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BLOW-UP OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN THREE SPACE DIMENSIONS

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Dedicated to Hans Lewy and Charles B. Morrey, Jr.

Let u(x,t) be a solution, $\square u \geq A|u|^p$ for $x \in \mathbb{R}_3$, $t \geq 0$ where $\square$ is the d'Alembertian, and A, p are constants with $A > 0$, $1 < p < 1 + \sqrt{2}$. It is shown that the support of u is contained in the cone $0 \leq t \leq t_0 - |x-x^0|$, if the "initial data" u(x,0), $u_t(x,0)$ have their support in the ball $|x-x^0| \leq t_0$. In particular "global solutions" of $u = A|u|^p$ with initial data of compact support vanish identically. On the other hand for $A > 0$, $p > 1 + \sqrt{2}$ global solutions of $\square u = A|u|^p$ exist, if the initial data are of compact support and $\|u\|$ is "sufficiently small" in a suitable norm. For $p = 2$ the time at which u becomes infinite is of order $\|u\|^{-2}$.

Let $\square$ denote the d'Alembertian

$$\square = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

acting on functions $u(x,t) = u(x_1,x_2,x_3,t)$. We are concerned here with global solutions of a nonlinear wave equation of the form

(1) $\quad \square u = \phi(u)$

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or of an inequality of the form

(2) $\square u \geq \phi(u)$

with a prescribed function ϕ and prescribed intitial data

(3) $u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad .$

A global solution of (1), respectively (2), shall mean a solution of class C^2 in the closed half-space $x \in \mathbb{R}_3$, $t \geq 0$, for which $f \in C^3$, $g \in C^2$ for $x \in \mathbb{R}_3$. "Blow-up" consists in non-existence of a global solution for given f, g, ϕ. In that case instead of global solutions there may still exist local solutions u defined for $x \in \mathbb{R}_3$ and sufficiently small t.

We associate with u the solution u^0 of the linear wave equation

(4) $\square u^0 = 0$

with the same initial data f, g as u. We introduce for any $(x^0,t_0) \in \mathbb{R}_4$ the forward and backward solid characteristic cones with vertex (x^0,t_0):

(5a) $\Gamma^+ (x^0,t_0) = \{(x,t): |x-x^0| \leq t-t_0,\ t \geq 0\}$

(5b) $\Gamma^- (x^0,t_0) = \{(x,t): |x-x^0| \leq t_0-t,\ t \geq 0\} \quad .$

Our first theorem shows that for certain ϕ, u^0 a solution of (2) either blows up or becomes identically zero for sufficiently large t:

Theorem I. *Let* A, p *denote constants with*

(6) $A > 0,\ 1 < p < 1 + \sqrt{2}$

and let u *be a global solution of*

(7) $\square u \geq A|u|^p \ .$

Let moreover for a certain point $(x^0,t_0) \in \mathbb{R}_4$

(8) $u^0(x,t) \geq 0$ *for* $(x,t) \in \Gamma^+(x^0,t_0)$.

Then u *has compact support and*

(9) $\operatorname{supp} u \subset \Gamma^-(x^0,t_0)$.

We list a number of immediate consequences of Theorem I in the form of corollaries. Since $\Gamma^-(x^0,t_0)$ is empty for $t_0 \leq 0$ we have:

Corollary Ia. *If* u *is a global solution of* (6), (7), *and if* $u^0(x,t) \geq 0$ *in some forward characteristic cone with vertex in the plane* $t = 0$, *then* u *vanishes identically.*

Since $u^0 \geq 0$ for $f = 0$, $g \geq 0$ (see formula (17b) below), we conclude

Corollary Ib. *A global solution* u *of* (6), (7) *vanishes identically if the initial data* f, g *satisfy*

$$f(x) = 0 \quad , \; g(x) \geq 0 \quad \textit{for all} \quad x \in \mathbb{R}_3 \; .$$

Let f and g have their support in a ball $|x-x^0| \leq \rho$. Then by the strong Huygens principle valid in 3 dimensions u^0 vanishes in the cone $\Gamma^+(x^0,\rho)$. Hence

Corollary Ic. *Let* f *and* g *have their support in a ball* $|x-x^0| \leq \rho$. *Then every global solution of* (6), (7) *with initial data* f, g *vanishes outside the bounded set* $\Gamma^-(x^0,\rho)$.

Corollary Id. *Let the initial data* f, g *have compact support, then a global solution* u *of* (6), (7) *can exist only if*

(10) $f(x) \geq 0$ *for all* $x \in \mathbb{R}_3$, $\int g(x)\,dx \leq 0$.

Indeed u has compact support by Corollary Ic. Let $v(x,t)$ be any solution of $\Box v = 0$ with initial data F, G. If here $v(x,t) \geq 0$ for $t \geq 0$ we have by Green's identity

$$0 \le \int_{t>0} Av|u|^p \,dxdt \le \int_{t>0} (v\Box u - u\Box v)\,dxdt$$
$$= -\int (Fg-Gf)\,dx \quad .$$

For $F = 1$, $G = 0$ we have $v = 1$, leading to the second inequality in (10). For $F = 0$ we have $v \ge 0$ for arbitrary non-negative G. This implies the first inequality.

Standard arguments based on energy inequalities[1] for the operator $\Box$ show that the solution of the initial value problem for an equation of the form (1) is unique, if the function ϕ is Lipschitz continuous. More precisely it is easy to show that a function $u(x,t)$ of class C^2 in a cone $\Gamma^-(x^0,t_0)$ with $t_0 > 0$, and satisfying there an inequality $|\Box u| \le M|u|$ with some fixed M, vanishes in that cone if its initial data f, g vanish for $|x-x^0| \le t_0$. Applying this theorem to the function $u(x,\rho-t)$ we conclude from Corrolary Ic. the following:

Corrolary II. *Let* u *be a global solution of the differential equation* (1), *where the function* $\phi(s)$ *satisfies*

$$\phi(s) \ge A|s|^p \quad \textit{for all } s, \textit{ with constants}$$

(11a) $\quad A > 0 \;,\; 1 < p < 1 + \sqrt{2}$

(11b) $\quad \phi(0) = 0 \;,\; \limsup_{s\to 0} \phi(s)/|s| < \infty$

Then u *vanishes identically if it has initial data of compact support*[2] .

Examples of global solutions: a) For $t_0 > 0$, $p > 1$ the function

(12a) $\quad u = a(t_0+t)^{-2/(p-1)} \quad$ with $a = \dfrac{A(p-1)^2}{2(p+1)}$

[1] See [1], p. 119.

[2] More generally this would apply to solutions of differential equations $\Box u = \phi$, in which ϕ depends on x, t, u and derivatives of u, as long as ϕ in its dependence on u satisfies conditions (11a,b), uniformly in the other variables. What matters in addition to (6), (7) is only that $(\Box u)/|u|$ is bounded for small $|u|$.

is a global solution of

(12b) $\Box u = A|u|^p$.

Here, f, g do not have compact support. Moreover

$$u_0 = (at_0)^{-2/(p-1)}\left(1 - \frac{2t}{(p-1)t_0}\right) < 0$$

for all x and sufficiently large t.

b) For $t_0 > 0$, $p > 3/4$ the function u defined by

(13a) $u = a[(t-t_0)^2 - |x-x^0|^2]^4$ with $80a^{1-p}t_0^{6-8p} = A$

for $(x,t) \in \Gamma^-(x^0,t_0)$, and by $u = 0$ elsewhere, is a global solution of

(13b) $\Box u = 80a^{1/4}|u|^{3/4} \geq A|u|^p$.

with initial data of compact support. Since $\phi(s)/s$ for $s \to 0$ is not bounded when $\phi(s) = 80a^{1/4}|s|^{3/4}$, the solution u does not have to vanish identically.[3]

Theorem II. *Let the function* $\phi(s)$ *belong to* $C^2(R)$, *satisfy* $\phi(0) = \phi'(0) = \phi''(0) = 0$, *and be Hölder continuous with exponent* $> \sqrt{2} - 1$ *for* $|s| < 1$. *Then global solutions* u *of* (1) *exist for any sufficiently regular initial data* f, g *with support in a ball of radius* ρ, *provided* $|D^\alpha f|$ *for* $|\alpha| \leq 2$ *and* $|D^\beta g|$ *for* $|\beta| \leq 1$ *do not exceed a certain positive number* δ, *that only depends on* ρ *and the choice of* $\phi(s)$.

Since the function $\phi(s) = A|s|^p$ for $A > 0$, $p > 1 + \sqrt{2}$ satisfies the assumptions of this theorem, global solutions of $\Box u = A|u|^p$ for $p > 1 + \sqrt{2}$ exist for any initial data f, g with compact support that are sufficiently small in a suitable norm, regardless of the sign of f and g. Thus

[3] More general global solution v of $\Box v \geq A^*|v|^p$ with support in $\Gamma^-(x^0,t_0)$ are obtained by forming $v = (1+\varepsilon w)u$ where $w(x,t) \in C^\infty$ and ε is sufficiently small.

Corollary III. *In Corollary II, and hence also in* Theorem I, *the constant* $1+\sqrt{2}$ *cannot be replaced by any larger one.*

Corollary II tells us that for certain ϕ a non-trivial solution u of (1) with initial data of compact support blows up after a finite time T, without however giving an estimate for T. For the special case $\phi(s) = s^2$ Theorem III below shows that T is of the precise order ε^{-2}, where ε is a measure for the magnitude of the initial data:

Theorem III: *Let the initial data be of the form*

$$(14) \quad f(x) = \varepsilon F(x) \ , \ g(x) = \varepsilon G(x)$$

for given $F \in C_0^3(\mathbb{R}_3)$, $G \in C_0^2(\mathbb{R}_3)$. *Let* $T = T(\varepsilon)$ *be the largest* T *such that a solution* $u(x,t)$ *of* $\Box u = u^2$ *with initial data* f, g *exists for* $x \in \mathbb{R}_3$, $0 \le t < T$. *There exist three positive constants* A, B, ε_0 *depending on* F, G *but not on* ε, *such that*

$$(15) \quad A\varepsilon^{-2} < T < B\varepsilon^{-2} \quad \text{for} \quad |\varepsilon| < \varepsilon_0 \ .$$

The literature on global existence, decay and blow-up of solutions of nonlinear hyperbolic equations is extensive[4]. The natural tool for deriving local existence theorems are energy inequalities leading to a priori estimates for L_2-norms. These by themselves prove inadequate to discuss behavior of solutions for very long times, except when the solutions can be shown to decay sufficiently rapidly in the maximum norm[5]. Additional information on global behavior can be obtained by establishing convexity properties or other differential inequalities for various integral expressions formed from the solution, as in the methods based on "logarithmic convexity"[6]. In the case where the

[4] See [2] - [31].

[5] See [17], [18], [27], [29], [6], [10].

[6] See [14], [11], [16], [19].

number of space dimensions does not exceed three (but not in higher dimensions), we have the very helpful fact that the inverse of the d'Alembert operator is positive: if $\square u \geq 0$ and u has vanishing initial data, then $u \geq 0$. This fact formed already the basis of J.B. Keller's classic investigation [9] of nonlinear wave equations, and also is the main tool in the present paper.

Many of the earlier results referred to overlap with the ones given here, but require the initial data to satisfy various inequalities[7]. Recently T. Kato [32], gave a very simple proof of an analogue of Corollary II in m space dimensions, with the bound $1 + \sqrt{2}$ in (11a) replaced by $(m+1)/(m-1)$ under the additional assumption

$$\int f(x)\,dx \geq 0, \int g(x)\,dx \geq 0, \text{ not both zero}$$

on the initial data. [Actually Kato's theorem refers to hyperbolic equations more general than (1)].
On the other hand some global existence theorems analogous to Theorem II have been proved in m dimensions. Independently S. Klainerman [10] and W. Strauss [33] have shown that for $m \geq 5$ global solutions of $\square u = u^2$ exist, whenever the initial data are of compact support and "sufficiently small" in some suitable sense.

Proof of Theorem I.

We make use of the classic explicit expression in three dimensions for the global solution u(x,t) of the linear problem

(16a) $\square u = w(x,t)$ for $t \geq 0,\ x \in \mathbb{R}_3$

(16b) $u = f(x),\ u_t = g(x)$ for $t = 0,\ x \in \mathbb{R}_3$

for given $f \in C^3(\mathbb{R}_3)$, $g \in C^2(\mathbb{R}_3)$ and $w \in C^2$ for $x \in \mathbb{R}_3$, $t \geq 0$. One finds that

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[7] See e.g. Glassey's paper [3].

(17a) $u = u^0 + Lw$

where

(17b) $u^0(x,t) = \frac{t}{4\pi}\int_{|\xi|=1} g(x+t\xi)d\omega_\xi + \frac{\partial}{\partial t}\frac{t}{4\pi}\int_{|\xi|=1} f(x+t\xi)\, d\omega_\xi$

is the solution of $\Box u^0 = 0$ with initial values f, g, and

(17c) $Lw(x,t) = \frac{1}{4\pi}\int_0^t (t-s)\, ds \int_{|\eta|=1} w(x+(t-s)\eta,s)\, d\omega_\eta$

is the solution of (16a) with zero initial data. We notice two often used consequences of these formulae, namely that $f = 0$, $g \geq 0$ implies $u^0 \geq 0$, and that $w \geq 0$ implies $Lw \geq 0$.

Assume now that $u(x,t)$ is a global solution of (6), (7) and that the corresponding u^0 satisfies (8) for some (x^0, t_0).

We associate with a function $w(x,t)$ its averages $\tilde{w}(r,t)$ on spheres of radius r about a point x^0:

(18) $\tilde{w}(r,t) = \frac{1}{4\pi}\int_{|\xi|=1} w(x^0+r\xi,t)\, d\omega_\xi$.

Expressing iterated spherical means by simple ones, we have from (17c) that for $r \geq 0$, $t \geq 0$

$$\widetilde{Lw}(r,t) = (4\pi)^{-2}\int_0^t (t-s)ds \int_{|\xi|=1} d\omega_\xi \int_{|\eta|=1} w(x^0+r\xi+(t-s)\eta,s)d\omega_\eta$$

$$= \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda\, d\lambda}{8\pi r}\int_{|\zeta|=1} w(x^0+\lambda\zeta,s)\, d\omega_\zeta$$

(18a) $$= \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r}\tilde{w}(\lambda,s)\, d\lambda \quad .$$

We write the identity in the form

(19) $\widetilde{Lw} = P\tilde{w}$

where the operator P acting on functions $\sigma(r,t)$ with domain $r \geq 0$, $t \geq 0$ is defined by

(20) $P\sigma(r,t) = \iint_{R_{r,t}} \frac{\lambda}{2r}\sigma(\lambda,s)\,d\lambda ds$,

and $R_{r,t}$ denotes the set

(21) $R_{r,t} = \{(\lambda,s): t-r<s+\lambda<t+r,\ s-\lambda<t-r,\ 0<s\}$

in the λs-plane. (See Fig. 1). Observe that $\sigma \geq 0$ implies $P\sigma \geq 0$ since $\lambda \geq 0$ in $R_{r,t}$.

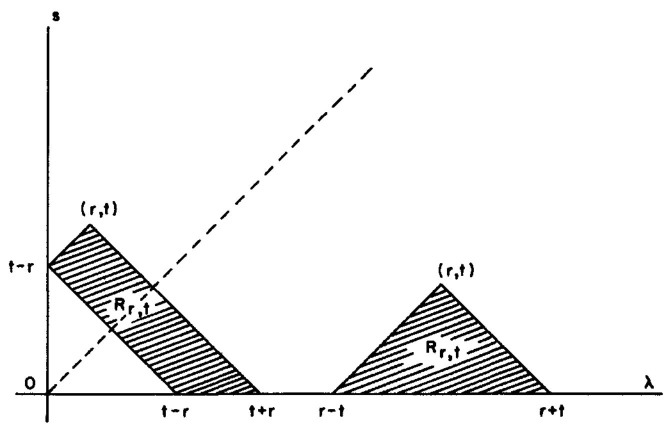

Fig. 1

Assume that u(x,t) is a global solution of (6), (7) and that the corresponding u^0 satisfies (8) for some (x^0,t_0). It follows from (17a) and (7) that

(22) $u \geq LA|u|^p$

for $(x,t) \in \Gamma^+(x^0,t_0)$. Suppose now that (9) does not hold. There exists then a point (x^1,t_1) for which

$$|x^1-x^0| \geq t_0-t_1\ ,\ t_1 \geq 0\ ,\ u(x^1,t_1) \neq 0\ .$$

Set $t_2 = t_1 + |x^1-x^0|$. Then

(23) $t_0 \leq t_2,\ 0 \leq t_1 \leq t_2,\ (x^0,t_2) \in \Gamma^+(x^0,t_0)$,

and thus by (22)

(24) $u(x^0,t_2) > 0$

since the point (x^1,t_1) lies in the domain of integration of L as defined by (17c) for the arguments

$$x = x^0,\ t = t_2,\ s = t_1,\ (t-t_1)\eta = x^1-x^0 \quad .$$

Since $\Gamma^+(x^0,t_2) \subset \Gamma^+(x^0,t_0)$, inequality (22) holds for $|x-x^0| < t-t_2$. Introducing the average $\tilde{u}$ of u we find from (22) that

(25) $\tilde{u} \geq AP\widetilde{|u|^p}$ for $0 \leq r \leq t - t_2$.

Because of the convexity of $|u|^p$ as a function of u for $p \geq 1$ we have generally

(26) $\widetilde{|u|^p} \geq |\tilde{u}|^p$.

Hence for (r,t) with $0 \leq r \leq t - t_2$

(27) $$\tilde{u}(r,t) \geq AP|\tilde{u}|^p = A \iint_{R_{r,t}} \frac{\lambda}{2r} |\tilde{u}(\lambda,s)|^p \, d\lambda ds \quad .$$

Moreover by (24)

(28) $\tilde{u}(0,t_2) > 0$.

We shall see that inequalities (27), (28) imply blow-up of u. This is proved by deriving larger and larger lower bounds for $\tilde{u}$.

By (28) we can find a positive δ so small that

(29) $\tilde{u}(\frac{1}{2}\delta,\ t_2 + \frac{1}{2}\delta) > 0$.

We define the regions

(30a) $T = \{(\lambda,s):\ t_2 + \delta \leq s + \lambda \leq t_2 + 2\delta,\ s-\lambda \leq t_2\}$

(30b) $S = \{(\lambda,s):\ t_2 + 2\delta \leq s + \lambda,\ t_2 \leq s-\lambda \leq t_2 + \delta\}$.

Then (see Fig. 2)

(31a) $T \subset R_{r,t}$ for $(r,t) \in S$; $(\frac{1}{2}\delta,\ t_2 + \frac{1}{2}\delta) \in T$;

(31b) $0 \leq r \leq t - t_2$ for $(r,t) \in S$

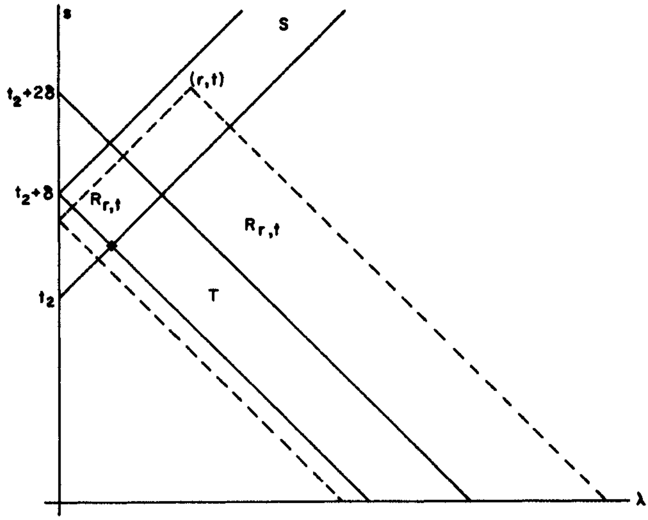

Fig. 2

It follows from (28) that for $(r,t) \in S$

(32) $$\tilde{u}(r,t) \geq A \iint_T \frac{\lambda}{2r} |\tilde{u}(\lambda,s)|^p \, d\lambda ds = \frac{c}{r}$$

where c by (29), (31a), is a positive constant.

Let Σ denote the set

(33a) $$\Sigma = \{(r,t): 0 \leq r \leq t - t_2 - 2\delta\} \quad .$$

We introduce for any $(r,t) \in \Sigma$ the sets (see Fig. 3)

(33b) $$S_{r,t} = \{(\lambda,s): t-r < \lambda+s < t+r \; ; \; t_2 < s-\lambda < t_2+\delta\}$$

(33c) $$T_{r,t} = \{(\lambda,s): t-r < \lambda+s < t+r \; ; \; t_2+2\delta < s-\lambda < t-r\}$$

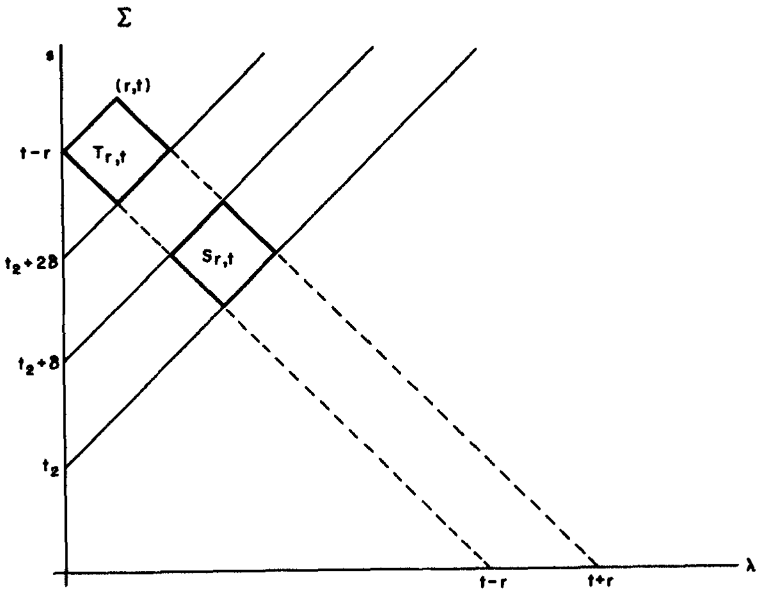

Fig. 3

Then for $(r,t) \in \Sigma$

(33d) $S_{r,t} \subset R_{r,t}$, $T_{r,t} \subset R_{r,t}$, $S_{r,t} \subset S$, $T_{r,t} \subset \Sigma$.

It follows from (27), (32) that for $(r,t) \in \Sigma$

$$\tilde{u}(r,t) \geq \frac{Ac^p}{2r} \iint_{S_{r,t}} \lambda^{1-p}\, d\lambda ds \quad .$$

Introducing new variables of integration

(34) $\alpha = s + \lambda$, $\beta = s - \lambda$

we find, since $p > 1$,

$$\tilde{u}(r,t) \geq \frac{1}{2r}\, Ac^p 2^{p-2} \int_{t_2}^{t_2+\delta} d\beta \int_{t-r}^{t+r} (\alpha-\beta)^{1-p}\, d\alpha$$

$$\geq \frac{1}{2r}\ \delta\ Ac^p 2^{p-2} \int_{t-r}^{t+r} (\alpha-t_2)^{1-p}\, d\alpha$$

(35) $$\geq \delta\ Ac^p 2^{p-2} (t+r-t_2)^{1-p}$$

for $(r,t) \in \Sigma$. Since for $(r,t) \in \Sigma$

$$\delta \leq t-r-t_2 \leq t+r-t_2 \leq t+r$$

we have

$$(t+r-t_2)^{1-p} \geq \begin{cases} \delta^{2-p}(t+r)^{-1} & \text{for } 1 < p \leq 2 \\ (t+r)^{1-p} & \text{for } 2 < p < 1+\sqrt{2} \end{cases} .$$

It follows from (35) that

(36) $$\tilde{u}(r,t) \geq C_0(r+t)^{-q_0} \quad \text{for } (r,t) \in \Sigma ,$$

where C_0, q_0 are positive constants not depending on (r,t), and in particular

(37) $$1 \leq q_0 = \begin{cases} 1 & \text{for } 1 < p \leq 2 \\ p-1 & \text{for } 2 < p < 1+\sqrt{2} \end{cases} .$$

Assume more generally that we have established an inequality of the form

(38a) $$u(r,t) \geq C(t+r)^{-q}(t-r-\tau)^a(t-r)^{-b} \quad \text{for } (r,t) \in \Sigma$$

where the constants C, q, a, b satisfy

(38b) $$C > 0,\ q \geq 1,\ a \geq 0,\ b \geq 0$$

and we have set

$$t_2 + 2\delta = \tau .$$

Then by (27), (33d), (34) for $(r,t) \in \Sigma$

$$\tilde{u}(r,t) \geq A \iint_{T_{r,t}} \frac{\lambda}{2r} \left|u(\lambda,s)\right|^p d\lambda ds$$

$$\geq \frac{A}{8r} \int_{\tau}^{t-r} d\beta \int_{t-r}^{t+r} (\alpha-\beta) C^p \alpha^{-qp} (\beta-\tau)^{pa} \beta^{-pb}\, d\alpha$$

$$\geq \frac{A}{8r} C^p (t-r)^{-pb} \int_{\tau}^{t-r} d\beta \int_{t-r}^{t+r} (\alpha-\beta)\alpha^{-qp}(\beta-\tau)^{pa}\, d\alpha$$

$$= \frac{AC^p (t-r-\tau)^{pa+1}}{8r(pa+1)(pa+2)(t-r)^{pb}} \int_{t-r}^{t+r} \frac{(pa+2)(\alpha-t+r)+(t-r-\tau)}{\alpha^{qp}}\, d\alpha$$

$$\geq \frac{AC^p (t-r-\tau)^{pa+2}}{4(pa+1)(pa+2)(t-r)^{pb}} I(r,t)$$

where

$$I(r,t) = \frac{1}{2r} \int_{t-r}^{t+r} \alpha^{-pq}\, d\alpha \quad .$$

Observe that $qp \geq p > 1$. If here $t-r > \frac{1}{2}(t+r)$, then by the mean value theorem

$$I(r,t) \geq (t+r)^{-pq} \geq 2^{1-pq}(t+r)^{-1}(t-r)^{1-qp} \quad .$$

While for $0 < t-r < \frac{1}{2}(t+r)$

$$I(r,t) = \frac{1}{2(qp-1)r} \left((t-r)^{1-qp} - (t+r)^{1-qp}\right)$$

$$\geq \frac{1}{2(qp-1)r} (1-2^{1-qp})(t-r)^{1-qp}$$

$$\geq \frac{1-2^{1-qp}}{2(qp-1)} (t+r)^{-1}(t-r)^{1-qp} \quad .$$

It follows that

(39a) $\tilde{u}(r,t) \geq C^*(t+r)^{-1}(t-r-\tau)^{a^*}(t-r)^{-b^*}$ for $(r,t) \in \Sigma$

where

(39b) $a^* = pa + 2,\ b^* = pb + qp - 1,\ C^* = \dfrac{C^p}{(pa+2)^2} D_q$

(39c) $D_q = \dfrac{A}{4} \operatorname{Min} \left(\dfrac{1-2^{1-pq}}{2(pq-1)},\ 2^{1-pq}\right) \quad .$

Using the values $a = b = 0$, $q = q_0$, $C = C_0$ corresponding to (36) we arrive at the inequality (39a) with

(40a) $a^* = 2,\ b^* = q_0 p-1\ ,\ C^* = \frac{1}{4} C_0^p D_{q_0} \quad .$

Define generally sequences a_k, b_k, C_k for $k = 1,2,3,\ldots$ by

(40b) $a_1 = a^*$, $b_1 = b^*$, $C_1 = C^*$ as given in (40a)

(40c) $a_{k+1} = pa_k+2$, $b_{k+1} = pb_k+p-1$, $C_{k+1} = (pa_k+2)^{-2}D_1\ C_k^p$.

Then (38) will hold with $C = C_k$, $a = a_k$, $b = b_k$, $q = 1$ for $k \geq 1$. Solving (40b, c) yields

$$a_k = \frac{2(p^k-1)}{p-1}\ ,\ b_k = q_0 p^k - 1$$

and thus

$$C_{k+1} = (a_{k+1})^{-2} D_1 C_k^p \geq \frac{1}{4(k+1)^2 p^{2k}} D_1 C_k^p$$

$$C_k \geq \exp\left[p^k\left(\frac{1}{p}\log C^* - \sum_{j=1}^{k-1} \frac{2\log(j+1)+2j\log p-\log(D_1/4)}{p^{j+1}}\right)\right] .$$

It follows that for k sufficiently large

$$C_k \geq \exp(Ep^k)\ ,$$

where

$$\text{(40d)}\quad E = \frac{1}{p}\log C^* - \sum_{j=1}^{\infty} \frac{2\log(j+1)+2j\log p-\log(D_1/4)}{p^{j+1}} .$$

Then by (38a) with C, q, a, b, replaced by C_k, 1, a_k, b_k respectively we have

$$\text{(41)}\quad \tilde{u}(r,t) \geq \frac{t-r}{(t+r)(t-r-\tau)^{2/(p-1)}} \exp\left[p^k J(r,t)\right] \quad \text{for } (r,t) \in \Sigma .$$

Here

$$\text{(42)}\quad J(r,t) = E + \frac{2}{p-1}\log(t-r-\tau) - q_0 \log(t-r) .$$

It is crucial[8] now that for the q_0 defined by (37)

$$\frac{2}{p-1} > q_0 \ .$$

For then $J(r,t) \to \infty$ for $t-r \to \infty$. It follows from (41) for

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[8] This is where the upper bound $1 + \sqrt{2}$ for p comes into play!

$k \to \infty$ that $u(r,t) = \infty$ for all r,t with $t-r$ sufficiently large. But then u cannot be a global solution.

Proof of Theorem II.

Our assumptions on the function $\phi(u)$ are that

(43a) $\phi \in C^2(\mathbb{R})$, $\phi(0) = \phi'(0) = \phi''(0) = 0$

and that there exists constants B, q with

(43b) $B > 0, \sqrt{2} - 1 < q \leq 1$

such that

(43c) $|\phi''(u) - \phi''(v)| \leq B|u-v|^q$ for $|u| \leq 1, |v| \leq 1$.

Assumptions (43a, b, c) imply that

$$|\phi''(u)| \leq Ap(p-1)|u|^{p-2}; \quad |\phi'(u)| \leq Ap|u|^{p-1} ;$$

(43d) $|\phi(u)| \leq A|u|^p$ for $|u| \leq 1$,

where

$$A = \frac{B}{(q+1)(q+2)} , \quad p = q + 2, \quad 1 + \sqrt{2} < p \leq 3 .$$

Conversely for the function

(44a) $\phi(u) = A|u|^p$ with $A > 0$, $p > 1 + \sqrt{2}$

(43a, b, c) are satisfied with

(44b) $B = Ap(p-1)$, $q = p-2$ when $1 + \sqrt{2} < p < 3$
$B = Ap(p-1)(p-2)$, $q = 1$ when $p \geq 3$.

We also assume that $f \in C_0^5(\mathbb{R}_3)$, $g \in C_0^4(\mathbb{R}_3)$, and more precisely that for certain x^0, ρ

(45a) $f(x) = g(x) = 0$ for $|x-x^0| > \rho$.

Under these circumstances u^0 and its derivatives of orders ≤ 3 will decay like $1/t$ for large t, as follows from the explicit expression (17b), when we convert the surface integrals to volume integrals[9]. More exactely define

$$(45b)\quad N_\nu = \sup_{\substack{x \\ |\alpha| \leq \nu+2 \\ |\beta| \leq \nu+1}} \left| \rho^{|\alpha|} |D^\alpha f(x) \quad , \quad \rho^{|\beta|+1} |D^\beta g(x)| \right| .$$

There exists a universal constant γ such that

$$(45c)\quad \rho^{|\alpha|} |D^\alpha u^O(x,t)| < \gamma \frac{\rho}{\rho+t} N_\nu \quad \text{for } x \in \mathbb{R}_3,\ t \geq 0, |\alpha| \leq \nu$$

when[10] $\nu = 0,1,2,3$. By the strong Huygens principle moreover

$$(45d)\quad u^O(x,t) = 0 \text{ unless } t - \rho < |x-x^O| < t+\rho \quad .$$

We shall show the existence of a global solution u of

$$(46a)\quad \Box u = \phi(u)$$

with prescribed initial data f, g, provided N_O(for given p, ρ) is sufficiently small. It suffices to find a u(x,t) for which the expressions $D^\alpha u(x,t)$ for $|\alpha| \leq 2$ exist and are continuous in x,t for $x \in \mathbb{R}_3$, $t \geq 0$ and which satisfies

$$(46b)\quad u = u^O + L\phi(u) \quad .$$

We only have to make use of the following lemma:

<u>Lemma I</u>: *Let* w(x,t) *be defined for* $x \in \mathbb{R}_3$, $t \geq 0$, *and let the* $D^\alpha w(x,t)$ *for* $|\alpha| \leq 2$ *also exist*[11] *and be continuous there. Then* Lw(x,t) *as a function of* x *and* t *is of class* C^2 *for* $x \in \mathbb{R}_3$, $t \geq 0$, *has initial data zero, and satisfies* $\Box u = w$.

To prove the Lemma we introduce

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[9] See [28].

[10] D^α shall always stand for the *space* differentiation

$$D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$$

[11] Observe that the existence of t-derivatives of w is not assumed.

(47a) $M(x,r,t) = \frac{r}{4\pi} \int_{|\eta|=1} w(x,+r\eta,t)\, d\omega_\eta$ for $x \in \mathbb{R}_3$, $r \geq 0$, $t \geq 0$.

Obviously $M(x,r,t)$ is of class C^2 in x,r uniformly in t. By Darboux's identity in 3-space (see [1], p. 104)

(47b) $M_{rr}(x,r,t) = \Delta_x M(x,r,t)$.

On the other hand by (17c)

$$Lw(x,t) = \int_0^t M(x,t-s,s)\, ds .$$

Obviously

$$\frac{\partial}{\partial t} Lw(x,t) = M(x,0,t) + \int_0^t M_r(x,t-s,s)\, ds = \int_0^t M_r(x,t-s,s)\, ds$$

$$\frac{\partial}{\partial t} D_k\, Lw(x,t) = \int_0^t D_k\, M_r(x,t-s,s)\, ds$$

$$\frac{\partial^2}{\partial t^2} Lw(x,t) = M_r(x,0,t) + \int_0^t M_{rr}(x,t-s,s)\, ds$$

$$= w(x,t) + \Delta_x \int_0^t M(x,t-s,s)\, ds = w + \Delta Lw.$$

We associate with a continuous function $w(x,t)$ with domain $x \in R_3$, $t \geq 0$ the function

(48a) $\bar{w}(r,t) = \sup_{\substack{x \\ |x-x^0|=r}} |w(x,t)|$

defined for $r \geq 0$, $t \geq 0$. We can estimate Lw in terms of $\bar{w}$:

<u>Lemma II</u>: *For* $|x-x^0| = r$, $t \geq 0$

(48b) $|Lw(x,t)| \leq \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \bar{w}(\lambda,s)\, d\lambda =$

$$= \iint_{R_{r,t}} \frac{\lambda}{2r} \bar{w}(\lambda,s)\, d\lambda ds$$

or

(48c) $\overline{Lw} \leq P\bar{w}$

where P *is the operator defined by* (20).

For the proof set

$$v(x,t) = \bar{w}(|x-x^0|,t) .$$

Then $|w| \leq v$ and thus also

(48d) $|Lw| \leq Lv$, $\overline{Lw} \leq \overline{Lv}$.

Since $v(x,t)$ depends only on $|x-x^0|$ and t and is non-negative we have $\bar{v} = \tilde{v}$ (where $\tilde{v}$ is defined by (18) with w replaced by v). But with v also Lv is invariant under rotations about x^0, as is evident from formula (17c) or from the invariance of the operator $\Box$. Thus also $\overline{Lv} = \widetilde{Lv}$. It follows from (48d), (19) that

$$\overline{Lw} \leq \widetilde{Lv} = P\tilde{v} = P\bar{v} = P\bar{w} \quad .$$

Existence of a solution u of (46b) will be proved by iteration. For that we have to introduce a suitable norm. For functions $u(x,t)$ which are continuous for $x \in \mathbb{R}_3$, $t \geq 0$ and have their support in $\Gamma^+(x^0,-\rho)$, we define

$$\|u\| = \sup_{\substack{r\geq 0\\ t\geq 0}} \rho^{1-p} (t+r+2\rho)(t-r+2\rho)^{p-2}\bar{u}(r,t)$$

(49a) $$= \sup_{\substack{x\\ t\geq 0}} \rho^{1-p} (t+|x-x^0|+2\rho)(t-|x-x^0|+2\rho)^{p-2}|u(x,t)|$$

The function u^0 and more generally the $D^\alpha u^0$ with $|\alpha| < 3$ have finite norms. Indeed by (45d) we have for (x,t) in the support of u^0

$$\rho^{1-p} (t+|x-x^0|+2\rho)(t-|x-x^0|+2\rho)^{p-2}$$
$$\leq \rho^{1-p}(2t+3\rho)(3\rho)^{p-2} \leq 3^{p-1}(t+\rho)\rho^{-1}$$

so that by (45c)

(49b) $\|D^\alpha u^0\| \leq \gamma 3^{p-1} \rho^{-\nu} N_\nu$ for $|\alpha| = \nu \leq 3$

The next lemma gives the basic estimate for the existence proof:

Lemma III. *There exists a constant* c, *only depending on* p, *such that*

(50a) $\|L|u|^p\| \leq c\rho^2 \|u\|^p$

for u(x,t) *that are continuous for* $x \in R_3$, $t \geq 0$ *and satisfy*

(50b) $u(x,t) = 0$ *for* $|x-x^0| > t + \rho$.

Applying the estimate (48b) to $L|u|^p$, inequality (50a) is proved if we can show that

(51a) $$\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \bar{u}^p(\lambda,s)d\lambda \leq \frac{cC^p \rho^{p+1}}{(t+r+2\rho)(t-r+2\rho)^{p-2}}$$

with a certain c depending only on p, provided we know that

(51b) $$0 \leq \bar{u}(\lambda,s) \leq \frac{C \rho^{p-1}}{(s+\lambda+2\rho)(s-\lambda+2\rho)^{p-2}}$$

and that (see (50b))

(51c) $\bar{u}(\lambda,s) = 0$ for $\lambda > s + \rho$.

We first prove (51a) in the case where

(52a) $t - \rho < r < t + \rho$.

Then also

$$s - \rho < \lambda < s + \rho$$

for (λ,s) in the region of integration where $\bar{u}(\lambda,s) \neq 0$. It follows from (51b) that

$$\bar{u}(\lambda,s) \leq \frac{C\rho}{2s+\rho}$$

and hence that

$$\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \bar{u}^p(\lambda,s)d\lambda \leq C^p \rho^p \int_0^t (2s+\rho)^{-p} k(s)\, ds$$

where

$$k(s) = \frac{1}{2r} \int_I \lambda \, d\lambda$$

and I is the intersection of the intervals $[|r-t+s|, r+t-s]$ and $[|r-t+s, s+\rho]$. Here for $r > 2\rho$ by (52a)

$$k(s) \le \frac{1}{2r} \int_{|r-t+s|}^{s+\rho} \lambda \, d\lambda = \frac{(\rho+t-r)(2s+\rho+r-t)}{4r} \le \frac{\rho(s+\rho)}{r}$$

$$\le \frac{2\rho(s+\rho)}{(t+\rho)+(r-2\rho)} \le \frac{12\rho(s+\rho)}{t+\rho}$$

while for $0 < r < 2\rho$

$$k(s) \le \frac{1}{2r} \int_{|r-t+s|}^{r+t-s} \lambda \, d\lambda = t-s \le t \le (t-r)+r \le 3\rho \le \frac{12\rho(s+\rho)}{t+\rho} .$$

Thus

$$\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \, \bar{u}^p(\lambda,s) d\lambda \le \frac{12C^p \rho^{p+1}}{t+\rho} \int_0^t (s+\rho)(2s+\rho)^{-p} ds$$

$$\le \frac{12C^p \rho^3}{t+\rho} \int_0^\infty (\mu+1)^{1-p} d\mu \le \frac{36 \cdot 3^{p-2} C^p \rho^{p+1}}{(p-2)(t+r+2\rho)(t-r+2\rho)^{p-2}} \tag{52b}$$

We used here that $p > 2$.

Consider next the case where

(52c) $0 < r < t - \rho$.

Set

(52d) $Q = p(p-2)-1$.

Then $Q > 0$ because of our assumption $p > 1 + \sqrt{2}$. Introducing the variables of integration α, β from (34) and using (51b), (51c) we have here

$$\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \, \bar{u}^p(\lambda,s) \, d\lambda$$

$$\le \frac{C^p \rho^{p(p-1)}}{8r} \int_{-\rho}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\alpha-\beta}{(\alpha+2\rho)^p (\beta+2\rho)^{p(p-2)}} d\alpha$$

$$\frac{c^p \rho^{p(p-1)}}{8r} \int_{-\rho}^{\infty} d\beta \int_{t-r}^{t+r} \frac{d\alpha}{(\alpha+2\rho)^{p-1}(\beta+2\rho)^{Q+1}}$$

$$= \frac{c^p \rho^{p+1}}{8rQ} \int_{t-r}^{t+r} (\alpha+2\rho)^{1-p}\, d\alpha \le \frac{2c^p \rho^{p+1}}{Q(p-2)(t+r+2\rho)(t-r+2\rho)^{p-2}}$$

as is seen again by distinguishing the cases $t-r+2\rho > \frac{1}{2}(t+r+2\rho)$ and $t-r+2\rho < \frac{1}{2}(t+r+2\rho)$. This completes the proof of Lemma III.

Definition (49a) for functions satisfying (50b) implies that

(53a) $|u(x,t)| \le \|u\|$

(53b) $\|\,|u|^{\Theta}|v|^{1-\Theta}\| \le \|u\|^{\Theta}\,\|v\|^{1-\Theta}$ for $0 \le \Theta \le 1$.

It follows then from (50a) that

(53c) $L(|u|^{\Theta p}|v|^{(1-\Theta)p}) \le c\rho^2\,\|u\|^{\Theta p}\,\|v\|^{(1-\Theta)p}$

for $0 \le \Theta \le 1$.

Let now Z denote the linear space of functions $u(x,t)$ for which the $D^{\alpha}u(x,t)$ for $|\alpha| \le 2$ are defined and continuous in x,t for $x \in \mathbb{R}_3$, $t \ge 0$, and which satisfy

(54a) $u(x,t) = 0$ for $|x-x^0| > t+\rho$

(54b) $\|D^{\alpha}u\| < \infty$ for $|\alpha| \le 2$.

In particular $u^0 \in Z$.

If $u \in Z$ and $\|u\| < 1$ we derive from (43d), (53a), (53c) the inequalities

(55a) $\|L\phi(u)\| \le A\,\|L|u|^p\| \le Ac\rho^2\,\|u\|^p$

$\|D_kL\phi(u)\| = \|L(\phi'(u)D_ku)\| \le Ap\,\|L(|u|^{\Theta p}|D_ku|^{(1-\Theta)p})$

(55b) $\le Apc\rho^2\,\|u\|^{p-1}\,\|D_ku\|$

(using $\Theta = (p-1/p)$ and

$\|D_jD_kL\phi(u)\| = \|L(\phi'(u)D_jD_ku + \phi''(u)(D_ju)(D_ku))\|$

$$\leq Apc\rho^2(\|u\|^{p-1}\ \|D_jD_ku\| + (p-1)\|u\|^{p-2}\ \|\sqrt{|(D_ju)(D_ku)|}\ \|^2$$

$$(55c)\quad \leq Apc\rho^2(\|u\|^{p-1}\ \|D_jD_ku\| + (p-1)\|u\|^{p-2}\ \|D_ju\|\ \|D_ku\|\ ),$$

where for the second term we have used first (53c) with $\Theta = (p-2)/p$ and then (53b) with $\Theta = 1/2$. In the same way, using (43c,e) one shows that for $u,v \in Z$, $\|u\| < 1$, $\|v\| < 1$, and $w(x,t) = \mathrm{Max}(|u(x,t)|,|v(x,t)|)$

$$(55d)\quad \|L(\phi(u)-\phi(v))\| \leq \|L(pw^{p-1}|u-v|)\| \leq Apc\rho^2\ \|w\|^{p-1}\|u-v\|$$

$$\|D_jL(\phi(u)-\phi(v))\| = \|L((\phi'(u)-\phi'(v))D_ju + \phi'(v)D_j(u-v))\|$$

$$(55e)\quad \leq Ap(p-1)c\rho^2\|w\|^{p-2}\|u-v\|\ \|D_ju\| + Apc\rho^2\|w\|^{p-1}\|D_j(u-v)\|$$

$$\|D_jD_kL((\phi(u)-\phi(v))\|$$

$$= \|L\Big[(\phi'(u)-\phi'(v))D_jD_ku + \phi'(v)D_jD_k(u-v) + \phi''(v)(D_jv)D_k(u-v)$$
$$+ \phi''(v)(D_ku)D_j(u-v) + (\phi''(u)-\phi''(v))(D_ju)(D_ku)\Big]\|$$

$$\leq Ap(p-1)c\rho^2\|w\|^{p-2}\Big[\|D_jD_ku\|\ \|u-v\| + \|D_ku\|\ \|D_j(u-v)\|$$
$$+ \|D_jv\|\ \|D_k(u-v)\|\Big]$$
$$+ Apc\rho^2\|w\|^{p-1}\|D_jD_k(u-v)\| + Ap(p-1)c\rho^2\|u-v\|^{p-2}\|D_ju\|\ \|D_ku\|$$

(55f)

Here

$$(56a)\quad \|w\| = \mathrm{Max}(\ \|u\|\ ,\ \|v\|\ )$$

Relations (55a,b,c) imply that

$$(56b)\quad L\phi(u) \in Z \quad \text{for } u \in Z,\ \|u\| < 1\ .$$

We define the sequence of functions u_n by

$$(56c)\quad u_0 = u^0,\ u_{n+1} = u^0 + L\phi(u_n) \text{ for } n \geq 0\ .$$

If here $\|u_n\| < 1$, we have by (55a)

$$(56d)\quad \|u_{n+1}\| \leq \|u^0\| + Ac\rho^2\ \|u_n\|^p\ .$$

Assume now that $\|u^0\|$ is so small that

(56e) $Ac\rho^2 \, \|u^0\|^{p-1} \le \frac{1}{p2^p}$, $\|u^0\| < \frac{1}{2}$

(by (49b) this will be the case for N_0 sufficiently small). We find by induction from (56d) that

(56f) $\|u_n\| \le 2\,\|u^0\| < 1$ for $n \ge 0$.

In what follows C_1, C_2, ... denote constants depending on A, p, ρ and the $\|D^\alpha u^0\|$ for $|\alpha| \le 2$, but not on n. It follows from (56e), (55b), (56f) that

$$\begin{aligned} \|D_k u_{n+1}\| &\le \|D_k u^0\| + Apc\rho^2\, \|u_n\|^{p-1}\, \|D_k u_n\| \\ &\le \|D_k u^0\| + \tfrac{1}{2}\, \|D_k u_n\| . \end{aligned}$$

Hence

(56g) $\|D_k u_n\| \le C_1$ for $n \ge 0$.

Then by (55c)

$$\begin{aligned} \|D_j D_k u_{n+1}\| &\le \|D_j D_k u^0\| \\ &\quad + Apc\rho^2 (\|u_n\|^{p-1} \|D_j D_k u_n\| \\ &\quad + (p-1)\, \|u_n\|^{p-2} \|D_j u_n\|\, \|D_k u_n\|) \\ &\leqq \|D_j D_k u^0\| + Ap(p-1)c\rho^2 2^{p-2}\, \|u^0\|^{p-2} C_1^{\,2} \\ &\quad + Apc\rho^2 2^{p-1} \|u^0\|^{p-1} \|D_j D_k u_n\| \le C_2 + \tfrac{1}{2}\, \|D_j D_k u_n\| . \end{aligned}$$

Hence

(56h) $\|D_j D_k u_n\| \le C_3$.

Then by (55d, e, f), (56a) with

$$u = u_n,\ v = u_{n-1},\ w = \mathrm{Max}(|u_n|, |u_{n-1}|),\ \|w\| \le 2\, \|u^0\|$$

we have for $n \ge 1$

$$\|u_{n+1} - u_n\| \le Apc\rho^2 2^{p-1}\, \|u^0\|^{p-1}\, \|u_n - u_{n-1}\| \le \tfrac{1}{2}\, \|u_n - u_{n-1}\|$$

and thus

(56i) $\|u_{n+1} - u_n\| \le 2^{-n} C_4$ for $n \ge 0$.

Thus for $n \geq 1$

$$\|D_j(u_{n+1}-u_n)\| \leq Ap(p-1)c\rho^2 2^{p-2}\ \|u^0\|^{\,p-2}\ \|u_n-u_{n-1}\|\ \|D_j u^0\|$$
$$+\ Apc\rho^2 2^{p-1}\ \|u^0\|^{\,p-1}\ \|D_j(u_n-u_{n-1})\|$$
$$\leq 2^{-n}C_5 + \frac{1}{2}\|u_n-u_{n-1}\| \quad ,$$

and thus

(56j) $\|D_j(u_{n+1}-u_n)\| \leq n2^{-n}C_6$ for $n \geq 0$.

Finally for $n \geq 1$

$$\|D_jD_k(u_{n+1}-u_n)\| \leq Ap(p-1)c\rho^2 2^{p-2}\|u^0\|^{\,p-2}\Big[\|D_jD_ku_n\|\ \|u_n-u_{n-1}\|$$
$$+\ \|D_ku_n\|\ \|D_j(u_n-u_{n-1})\| + \|D_ju_{n-1}\|\ \|D_k(u_n-u_{n-1})\|\Big]$$
$$+\ Apc\rho^2 2^{p-1}\ \|u^0\|^{\,p-1}\ \|D_jD_k(u_n-u_{n-1})\|$$
$$+\ Ap(p-1)c\rho^2\ \|u_n-u_{n-1}\|^{\,p-2}\ \|D_ju_n\|\ \|D_ku_n\|$$
$$\leq (n2^{-n} + 2^{-(p-2)n})C_7 + \frac{1}{2}\|D_jD_k(u_n-u_{n-1})\|$$

where $1 < 2^{p-2} \leq 2$ by (43e). Thus

(56k) $\|D_jD_k(u_{n+1}-u_n)\| \leq (n^2 2^{-n} + n2^{-(p-2)n})C_8$.

Relations (56i, j, k) clearly imply that the $D^\alpha u_n$ for $|\alpha| \leq 2$ converge uniformly for $n \to \infty$ towards functions $D^\alpha u$, which are continuous in x,t, where u is a solution of (46b). This completes the proof of Theorem II.

<u>Proof of Theorem III</u>.

The proofs of Theorems I and II given above easily lead to some upper and lower bounds for the time T at which blow-up occurs. To obtain however bounds of the correct order of magnitude some refinement of the estimates is needed. We restrict ourselves to the equation

(57a) $\Box u = u^2$

with initial conditions of the form

(57b) $u = f(x) = \varepsilon F(x)$, $u_t = g(x) = \varepsilon G(x)$.

Here $F \in C_0^5(\mathbb{R}_3)$, $G \in C_0^4(\mathbb{R}_3)$ are prescribed functions with

(57c) $F(x) = G(x) = 0$ for $|x-x^0| > \rho$,

while ε is varied. By Corollary II for $p = 2$ we are sure to have $T < \infty$ unless F, G vanish identically. The aim is to prove that $\varepsilon^2 T$ lies between fixed positive bounds, depending only on F, G for sufficiently small ε. One could further explore the dependence of these bounds on F, G; the lower bound derived for T will be seen to be actually of order $\|u^0\|^{-2}$, while the upper bound involves lower bounds for $|u^0|$ in a ball of sufficient size. However in order to keep things simpler we just study the dependence of T on ε.

Proving first existence of a solution u of (57a,b) for sufficiently small t we are led to consider for a positive τ the class Z_τ of functions continuous in x,t for $x \in \mathbb{R}_3$, $0 \leq t < \tau$, for which

(58a) $u(x,t) = 0$ for $|x-x^0| > t+\rho$.

The first aim is to find an estimate of type (50a) for Lu^2 in terms of u for $u \in Z_\tau$. For this purpose it is appropriate to work with a more complicated norm on Z_τ than the one furnished by (49a) for p = 2.

We introduce (for fixed τ,ρ) the function

$$\text{(58b)}\quad z(r,t) = \begin{cases} \rho(t+\rho)^{-1} & \text{for } t-\rho < r < t+\rho \\ \tau^{-1/2}\rho^{3/2}r^{-1}\log\dfrac{t+r+\rho}{t-r+\rho} & \text{for } 0 \leq r < t-\rho \, , \\ 0 & \text{for } t+\rho < r \end{cases}$$

and now define for $u \in Z_\tau$ and $\bar{u}$ given by (48a)

$$\text{(58c)}\quad \|u\| = \sup_{\substack{r \geq 0 \\ 0 \leq t < \tau}} z^{-1}(|x-x^0|,t)\,|u(x,t)|$$

For $|x-x^0| = r,\ 0 \le t < \tau$

$$Lu^2(x,t) \le \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r}\,\bar{u}^2(\lambda,s)\,d\lambda$$

$$\le \|u\|^2 \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r}\,z^2(\lambda,s)\,d\lambda \quad . \tag{59a}$$

If here $t-\rho < r < t+\rho$, then also $s-\rho < \lambda < s+\rho$, $z(\lambda,s) = \rho^{-1}(s+\rho)$ in the region of integration, and (see (52b))

$$Lu^2(x,t) \le \frac{12\,\|u\|^2\rho^3}{t+\rho} \int_0^t (s+\rho)^{-1} ds \quad \frac{12\,\|u\|^2\rho^3}{t+r} \log \frac{t+\rho}{\rho}$$

$$\le 12\,\|u\|^2\rho^2 z(r,t) \log \frac{\tau+\rho}{\rho} \quad . \tag{59b}$$

Let next $0 \le r < t-\rho$. Then using the variables of integration α,β from (34)

$$Lu^2(x,t) \le \frac{\|u\|^2}{8r} \int_{-\rho}^{t-r} d\beta \int_{t-r}^{t+r} (\alpha-\beta)z^2(\lambda,s)\,d\alpha = (I_1+I_2)\,\|u\|^2,$$

where by (58b)

$$I_1 = \frac{\rho^2}{2r} \int_{-\rho}^{\rho} d\beta \int_{t-r}^{t+r} (\alpha-\beta)(\alpha+\beta+2\rho)^{-2}\,d\alpha$$

$$I_2 = \frac{\rho^3}{2r\tau} \int_{\rho}^{t-r} d\beta \int_{t-r}^{t+r} (\alpha-\beta)^{-1} \log^2\left(\frac{\alpha+\beta}{\beta+\rho}\right) d\alpha \quad .$$

Here

$$I_1 \le \frac{\rho^2}{2r} \int_{-\rho}^{\rho} d\beta \int_{t-r}^{t+r} (\alpha+\beta)(\alpha+\rho)^{-2}\,d\alpha$$

$$= \frac{\rho^3}{2r} \log \frac{t+r+\rho}{t-r+\rho} = \rho^{3/2}\tau^{1/2}\,z(r,t) \quad . \tag{59c}$$

Introducing new variables of integration σ,Θ in I_2 by

$$\alpha + \rho = (t-r+\rho)\sigma, \quad \beta + \rho = (t-r+\rho)\Theta\sigma$$

and setting

$$A = t+r+\rho \quad , \quad B = t-r+\rho$$

we have

$$I_2 = \frac{\rho^3 B}{2r} \int_1^{A/B} d\sigma \int_{2\rho/B\sigma}^{1/\sigma} \frac{\log^2\Theta}{1-\Theta}\, d\Theta$$

$$\leq \frac{(A-B)\,\rho^3}{2r\tau} \int_0^1 \frac{\log^2\Theta}{1-\Theta}\, d\Theta \leq \frac{\rho^3 A \log(A/B)}{2r\tau} \int_0^1 \frac{\log^2\Theta}{1-\Theta}\, d\Theta$$

(59d) $$\leq \frac{1}{2}\, \rho^{3/2}\tau^{1/2} \int_0^1 \frac{\log^2\Theta}{1-\Theta}\, d\Theta\; z(r,t) \quad .$$

It follows altogether from (59b,c,d) that

(60) $$\|Lu^2\| \leq c\tau^{1/2}\, \rho^{3/2}\, \|u\|^2 \quad ,$$

where

(61) $$c = 1 + \frac{1}{2}\int_0^1 \frac{\log^2\Theta}{1-\Theta}\, d\Theta + 12 \operatorname*{Max}_{\mu>0} \left(\mu^{-1/2} \log(1+\mu)\right) \quad .$$

From here on the existence proof is exactly the same as the one for theorem II, the only difference being that in the basic inequality (50a) taken for $p = 2$ we have to replace c by $c\tau^{1/2}\,\rho^{-1/2}$. The essential restriction (56e) on u^0 becomes

(26) $$c\rho^{3/2}\,\tau^{1/2}\,\|u^0\| < \frac{1}{4}$$

since ϕ'' for $\phi(u) = u^2$ is uniformly Hölder continuous. Obviously

(63a) $$u^0 = \varepsilon U^0 \quad ,$$

where U^0 is the solution of

(63b) $$\Box U^0 = 0$$

(63c) $$U^0 = F,\; U^0_t = G \quad \text{for} \quad t = 0 \; .$$

Condition (62) then reads

(64a) $$16c^2\, \rho^3\tau\, \varepsilon^2\, \|u^0\|^2 < 1 \quad .$$

If τ satisfies this inequality the solution u of (57a,b) exists for $0 \leq t < \tau$. This implies the lower bounds $A\varepsilon^{-2}$ with

(65b) $$A = (16c^2\, \rho^3\, \|U^0\|^2)^{-1}$$

for the time T at which the solution blows up.

We next determine an upper bound for T. Assume that u is a solution of (57a,b) for $x \in \mathbb{R}_3$, $0 \leq t < T$. We exclude the trivial case where the solution U^0 of (63b,c) vanishes identically, which only leads to $u = 0$. We make the further assumption that

$$(66) \quad U^0(x^0,t) \not\equiv 0 \quad \text{for some } t > 0.$$

(This is not an essential restriction since either F(x) or G(x) is different from zero at some x^1 with $|x^1-x^0| < \rho$. If necessary we replace x^0 by x^1 and ρ by 2ρ in (57c), which does not change anything in the theorem to be proved.)

Take any τ with $0 < \tau < T$. Then u satisfies

$$(67a) \quad u = u^0 + Lu^2 \quad \text{for } 0 \leq t < \tau \quad .$$

Consider first the restriction of u to the set

$$(67b) \quad K_\tau = \{(x,t): t-\rho<|x-x^0| < t+\rho,\ 0 < t < \tau\} .$$

For $(x,t) \in K_\tau$ the integral for Lu^2 only involves points of K_τ, and we find from (59b), (58b) that

$$Lu^2(x,t) \leq \frac{12\,\|u\|^2_\tau\,\rho^3}{t+\rho} \log \frac{\tau+\rho}{\rho}$$

where for the restriction of u to K_τ we define

$$(67c) \quad \|u\|_\tau = \sup_{K_\tau} \frac{t+\rho}{\rho} |u(x,t)| \quad ;$$

thus

$$(67d) \quad \|Lu^2\|_\tau \leq 12\rho^2 \left(\log\left(\frac{\tau+\rho}{\rho}\right)\right) \|u\|_\tau^2 \quad .$$

It follows from (67a) that

$$(67e) \quad \|u\|_\tau \leq \|u^0\|_\tau + 12\rho^2(\log(\frac{\tau+\rho}{\rho}) \|u\|_\tau^2 \quad .$$

For sufficiently small τ we have

$$(67f) \quad \|u\|_\tau < 2\,\|u^0\|_\tau \quad .$$

If τ is the smallest value for which $\|u\|_\tau = 2\|u^0\|_\tau$ we would find from (67e) that

$$48\rho^2 \log(\frac{\tau+\rho}{\rho}) \|u^0\|_\tau \leq 1,$$

where of course

$$(67g) \quad \|u^0\|_\tau \leq \|u^0\| = \sup_{\substack{x \\ t\geq 0}} (\frac{t+\rho}{\rho}|u^0(x,t)|) \quad .$$

It follows that the restriction of u to K_τ satisfies

$$(67h) \quad \|u\|_\tau \leq 2\|u^0\|$$

$$(67i) \quad \|u-u^0\|_\tau = \|Lu^2\|_\tau \leq 48\rho^2 (\log(\frac{\tau+\rho}{\rho})) \|u^0\|^2$$

that is

$$(67j) \quad |u(x,t)-u^0(x,t)| \leq \frac{48\rho^3}{t+\rho}(\log(\frac{\tau+\rho}{\rho}))\varepsilon^2\|U^0\|^2 \text{ for } (x,t)\in K_\tau$$

as long as τ is so small that

$$(67k) \quad 48\rho^2(\log(\frac{\tau+\rho}{\rho}))\varepsilon\|U^0\| < 1 \quad .$$

We go over to the spherical averages $\tilde{u}(r,t)$ of u, which by (27) satisfy

$$(68a) \quad \tilde{u}(r,t) \geq P\tilde{u}^2 = \iint_{R_{r,t}} \frac{\lambda}{2r}\tilde{u}^2(\lambda,s)\, d\lambda ds$$

for $0 < r < t-\rho$. Taking the spherical means of inequality (67j) we find that for $t-\rho < r < t+\rho$

$$(68b) \quad |\tilde{u}(r,t)-\tilde{u}^0(r,t)| \leq \frac{48\rho^3}{t+\rho}(\log\frac{\tau+\rho}{\rho})\varepsilon^2\|U^0\|^2 \quad .$$

The function $\tilde{U}^0(r,t)$ as a solution of $(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2})rU^0 = 0$ (see (47a,b)) is of the form

$$\tilde{U}^0(r,t) = \frac{H(t+r) - H(t-r)}{2r},$$

$$H(s) = \frac{1}{2}\tilde{F}(s) - \frac{1}{2}\int_s^\infty \sigma\tilde{G}(\sigma)\, d\sigma$$

where $H(s) = 0$ for $s > \rho$. Here

$$U^O(x^O,t) = \tilde{U}^O(0,t) = H'(t) \ .$$

It follows from (66a) that there exists a t between 0 and ρ for which $H'(t) \neq 0$. There exists then a $\mu > 0$ and a,b with $0 < a < b < \rho$ such that

$$|H(s)| > \mu \quad \text{for} \quad a < s < b \ .$$

Then for $t+r > \rho$, $a < t-r < b$

$$|\tilde{U}^O(r,t)| = \frac{|H(t-r)|}{2r} \geqq \frac{\mu}{2r} \quad \text{for } t+r > \rho, \ a < t-r < b$$

and by (68b)

$$\text{(68c)} \quad |u(r,t)| > \mu\frac{\varepsilon}{2r} - \frac{48\ \rho^3}{r} \log\frac{\tau+\rho}{\rho}\, \varepsilon^2 \,\|U^O\|^2 > \mu\frac{\varepsilon}{4r}$$

if

$$\text{(68d)} \quad 196\varepsilon\ \rho^3\left(\log\frac{\tau+\rho}{\rho}\right) \|U^O\|^2 < \mu .$$

Let now (r,t) belong to the set

$$\Sigma = \{(r,t):\ 0 < r < t-2\rho\}$$

identifying the t_2,δ in (33a) with the present $0,\rho$. Then $R_{r,t}$ contains the set

$$S^* = \{(\lambda,s):\ 2\rho < s+\lambda,\ 0 < a < s-\lambda < b < \rho\}$$

in which

$$|u(\lambda,s)| > \frac{\mu\varepsilon}{4r} \ .$$

It follows that for $(r,t) \in \Sigma$

$$u(r,t) > \frac{\mu^2\varepsilon^2}{32r}\int_a^b d\beta \int_{t-r}^{t+r} \frac{d\alpha}{\alpha-\beta}$$

$$> \frac{\mu^2\alpha^2(b-a)}{16}\,\frac{1}{t+r} \ .$$

This is an inequality of the form (38a) with

$$q = 1\ ,\ a = b = 0\ ,\ C = \frac{\mu^2(b-a)}{16}\,\varepsilon^2 \ .$$

It leads to the inequality (41) where J, E are defined by (40d), (42) and here

$$C^* = \frac{C^2}{64} \quad .$$

Then $u(r,t) = \infty$ when

$$J = E + \log \frac{(t-r-2\rho)^2}{t-r} > 0 \quad .$$

But here E differs from $\log(\varepsilon^2)$ only by a constant. It follows that $u(0,\tau) = \infty$ when $\varepsilon^2\tau$ is sufficiently large. This is still entirely consistent with the restriction (68d) on τ when ε is sufficiently small. Hence $T < B\varepsilon^{-2}$ with a suitable constant B for ε sufficiently small, proving Theorem III.

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