) := \frac{\partial \Phi}{\partial \varepsilon}(0,t) \qquad \forall t \in [0,\rho] ,$$

is a Jacobi field along σ satisfying

$$X(0) = v_1^{nor} \quad , \quad X(\rho) = v_2^{nor} .$$

Therefore, from Synge's formula ([4],§4.1), we obtain

$$\begin{aligned} \nabla^2 \mathrm{dist}(v,v) &= \frac{\partial^2}{\partial \varepsilon^2} \, \mathrm{length}(\sigma_\varepsilon)|_{\varepsilon=0} \\ &= \int_0^\rho (|X'|^2 - \langle X, R(X,\sigma')\sigma' \rangle)\, dt . \end{aligned}$$

Taking Jacobi's differential equation (17) and Lemma 2 into account, we get

$$\begin{aligned} \nabla^2 \mathrm{dist}(v,v) &= \int_0^\rho (|X'|^2 + \langle X, X'' \rangle)\, dt \\ &= \langle X, X' \rangle |_0^\rho \\ &\geq \frac{s_\kappa'(\rho)}{s_\kappa(\rho)} \sum_{i=1}^{2} |v_i^{nor}|^2 - \frac{2}{s_\kappa(\rho)} \cdot |v_1^{nor}| \cdot |v_2^{nor}| . \end{aligned}$$

This inequality together with (23) and (21) yields in case $\rho > 0$

(24) $$\nabla^2 Q_\kappa(v,v) \geq s_\kappa'(\rho) \langle e_1(y) \oplus e_2(y), v_1 \oplus v_2 \rangle^2 +$$

$$(24) \qquad + s_\kappa'(\rho) \sum_{i=1}^{2} |v_i^{nor}|^2 - 2|v_1^{nor}|\cdot|v_2^{nor}| .$$

Using

$$\left(\sum_i \langle e_i(y), v_i\rangle\right)^2 \leq \left(\sum_i |v_i^{tan}|\right)^2 \leq 2 \sum_i |v_i^{tan}|^2 ,$$

we get

$$\begin{aligned}
\nabla^2 Q_\kappa(v,v) &\geq s_\kappa'(\rho)\left(\sum_i \langle e_i(y), v_i\rangle\right)^2 \\
&\quad - (1-s_\kappa'(\rho)) \sum_i |v_i^{nor}|^2 \\
&= s_\kappa'(\rho)\left(\sum_i \langle e_i(y), v_i\rangle\right)^2 \\
&\quad - (1-s_\kappa'(\rho)) \sum_i |v_i|^2 + (1-s_\kappa'(\rho)) \sum_i |v_i^{tan}|^2 \\
&\geq \left(s_\kappa'(\rho) + \tfrac{1}{2}(1-s_\kappa'(\rho))\right)\left(\sum_i \langle e_i(y), v_i\rangle\right)^2 \\
&\quad - (1-s_\kappa'(\rho)) \sum_i |v_i|^2 \\
&\underset{(22)}{=} \tfrac{1}{2}(1+s_\kappa'(\rho)) \cdot s_\kappa(\rho)^{-2} \langle \nabla Q_\kappa(y), v\rangle^2 \\
&\quad - (1-s_\kappa'(\rho)) \sum_i |v_i|^2 \\
&= (2Q_\kappa(y))^{-1} \langle \nabla Q_\kappa(y), v\rangle^2 - \kappa Q_\kappa(y) \sum_i |v_i|^2 .
\end{aligned}$$

Observing (23) for $\rho = 0$, we have proved the first inequality. In order to prove the second one, we conclude from (24) in case $v_1 = 0$, $v_2 = u$

$$\begin{aligned}
\nabla^2 Q_\kappa(v,v) &\geq s_\kappa'(\rho) \langle e_2(y), u\rangle^2 + s_\kappa'(\rho) |u^{nor}|^2 \\
&= s_\kappa'(\rho) |u^{tan}|^2 + s_\kappa'(\rho) |u^{nor}|^2 \\
&= s_\kappa'(\rho) |u|^2 \\
&= (1 - \kappa Q_\kappa(y)) |u|^2
\end{aligned}$$

for $\rho > 0$ and due to (23), this remains true also for $\rho = 0$. □

Let $f_1, f_2 : \overset{\circ}{M} \longrightarrow N$ be harmonic mappings and assume that there is an open ball $B_r(a) \subset N$ which contains the traces $f_1(\mathring{M}), f_2(\mathring{M})$ and fulfills

$$r < \pi/(2\sqrt{\kappa}).$$

We define maps

$$\psi, \psi_1, \psi_2 : \mathring{M} \longrightarrow \mathbb{R}$$

by

$$(25) \quad \begin{cases} \psi(x) := q_\kappa \circ \mathrm{dist}(f_1(x), f_2(x)) \\ \psi_i(x) := q_\kappa \circ \mathrm{dist}(a, f_i(x)) \qquad i = 1,2. \end{cases}$$

ψ is C^2-differentiable if $B_r(a)$ satisfies the cut locus condition, and ψ_1, ψ_2 are C^2-differentiable if any point of $B_r(a)$ can be joined by exactly one minimizing geodesic arc. We denote by

$$(26) \quad K := \{ x \in \mathring{M} : f_1(x) = f_2(x) \}$$

the coincidence set of f_1 and f_2.

<u>Lemma 4</u>. <u>Suppose that</u> $\psi, \psi_1, \psi_2 : \mathring{M} \longrightarrow \mathbb{R}$ <u>are of class</u> C^2, <u>then</u>

$$(27) \quad \Delta\psi \geq \begin{cases} \sum_{i=1}^{2} |df_i|^2 & \underline{\text{on}} \ K \\ \frac{|\nabla\psi|^2}{2\psi} - \kappa \cdot \psi \cdot \sum_{i=1}^{2} |df_i|^2 & \underline{\text{on}} \ \mathring{M} - K, \end{cases}$$

$$(28) \quad \nabla\psi = 0 \ \underline{\text{on}} \ K,$$

<u>and</u>

$$(29) \quad \Delta\psi_i \geq (1 - \kappa\,\psi_i)|df_i|^2, \quad i = 1,2.$$

<u>Proof</u>. Introducing the notations

$$F(x) := (f_1(x), f_2(x)), \quad F_i(x) := (a, f_i(x)) \quad i = 1,2,$$

we can write

$$\psi = Q_\kappa \circ F \quad , \quad \psi_i = Q_\kappa \circ F_i \qquad i=1,2$$

(Q_κ is defined as in Lemma 3) and due to (19), the mappings

$$F, F_1, F_2 : \overset{\circ}{M} \longrightarrow N \times N$$

are also harmonic.
We fix a point $x \in \overset{\circ}{M}$ and choose a co-ordinate map around x with

$$g^{\alpha\beta}(x) = \delta_{\alpha\beta} \quad .$$

From the identities (11),(1o) and the harmonicity of F, F_1, F_2 we obtain the formulas at x suppressing the argument x in the formulas

$$\text{(3o)} \qquad |\nabla\psi|^2 = |\nabla(Q_\kappa \circ F)|^2 = \sum_\alpha \langle(\nabla Q_\kappa)\circ F, \partial_\alpha F\rangle^2 \quad ,$$

$$|\nabla\psi_i|^2 = |\nabla(Q_\kappa \circ F_i)|^2 = \sum_\alpha \langle(\nabla Q_\kappa)\circ F_i, \partial_\alpha F_i\rangle^2 \qquad i=1,2,$$

$$\Delta\psi = \Delta(Q_\kappa \circ F) = \sum_\alpha \nabla^2 Q_\kappa(\partial_\alpha F, \partial_\alpha F) \quad ,$$

$$\Delta\psi_i = \Delta(Q_\kappa \circ F_i) = \sum_\alpha \nabla^2 Q_\kappa(\partial_\alpha F_i, \partial_\alpha F_i) \qquad i=1,2.$$

The application of Lemma 3 yields

$$\Delta\psi \geqq \sum_\alpha |\partial_\alpha F|^2 = |dF|^2$$

if $x \in K$ ($\Longleftrightarrow \psi(x) = Q_\kappa \circ F(x) = 0$), and in case $x \in \overset{\circ}{M} - K$:

$$\Delta\psi \geqq \sum_\alpha \left(\frac{\langle(\nabla Q_\kappa)\circ F, \partial_\alpha F\rangle^2}{2\, Q_\kappa \circ F} - \kappa\cdot(Q_\kappa \circ F)|\partial_\alpha F|^2 \right)$$

$$= \frac{|\nabla\psi|^2}{2\psi} - \kappa\cdot\psi\cdot|dF|^2 \quad .$$

Similarly we derive from the second inequality in Lemma 3

$$\Delta\psi_i \geqq (1-\kappa\psi_i)|dF_i|^2 \qquad i=1,2 \quad ,$$

and observing

$$|dF|^2 = |df_1|^2 + |df_2|^2 \ , \ |dF_i|^2 = |df_i|^2 \quad i=1,2,$$

we get the inequalities (27) and (29). The third assertion (28) follows immediately from (3o) and (22). □

Proof of Theorem A. Let the assumptions of Theorem A be satisfied. We use the notations (25) and (26):

$$\psi = q_\kappa \circ \mathrm{dist}(f_1,f_2) \quad , \quad \psi_i = q_\kappa \circ \mathrm{dist}(a,f_i) \quad i=1,2,$$

$$K = \{ x \in M : f_1(x) = f_2(x) \} = \psi^{-1}\{0\} \ .$$

We define

$$\omega : [0,\kappa^{-1}[\longrightarrow \mathbb{R} \ , \quad t \longmapsto -\lg(1-\kappa t) \ ,$$

and construct the differential operator $\mathcal{L} = \operatorname{div} e^{-\Phi}\nabla$ on M by (14) using the functions ψ_1, ψ_2, ω (k=2) : We have

$$\omega'(t) = \frac{\kappa}{1-\kappa t} \quad , \quad \omega''(t) = \frac{\kappa^2}{(1-\kappa t)^2} = \omega'^2(t) \ ,$$

$$\begin{aligned} \Phi &= (\omega\circ\psi_1) + (\omega\circ\psi_2) = -\lg(1-\kappa\psi_1)(1-\kappa\psi_2) \\ &= -\lg(\cos(\sqrt{\kappa}\,\mathrm{dist}(a,f_1))\cdot\cos(\sqrt{\kappa}\,\mathrm{dist}(a,f_2))) \ , \end{aligned}$$

and

$$\theta = \theta(f_1,f_2) = e^\Phi\cdot\psi \ .$$

Because of (28) and $\omega'' = \omega'^2$, Lemma 1 is applicable and gives

$$\begin{aligned} \mathcal{L}(\theta) &= \mathcal{L}(e^\Phi\cdot\psi) \\ &\geq \begin{cases} \Delta\psi & \text{on } K \\ \Delta\psi - \dfrac{|\nabla\psi|^2}{2\psi} + \psi \sum\limits_{i=1}^{2} \dfrac{\kappa}{1-\kappa\psi_i}\Delta\psi_i & \text{on } \mathring{M} - K \ . \end{cases} \end{aligned}$$

Application of the estimates (27) and (29) in Lemma 4 yield

$$\mathcal{L}(\theta) \geq 0 \quad \text{on} \quad \mathring{M} \ .$$

Therefore, the function $\theta \in C^0(M,\mathbb{R}) \cap C^2(\mathring{M},\mathbb{R})$ is a subsolution of the elliptic second order operator $\mathcal{L}$, thus by Hopf's maximum principle the assertion

$$\sup_M \theta \leq \sup_{\partial M} \theta$$

follows. □

Proof of Theorem B. Let the assumptions of Theorem B be fulfilled. The function

$$\psi := |X|^2 : M \longrightarrow \mathbb{R}$$

is C^2 on $\overset{\circ}{M}$. After choice of a co-ordinate chart for $\overset{\circ}{M}$ we get

$$(31) \qquad \begin{aligned} \partial_\beta \psi &= \partial_\beta |X|^2 = 2 \langle D_\beta X, X \rangle , \\ \partial_\alpha \partial_\beta \psi &= 2 \langle D_\alpha D_\beta X, X \rangle + 2 \langle D_\alpha X, D_\beta X \rangle , \end{aligned}$$

hence

$$\begin{aligned} \Delta \psi &= \sum_{\alpha,\beta} g^{\alpha\beta} (\partial_\alpha \partial_\beta \psi - \sum_\gamma \Gamma_{\alpha\beta}{}^\gamma \partial_\gamma \psi) \\ &= 2 \sum_{\alpha,\beta} g^{\alpha\beta} (\langle D_\alpha D_\beta X, X \rangle + \langle D_\alpha X, D_\beta X \rangle \\ &\qquad - \sum_\gamma \Gamma_{\alpha\beta}{}^\gamma \langle D_\gamma X, X \rangle). \end{aligned}$$

Inserting Jacobi's differential equation (7), we get

$$(32) \qquad \begin{aligned} \Delta \psi &= 2 \sum_{\alpha,\beta} g^{\alpha\beta} \langle D_\alpha X, D_\beta X \rangle - 2 \langle R_f(X), X \rangle \\ &\underset{(2)}{=} 2 |DX|^2 - 2 \langle R_f(X), X \rangle . \end{aligned}$$

Let us fix a point $x \in \overset{\circ}{M}$ and choose a chart around x such that

$$g^{\alpha\beta}(x) = \delta_{\alpha\beta} .$$

Since κ is an upper bound for the sectional curvature of N , we have at the point x

$$\langle R_f(X), X \rangle \underset{(5)}{=} \sum_\alpha \langle R(X, \partial_\alpha f) \partial_\alpha f, X \rangle$$

$$\leq \sum_\alpha \kappa\cdot(|X|^2|\partial_\alpha f|^2 - \langle X,\partial_\alpha f\rangle^2)$$

$$\leq \kappa\cdot|X|^2 \sum_\alpha |\partial_\alpha f|^2$$

$$= \kappa\,\psi\,|df|^2 ,$$

furthermore, from (31) we get at x

$$|\nabla\psi|^2 = \sum_\alpha (\partial_\alpha\psi)^2 = 4\sum_\alpha \langle D_\alpha X,X\rangle^2$$

$$\leq 4\sum_\alpha |D_\alpha X|^2|X|^2 \underset{(2)}{=} 4\,\psi\,|DX|^2 .$$

From (32) and the last two inequalities we conclude:

(33) $\nabla\psi = 0 \quad , \quad \Delta\psi \geq 0 \quad \text{on} \quad \psi^{-1}\{0\}$,

and

(34) $\Delta\psi \geq \dfrac{|\nabla\psi|^2}{2\psi} - 2\kappa\psi\,|df|^2 \quad \text{on} \quad \mathring{M} - \psi^{-1}\{0\}$.

Now we define

$$\psi_1 := q_\kappa\circ\mathrm{dist}(a,f) : M \longrightarrow \mathbb{R} ,$$

$$\omega : [0,\kappa^{-1}[\longrightarrow \mathbb{R} \quad , \quad t \longmapsto -\lg(1-\kappa t) \quad ,$$

and construct the differential operator $\mathcal{L} = \mathrm{div}\, e^{-\Phi}\nabla$ on M by (14) using ψ_1,ω (k=1). Then we have

$$\Phi = 2\,\omega\circ\psi_1 = -2\lg(1-\kappa\psi_1) = -\lg\cos^2\sqrt{\kappa}\,\mathrm{dist}(a,f)$$

and

$$\theta = \theta_f(X) = e^{\Phi}\cdot|X|^2 = e^{\Phi}\cdot\psi .$$

The application of Lemma 1 to $\mathcal{L}$ and $\psi = |X|^2$ yields

$$\mathcal{L}(\theta) = \mathcal{L}(e^{\Phi}\cdot\psi) = \Delta\psi \geq 0 \quad \text{on} \quad \psi^{-1}\{0\} ,$$

and, in virtue of (34) and (29)

$$\mathcal{L}(\theta) \geq \Delta\psi - \frac{|\nabla\psi|^2}{2\psi} + 2\psi\frac{\kappa}{1-\kappa\psi_1}\Delta\psi_1$$

$$\geq -2\kappa\psi|df|^2 + 2\kappa\psi|df|^2 = 0 \quad \text{on} \quad \mathring{M} - \psi^{-1}\{0\} .$$

Therefore, $\theta : M \longrightarrow \mathbb{R}$ is a subsolution of the elliptic operator $\mathcal{L}$ and by Hopf's maximum principle, the assertion follows. □

In order to treat the parabolic case, we set for $T > 0$

$$M_T := [0,T]\times M \quad , \quad \partial_T M := (\{0\}\times M) \cup ([0,T]\times\partial M)$$

and denote by M_T^o the interior of M_T.
Suppose that

$$f_1, f_2 : M_T^o \longrightarrow N$$

are solutions of the heat equation $(\Delta - \partial_t) f = 0$ such that

$$f_1(M_T^o) , f_2(M_T^o) \subset B_r(a) \quad , \quad r < \pi/(2\sqrt{\kappa})$$

and $B_r(a)$ satisfies the cut locus condition. Quite analogously to the case of harmonic mappings, we define

$$\psi := q_\kappa \circ \mathrm{dist}(f_1, f_2) : M_T^o \longrightarrow \mathbb{R} \quad ,$$

$$\psi_i := q_\kappa \circ \mathrm{dist}(a, f_i) : M_T^o \longrightarrow \mathbb{R} \qquad i = 1,2,$$

and

$$K := \{ (t,x) \in M_T^o : f_1(t,x) = f_2(t,x) \} \quad .$$

Lemma 5.

$$(\Delta - \partial_t)\psi \geq \begin{cases} \sum_{i=1}^{2} |df_i|^2 & \text{on } K \\ \dfrac{|\nabla\psi|^2}{2\psi} - \kappa\psi \sum_{i=1}^{2} |df_i|^2 & \text{on } M_T^o - K , \end{cases}$$

$$\nabla\psi = 0 \quad \text{on} \quad K,$$

and

$$(\Delta - \partial_t)\psi_i \geq (1 - \kappa\psi_i)|df_i|^2 \quad , \quad i = 1,2.$$

(the operators d, ∇, Δ are acting on the variables of M)

Proof. We proceed as in the proof of Lemma 4 defining

$$F, F_1, F_2 : M_T^o \longrightarrow N \times N$$

correspondingly, such that we can write

$$\psi = Q_\kappa \circ F \quad , \quad \psi_i = Q_\kappa \circ F_i \ .$$

In virtue of (19), the mappings F, F_1, F_2 are also solutions of the heat equation.
We fix a point $(t,x) \in M_T^o$ and choose a chart around x such that

$$g^{\alpha\beta}(x) = \delta_{\alpha\beta} \ .$$

From the identity (9) follows

$$\partial_t \psi = \partial_t (Q_\kappa \circ F) = \langle (\nabla Q_\kappa) \circ F, \partial_t F \rangle \ ,$$

and from (11) and (1o) we get at x

$$(\Delta - \partial_t)\psi = \sum_\alpha \nabla^2 Q_\kappa (\partial_\alpha F, \partial_\alpha F)$$

and similarly

$$(\Delta - \partial_t)\psi_i = \sum_\alpha \nabla^2 Q_\kappa (\partial_\alpha F_i, \partial_\alpha F_i) \qquad i = 1,2.$$

The right hand sides of these both equations can be estimated in the same way as in Lemma 4. □

Proof of Theorem C. Here we follow the lines of the proof of Theorem A. We take the same function

$$\theta = \theta(f_1, f_2) = e^\Phi \cdot \psi : M_T \longrightarrow \mathbb{R} \ ,$$

with M replaced by M_T, but apply now the parabolic operator

$$u \longmapsto \mathcal{P} u := \mathcal{L} u - e^{-\Phi} \partial_t u \ ,$$

where $\mathcal{L}$ is the same elliptic operator as in Theorem A acting on the M-variables of $u : M_T \longrightarrow \mathbb{R}$. From Lemma 1 we obtain

$$\mathfrak{P}(\theta) = \mathfrak{P}(e^{\Phi}\cdot\psi)$$

$$\geqq \begin{cases} (\Delta-\partial_t)\psi & \text{on } K=\psi^{-1}\{0\} \\ (\Delta-\partial_t)\psi - \frac{|\nabla\psi|^2}{2\psi} + \psi \sum_{i=1}^{2} \frac{\kappa}{1-\kappa\psi_i}(\Delta-\partial_t)\psi_i & \text{otherwise.} \end{cases}$$

Taking the estimates of Lemma 5 into account, we get

$$\mathfrak{P}(\theta) \geqq 0 \quad \text{on} \quad M_T^o .$$

Since $\mathfrak{P}$ is a parabolic operator on M_T satisfying the assumptions of the maximum principle, we obtain the assertion

$$\sup_{M_T} \theta \leqq \sup_{\partial_T M} \theta . \square$$

References

[1] Eells,J.,Lemaire,L.: A report on harmonic maps. Bull. London Math.Soc. 1o. 1-68(1978)

[2] ——————————: Deformation of metrics and associated harmonic maps. To appear in the V.K.Patodi Memorial Volume. Tata Institute Bombay

[3] Eells,J,.Sampson,J.H.: Harmonic mappings of Riemannian manifolds. Amer.J.Math. 86. 1o9-16o(1964)

[4] Gromoll,D.,Klingenberg,W.,Meyer,W.: Riemannsche Geometrie im Großen. Berlin, Heidelberg, New York: Springer 1968

[5] Hartman,P.: On homotopic harmonic maps. Canad.J.Math. 19. 673-687(1967)

[6] Hildebrandt,S.,Kaul,H.,Widman,K.O.: An existence theorem for harmonic mappings of Riemannian manifolds. Acta Math. 138. 1-16(1977)

[7] Jäger,W.,Kaul,H.: Uniqueness of harmonic mappings and of solutions of elliptic equations on Riemannian manifolds. To appear in Math.Ann.

[8] Karcher,H.: Riemannian center of mass and mollifier smoothing. Comm.Pure Appl.Math. 3o. 5o9-541(1977)

Willi Jäger
Institut für Angewandte Mathematik
Im Neuenheimer Feld 294
D-6900 Heidelberg, Germany

Helmut Kaul
Mathematisches Institut
Auf der Morgenstelle 1o
D-7400 Tübingen, Germany

(Received March 13, 1979)