manuscripta math. 28, 159 - 183 (1979)

manuscripta mathematica

ON THE STRUCTURE OF MANIFOLDS WITH POSITIVE SCALAR CURVATURE

R. Schoen and S.T. Yau*

Dedicated to Hans Lewy and Charles B. Morrey, Jr.

In this paper, we study the question of which compact manifolds admit a metric with positive scalar curvature. Scalar curvature is perhaps the weakest invariant among all the well-known invariants constructed from the curvature tensor. It measures the deviation of the Riemannian volume of the geodesic ball from the euclidean volume of the geodesic ball. As a result, it does not tell us much of the behavior of the geodesics in the manifold.

Therefore it was remarkable that in 1963, Lichnerowicz [1] was able to prove the theorem that on a compact spin manifold with positive scalar curvature, there is no harmonic spinor. Applying the theorem of Atiyah-Singer, it then follows that the $\hat{A}$-genus of the manifold is zero. Later, Hitchin [2] found that the vanishing theorem of Lichnerowicz can also be used to prove the other KO-characteristic numbers defined by Milnor [3] are zero.

For a while, it was not clear whether these are the only topological obstructions for the existence of metrics with positive scalar curvature. It was not until 1977 that the authors found

* Research supported in part by NSF MCF 7804872.

0025-2611/79/0028/0159/$05.00

another topological obstruction in connection with some problems in general relativity. (See [4], [5], [6].) At that time, we restricted our attention to three dimensional manifolds. We found that if the fundamental group of a compact three dimensional manifold with non-negative scalar curvature contains a subgroup isomorphic to the fundamental group of a compact surface with genus ≥ 1, then the manifold is flat. Assuming the (topological) conjecture of Waldhausen (see [7]), one can then prove that the only possible candidates for compact three dimensional orientable manifolds with non-negative scalar curvature are flat manifolds and manifolds which can be decomposed as the connected sum

$$M_1 \# \cdots \# M_n \# k\cdot(S^2 \times S^1)$$

where each M_i is covered by a homotopy sphere and $k\cdot(S^2 \times S^1)$ is the connected sum of k copies of $S^2 \times S^1$. It is a conjecture that M_i is in fact the quotient of the three sphere by a finite group of orthogonal transformations. If so, this will give a complete answer to our question for three dimensional manifolds because one can prove that, conversely, the above manifolds do admit metrics with positive scalar curvature. At this point, one should mention that the proof of the positive mass conjecture [6] is much more delicate than the case of a compact manifold because the topology of the space does not help, and the analysis at infinity is much more involved.

In August of 1978, the second author visited Professor Hawking in Cambridge who indicated that a generalization of the above works to four space would be of great interest in quantum gravity. After a month, using an important idea by the first author, we were able to settle the positive action conjecture of Hawking. As an easy consequence of this proof, we gain some understanding of compact manifolds with positive scalar curvature in higher dimensions.

Defining a class of manifolds to be class C, in Section 1, we prove that for dimension ≤ 7, any compact manifold with positive scalar curvature must belong to class C. One of the key features that occurs here is that if the manifold has "enough" codimension one homology classes to intersect non-trivially, then it does not admit any metric with positive scalar curvature. For example, the connected sum of any manifold with the torus or the solvamanifold admits no metric with positive scalar curvature. The topological condition that we find here has an analogue with the condition that the second author [8] used in studying the circle action.

We feel that the topological conditions that we find here are quite satisfactory in low dimensions. Since we use the regularity theorems of minimal hypersurfaces, we have to restrict ourselves to manifolds with dimension ≤ 7. This is rather unsatisfactory because one feels that the singularity of the minimal hypersurface should not be the obstruction for the proof of the theorem. After we got these results, Professors Gromov and Lawson were able, by a beautiful argument, to find a topological condition which works for all dimensions. Namely they find that our condition is closely related to the Novikov signature. They replace the signature operator in the Lusztig proof of the Novikov conjecture by the Dirac operator. Applying the Lichnerowicz vanishing theorem to families of operators, they can prove the vanishing of a generalized $\hat{A}$-genus for spin manifolds. Exploiting a similar idea, they can also deal with a class of manifolds which they call enlargable manifolds. However, they have to restrict to the class of spin manifolds. Even in the case of spin manifolds, their condition does not include ours. Therefore it would certainly be of interest to combine the two conditions and also generalize our condition to arbitrary dimensions. Concerning this last part, we have some definite progress and we hope to report later.

At the same time we were finding the topological conditions, we were also working on the construction of manifolds with positive scalar curvature. As was mentioned above, in the class of three dimensional manifolds, we can connect two manifolds with positive scalar curvature to form another manifold with positive scalar curvature. As the theory of classification of manifolds is based on surgeries on manifolds, we generalize the procedure for connected sum to general surgeries. It turns out that if one does surgeries with codimension ≥ 3 on manifolds with positive scalar curvature, one always obtains manifolds with positive scalar curvature. In fact, a more general way of constructing manifolds with positive scalar curvature is provided in Section 2. With these surgery results, we believe that manifolds with positive scalar curvature may be classified soon because the geometric problem has been essentially reduced to a topological one.

Finally we should mention that recently we learned that Professors Gromov and Lawson have some form of our surgery result also. We wish to thank S.Y. Cheng for his interest in our work on connected sums since the time we announced our first results in this direction at Berkeley in March of 1978.

1. Integrability conditions for the existence of a metric with non-negative scalar curvature

In this section, we use the theory of minimal currents to give a topological restriction for manifolds to admit metrics with non-negative scalar curvature.

To see the precise statement of this topological restriction, we proceed to define inductively a class of manifolds in the following manner. Let C_3 be the class of compact orientable three dimensional manifolds M such that for any finite covering manifold $\tilde{M}$ of M, $\Pi_1(\tilde{M})$ contains no subgroup which is isomorphic to the fundamental group of a compact surface of genus ≥ 1. In general, we say that an n-dimensional compact orientable manifold M with $n \geq 4$ is of class C_n if either M is spin and the $\hat{A}$-genus of M is zero or for any finite covering space of M, every codimension one (real) homology class can be represented by an embedded compact hypersurface of class C_{n-1}.

<u>Theorem 1</u>. Suppose M is a compact orientable manifold with non-negative scalar curvature whose dimension is not greater than seven. Then either M has zero Ricci curvature or M is of class C_n.

<u>Proof</u>. First of all, we observe that we can assume the scalar curvature of M is everywhere positive. Otherwise the arguments in [9] show that the Ricci curvature of M is identically zero.

The proof is done by induction on dimension. For $n = 3$, this was the theorem proved in [5]. For $n \geq 4$, we proceed as follows. If M were not of class C_n, then the Lichnerowicz vanishing theorem and the Atiyah-Singer index theorem show that the $\hat{A}$-genus of M is zero. Therefore, by definition of C_n, for some finite covering space $\tilde{M}$ of M, some codimension one (real) homology class cannot be represented by any compact embedded hypersurface of class C_{n-1}.

On the other hand, by geometric measure theory (see [10]), one can show that this homology class can be represented by an orientable closed embedded hypersurface H of minimum area (compared with all other closed hypersurfaces in the homology class). The regularity theory guarantees that H is regular if $n \leq 7$.

We claim that H admits a metric with positive scalar curvature. Indeed, let $R_{ijk\ell}$ be the curvature tensor of M and Π_{ij} be the second fundamental form of H. Then we can compute the second variation of the area of H as follows. Let e_n be the unit normal vector of H and ϕ be an arbitrary smooth function defined on H. Then if we deform the hypersurface H along the direction ϕe_n, the second derivative of the area is given by (see [11])

$$(1.1) \qquad -\int_H \left(R_{nn}\phi^2 + \sum_{i,j} \Pi_{ij}^2 \phi^2\right) + \int_H |\nabla\phi|^2$$

where R_{nn} is the Ricci curvature of M in the direction of e_n.

By the minimality of H, this last quantity must be non-negative for all ϕ. In order to make use of this fact, we use the Gauss curvature formula as follows. Let $\tilde{R}_{ijk\ell}$ be the curvature tensor of H with respect to the induced metric and $e_1, \ldots, e_{n-1}$ be a local orthonormal frame in H. Then the Gauss curvature equation says

$$(1.2) \qquad \tilde{R}_{ijij} - R_{ijij} = \Pi_{ii}\,\Pi_{jj} - \Pi_{ij}^2$$

for i, j < n.

Summing (1.2), we have

$$(1.3) \qquad \sum_{i,j<n} \tilde{R}_{ijij} = \sum_{i,j<n} R_{ijij} + \left(\sum_i \Pi_{ii}\right)^2 - \sum_{i,j} \Pi_{ij}^2$$

Therefore, by the minimality of H, the scalar curvature of M is

$$(1.4) \qquad \begin{aligned} R &= \sum_{i,j\le n} R_{ijij} \\ &= 2\sum_i R_{nini} + \sum_{i,j<n} R_{ijij} \\ &= 2\,R_{nn} + \tilde{R} + \sum_{i,j} \Pi_{ij}^2 \end{aligned}$$

where $\tilde{R}$ is the scalar curvature of H.

Therefore, putting (1.4) into (1.1), we have

$$(1.5)\quad \int_H \frac{R\phi^2}{2} - \int_H \frac{\tilde{R}\phi^2}{2} + \frac{1}{2}\int_H \Big(\sum_{i,j} \Pi_{ij}^2\Big)\phi^2 \leq \int_H |\nabla\phi|^2$$

for all smooth functions ϕ defined on H.

Since $R > 0$ on H, we conclude that

$$(1.6)\quad -\int_H \frac{\tilde{R}\phi^2}{2} < \int_H |\nabla\phi|^2$$

for all non-zero smooth functions ϕ.

Let Δ be the Laplace operator of H. Then (1.6) implies that for $\lambda \geq 0$, the only solution of the equation

$$(1.7)\quad \Delta\phi = \frac{(n-3)}{4(n-2)}\tilde{R}\phi + \lambda\phi$$

is the zero function.

Otherwise multiple (1.7) by ϕ and integrating, we have

$$(1.8)\quad \frac{2(n-2)}{n-3}\int_H |\nabla\phi|^2 = -\frac{1}{2}\int_H \tilde{R}\phi^2 - \frac{2\lambda(n-2)}{n-3}\int_H \phi^2 < \int_H |\nabla\phi|^2$$

which is impossible.

The fact that (1.7) has no non-trivial solution means that all the eigenvalues of the operator $\Delta - \frac{(n-3)}{4(n-2)}\tilde{R}$ are positive. It is well-known that the first eigenfunction for operators of this form cannot change sign. If u is the first (positive) eigenfunction of the operator, then

$$\Delta u - \frac{n-3}{4(n-2)} \tilde{R} u = - \lambda u \tag{1.9}$$

where $\lambda > 0$.

If we multiply the metric of H by $u^{\frac{4}{n-3}}$, then the scalar curvature of H is changed to

$$u^{-\frac{4}{n-3} - 1} (\tilde{R}u - \frac{4(n-2)}{n-3} \Delta u$$

(Note that dim H = n-1). Therefore (1.9) shows that H admits a metric with positive scalar curvature. By the inductive hypothesis, H is of class C_{n-1} which is a contradiction.

Remark. We found the argument of using the first eigenfunction of the operator $\Delta - \frac{n-3}{4(n-2)} \tilde{R}$ from the paper of Kazdan-Warner [9].

In order to find a class of manifolds which behave well under maps of non-zero degree, we consider the following class. Let C_3' be the class of compact three dimensional manifolds which do not admit any non-zero degree map to a compact three dimensional manifold M such that $\Pi_2(M) = 0$ and M contains a two-sided incompressible surface with genus ≥ 1. For $n \geq 4$, let C_n' be those n-dimensional compact manifolds M such that every codimension one homology class of M can be represented, up to some non-zero integer, by a map from a manifold of class C_{n-1}'. It is clear from these definitions that if M is of class C_n' and if there is a non-zero degree map from M onto M', then M' is of class C_n'.

Theorem 2. If M is a compact n-dimensional manifold with non-negative scalar curvature and if $n \leq 7$, then either the Ricci curvature of M is identically zero or M is of class C_n'.

Proof. The proof is almost identical to that of Theorem 1 except that we have to notice the following remark. If M is of class C_3', then using a well-known lemma in topology (see [7] p. 62) one can show that M is of class C_3.

Corollary 1. Let M_n be an n-dimensional compact manifold with $n \leq 7$ such that for some compact manifolds M_i with dimension i for $3 \leq i \leq n$ there are maps from M_i onto M_{i-1} which pull the fundamental class in $H^{i-1}(M_{i-1})$ back to a non-trivial class in M_i. Suppose that $\Pi_2(M_3) = 0$ and M_3 contains an incompressible surface of genus ≥ 1 with trivial normal bundle. Then M_n admits no metric with non-negative scalar curvature except those with zero Ricci curvature.

Proof. This follows easily from the duality between cohomology and homology. The following corollary is an easy consequence of Corollary 1.

Corollary 2. Let M be a compact manifold which admits a non-zero degree map to the n-dimensional torus. Then for $n \leq 7$, the only possible metric with non-negative scalar curvature on M is the flat metric.

Proof. It is easy to check that M is not of class C_n' and hence the only possible metric with non-negative scalar curvature on M has zero Ricci curvature. On the other hand, the assumption on M guarantees that there are n linearly independent harmonic one-forms on M. As M has zero Ricci curvature, the familiar Bochner method shows that these forms are parallel forms. It is then easy to show that M is flat.

Remark. 1. Note that Corollary 2 remains valid if we replace the torus by a compact solvamanifold because the latter manifold is clearly not of class C_n'.

2. In the above theorems, part of the conclusion is that the Ricci curvature of the manifold is zero. This is a restrictive class of manifolds as was shown by Cheeger and Gromoll [12]. They prove that compact manifolds with zero Ricci curvature are covered isometrically by the product of a euclidean space and a compact simply connected manifold with zero Ricci curvature.

2. Construction of metrics of positive scalar curvature

In this section we construct a large class of compact manifolds of positive scalar curvature by showing that one can do surgeries on submanifolds of codimension at least three in the category of positive scalar curvature. Let M be an n-dimensional compact manifold with scalar curvature $R > 0$ and metric ds^2. Let N be a compact k-dimensional embedded submanifold with $k < n - 2$. In the appendix we have shown the existence of a positive function u on $M \sim N$ satisfying

$$\Delta u - \frac{n-2}{4(n-1)} Ru = 0 \quad \text{on} \quad M \sim N \tag{2.1}$$

$$u = \begin{cases} r^{2+k-n} + 0(r^{3+k-n}) & \text{if } k < n-3 \\ r^{2+k-n} + 0(\log r^{-1}) & \text{if } k = n-3 \end{cases}$$

where r is the distance function to N measured with respect to ds^2 and Δ is the Laplacian with respect to ds^2. (Here $0(f(r))$ means that after differentiating i times, the function is bounded by $d^i f/dr^i$.) For a positive function ϕ on N, we recall the followin standard formula for the scalar curvature of the metric

$$ds'^2 = \phi^{\frac{4}{n-2}} ds^2$$

$$R' = \phi^{-\frac{n+2}{n-2}} (R\phi - \frac{4(n-1)}{n-2} \Delta\phi) \quad . \tag{2.2}$$

Formulas (2.1) and (2.2) show that the metric $u^{\frac{4}{n-2}} ds^2$ has zero scalar curvature on $M \sim N$. We now let $h(r)$ be a smooth function which is zero on $[a,\infty)$ for a small number $a > 0$, and we define a metric $\overline{ds}^2 = ds^2 + dh \otimes dh$. Thus, if we choose coordinates $\{x^i\}$ on M, and suppose $ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$, then $\overline{ds}^2 = \sum_{i,j} \bar{g}_{ij} dx^i dx^j$ where $\bar{g}_{ij} = g_{ij} + h_{x^i} h_{x^j}$. The Christoffel symbols $\bar{\Gamma}^k_{ij} = \frac{1}{2} \bar{g}^{k\ell} (\bar{g}_{\ell i,j} + \bar{g}_{\ell j,i} - \bar{g}_{ij,\ell})$ are then easily seen to be

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + (1 + |\nabla h|^2)^{-1} h^k h_{ij} \tag{2.3}$$

where we use the metric ds^2 to raise and lower indices, and h_{ij} is the covariant hessian of h taken with respect to ds^2. Direct computation then shows that the scalar curvature of $\overline{ds}^2$ is

$$\bar{R} = R - 2(1 + |\nabla h|^2)^{-1} R_{ij}h^i h^j$$
$$+ (1 + |\nabla h|^2)^{-1} \bar{g}^{ik}\bar{g}^{j\ell}(h_{ik}h_{j\ell} - h_{i\ell}h_{jk})$$

where $\bar{g}^{ij} = g^{ij} - \dfrac{h^i h^j}{1+|\nabla h|^2}$ is the inverse matrix of $\bar{g}_{ij}$ and R_{ij} is the Ricci tensor of ds^2. Near N, one then sees

$$\begin{aligned}\bar{R} &= R + (r^{-2}+0(r^{-1}))(n-k-1)(n-k-2)(h')^2(1+(h')^2)^{-1}\\ &+ 2(r^{-1} + 0(1))(n-k-1)\frac{h'h''}{(1+(h')^2)^2}\end{aligned} \tag{2.4}$$

To compute this formula one uses the fact that $H(r)$, the Hessian of r, has the following form: If $x_1,\ldots,x_k,x_{k+1},\ldots,x_n$ is a local coordinate system such that $x_{k+1} = \ldots = x_n = 0$ defines N, and $r^2 = \sum_{i=k+1}^{n} x_i^2$, then

$$H(r)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} + 0(1)$$

for $i,j \geq k + 1$

$$H(r)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0(1)$$

otherwise.

(For the reader's convenience, we remark that with respect to an orthonormal frame field

$$\bar{R} = R - 2(1+|\nabla h|^2)^{-1}(\sum R_{ij}h_i h_j) + ((\Delta h)^2 - \sum h_{ij}^2)(1+|\nabla h|^2)^{-1}$$
$$- 2[(\Delta h)\sum h_i h_{ij} h_j - \sum h_i h_{ij} h_{jk} h_k](1 + |\nabla h|^2)^{-2}$$

$$= R + (1+|\nabla h|^2)^{-1}[(n-k-1)(n-k-2)r^{-2}|h'|^2 + 2(n-k-1)r^{-1}h'h''$$

$$+ 0(r^{-1})|h'|^2 + 0(1)h'h''] - 2(1+|\nabla h|^2)^{-2}h'^2$$

$$[0(r^{-1})h'^2 + (n-k-1)r^{-1}h'h'' + 0(1)h'h''])$$

We compute the scalar curvature $\bar{\bar{R}}$ of the metric $\overline{\overline{ds}}^2 = u^{\frac{4}{n-2}}\overline{ds}^2$. From (2.2) we see that

$$\bar{\bar{R}} = u^{-\frac{4}{n-2}}(\bar{R} - \frac{4(n-1)}{n-2}u^{-1}\bar{\Delta}u) \tag{2.5}$$

where barred quantities are taken with respect to $\overline{ds}^2$. We see from (2.3) that

$$\bar{\Delta}u = \Delta u - (1+|\nabla h|^2)^{-1}\bar{g}^{ij}h_{ij}h^k u_k - (1+|\nabla h|^2)^{-1}h^i h^j u_{ij} .$$

Near N we use (2.1) to get the expansion

$$u^{-1}\bar{\Delta}u = u^{-1}\Delta u - (2+k-n)(1+|\nabla h|^2)^{-1}(\Delta h)(r^{-1}+0(1))h'$$

$$+ (2+k-n)(1+|\nabla h|^2)^{-2}h_i h_j h_{ij}(r^{-1}+0(1))h'$$

$$- (1+|\nabla h|^2)^{-1}h^i h^j((2+k-n)r^{-1}r_{ij}$$

$$+ (2+k-n)(1+k-n)r^{-2}r_i r_j) .$$

Since $\Delta h = h'\Delta r + h'' = h'(n-k-1)(r^{-1}+0(1)) + h''$ and $h^i = h'(r_i+0(r))$ we conclude

$$u^{-1}\bar{\Delta}u = u^{-1}\Delta u + (n-k-2)\frac{h'h''}{(1+(h')^2)^2}(r^{-1}+0(1)) + 0(r^{-1})\frac{(h')^2}{1+(h')^2}$$

Substituting this and (2.4) into (2.5) and using (2.1) we have

$$\bar{R} = u^{-\frac{4}{n-2}}[(n-k-1)(n-k-2)(r^{-2}+0(r^{-1}))(h')^2(1+(h')^2)^{-1}$$

(2.6)

$$-2(\frac{n(n-k-2)}{n-2}-1)(r^{-1}+0(1))h'h''(1+(h')^2)^{-2}]$$

Note also that for $n > k+2$ and $n \geq 3$, we have

$$\frac{n(n-k-2)}{n-2} - 1 > 0 \quad .$$

Equation (2.6) shows that if a is sufficiently small we have $\bar{R} \geq 0$ provided h is chosen so that $h' \leq 0$ and $h'' \geq 0$. If we choose coordinates $x^1,\ldots,x^k$ locally on N and we choose a local orthonormal frame $\nu_{k+1},\ldots,\nu_n$ for the normal space, then we can define a coordinate system in an open set of M as follows: For each $y = (y^{k+1},\ldots,y^n) \in R^{n-k}$ with $|y|$ sufficiently small, set $F(x,y) = \exp_x(\sum_{\alpha=k+1}^{n} y^\alpha \nu_\alpha)$. If we take (x,y) as coordinates for M, ds^2 has the following form

$$ds^2 = \sum_{i,j=1}^{k} g_{ij}(x,y)dx^i dx^j + \sum_{\alpha,\beta}(\delta_{\alpha\beta}+0(r^2))dy^\alpha dy^\beta$$

(2.7)

$$+ \sum_{i,\alpha,\beta,\gamma}(y^\beta \Gamma^\gamma_{i\beta} g_{\alpha\gamma}+0(r^2))dx^i dy^\alpha \quad .$$

Note that $r^2 = \sum_\alpha (y^\alpha)^2$. Let δ be chosen so that $0 < \delta < a$ and choose the curve $t = h(r)$ for $\delta < r < a$ so that $h' < 0$, $h'' > 0$, and the curve joins the line $r = \delta$ in a smooth manner at the point (δ, t_o) in the rt-plane for some $t_o > 0$. If we write ds^2 in terms of t, we see that r tends to δ and

$dr = \frac{dr}{dt}\,dt$ tends to zero as t tends to t_o. It follows that $\overline{ds}^2 = ds^2 + dh \otimes dh = dt^2 + ds^2$ joins smoothly to the metric $dt^2 + ds_\delta^2$ at $t = t_o$ where ds_δ^2 is the metric induced by ds^2 on the hypersurface $S_\delta = \{r=\delta\}$.

Thus we may take

$$\overline{ds}^2 = u^{\frac{4}{n-2}}\,(dt^2 + ds_\delta^2) \qquad \text{for } t \geq t_o \tag{2.8}$$

which is a function times the product metric on $R \times S_\delta$.

We let $d\sigma^2$ be the metric on the total space of the normal bundle of N gotten by lifting ds_N^2 via the induced Riemannian normal connection. Note that in terms of the coordinates (x,y) we have

$$d\sigma^2 = \sum_{i,j} g_{ij}(x,0)\,dx^i dx^j + \sum_{\alpha} (dy^\alpha)^2 + \sum_{i,\alpha,\beta,\gamma} y^\beta \Gamma^\gamma_{i\beta}(x,0)\,\delta_{\alpha\gamma}\,dx^i dy^\alpha \; .$$

Let $d\sigma_\delta^2$ be the induced metric on the normal sphere bundle of radius δ. For $\varepsilon > 0$, we pull back the metric $ds^2_{\varepsilon\delta}$ from $S_{\varepsilon\delta}$ to S_δ via the map $(x,y) \to (x,\varepsilon y)$, and then multiply the radius of the fiber sphere by ε^{-2}. We denote this new metric $q_{\varepsilon,\delta}$. From (2.7) we have

$$\lim_{\varepsilon\to 0+} q_{\varepsilon,\delta} = d\sigma_\delta^2 \tag{2.9}$$

in smooth norm. Thus if we let $\varepsilon(t)$ be a smooth function which changes from 1 for $t \leq 2t_o$ to 0 for $t \geq 2t_o + b$ for $b > 0$ to be chosen, we may redefine $\overline{ds}^2$ as

$$\overline{ds}^2 = (\varepsilon(t)u + 1-\varepsilon(t))^{\frac{4}{n-2}} (dt^2 + q_{\varepsilon(t),\delta}) \text{ for } t < 2t_o + b$$

(2.10)

$$\overline{ds}^2 = dt^2 + d\sigma_\delta^2 \quad \text{for } t \geq 2t_o + b \ .$$

It is a straightforward calculation to see that a metric of the form $u_t^{4/(n-2)}(dt^2+q(t))$ has scalar curvature of the form

(2.11) $$u_t^{-\frac{4}{n-2}}(R_q+0(|\dot{q}|^2)+0(|\ddot{q}|) + u_t^{-1}\Delta_q u_t + 0(|\ddot{u}_t|) + 0(|\dot{q}||\dot{u}_t|))$$

where dot means differentiation in t, and $|\dot{q}|$, $|\ddot{q}|$ can be taken as the maximum matrix entry in some coordinate system. It is easy to check that for each fixed ε with $0 < \varepsilon < 1$ the metric $q_{\varepsilon,\delta}$ has scalar curvature bounded below by a fixed positive constant times δ^{-2}. (This is essentially a special case of (2.7).

Since $r \equiv \delta$ is constant on S_δ, it follows from (2.1) that $u_t^{-1}\Delta_q u_t = 0(\delta^{-1})$. It then follows from (2.9), (2.10), and (2.11) that $\overline{ds}^2$ has non-negative scalar curvature if b is large and $|\dot{\varepsilon}|$, $|\ddot{\varepsilon}|$ small. We can now prove the following theorem.

<u>Theorem 3</u>. Let N be a k-dimensional compact embedded (not necessarily connected) submanifold of a compact n dimensional manifold M of positive scalar curvature. Suppose $k < n-2$. Given any metric on N and any connection on the normal bundle ν of N we define a metric P on the total space $T\nu$ by using the connection to lift the metric from N. For $\delta > 0$ sufficiently small, there is a neighborhood V of N and a metric Q on $M \sim V$ so that in a neighborhood of ∂V, Q is a

product of a line with the sphere bundle of radius δ with the metric induced by P.

Proof. We simply connect the prescribed metric and connection to $d\sigma^2$ and the Riemannian connection with one parameter family of metrics and connections parametrized suitably by t. As above, if δ is sufficiently small, each δ-sphere bundle in the family will have scalar curvature bounded below by a positive constant times δ^{-2}. We may then choose the metrics to move very slowly with respect to t and apply (2.11) to finish the proof.

In the case in which the normal bundle is trivial we can show

Theorem 4. Let M, N, k, n be as in Theorem 3. There is a neighborhood V of N and a metric P of non-negative scalar curvature on $M \sim V$ which in a neighborhood of ∂V is a product of a line with $N \times S^{n-k-1}(\delta)$, N having any preassigned metric and $S^{n-k-1}(\delta)$ being the standard sphere of radius δ for a small $\delta > 0$.

Theorem 4 follows by applying Theorem 3 with a trivial connection.

Corollary 3. The connected sum of two compact manifolds of positive scalar curvature has a metric of positive scalar curvature.

<u>Corollary 4</u>. Let M_1, M_2 be compact n-dimensional manifolds of positive scalar curvature and N_1, N_2 compact k-dimensional submanifolds with $k < n - 2$. Suppose there is a fiber preserving diffeomorphism F of the normal bundle of N_1 to that of N_2. The new manifold formed by removing tubular neighborhoods of N_1 and N_2 and identifying the boundary sphere bundles via F has a metric of positive scalar curvature.

<u>Corollary 5</u>. The connected sum of two conformally flat manifolds of positive scalar curvature has a conformally flat metric of positive scalar curvature.

<u>Corollary 6</u>. If M is a compact manifold of positive scalar curvature, then any manifold which can be obtained from M by surgeries of codimension at least three also has a metric of positive scalar curvature.

<u>Corollary 7</u>. Let M, N be as in Theorem 3, and suppose N is the boundary of a k+1-dimensional manifold $\bar{N}$. Suppose the normal sphere bundle of N extends to a sphere bundle over $\bar{N}$. Then the manifold which is gotten by replacing a neighborhood of N in M with the extended sphere bundle of $\bar{M}$ has a metric of positive scalar curvature.

<u>Remark 2</u>. We have learned recently that Gromov and Lawson have independently obtained some form of our surgery results.

The proofs of the corollaries follow easily from Theorems 3 and 4. Corollary 4 follows by applying Theorem 3 to M_1, N_1 with metric and connection gotten by pulling back those from N_2. Note that by choosing δ smaller if necessary we can always make the radii of the boundary sphere bundles of Theorem 3 coincide for N_1 and N_2. Corollary 5 follows because it is easy to see that a metric of the form

$$u^{\frac{4}{n-2}}\,(dr^2+r^2d\theta^2+dh(r)\otimes dh(r))$$

is conformally flat where $d\theta^2$ is the unit sphere metric. To prove Corollary 6, note that if S^k is an embedded k-sphere with trivial normal bundle, then by Theorem 4 we have a metric of non-negative scalar curvature on $M \sim (S^k \times D_\delta^{n-k})$ which on the boundary can be taken as a product $S^k(1) \times S^{n-k-1}(\delta)$ of standard spheres. Take a metric on D^{k+1} which is a product of a line with $S^k(1)$ near the boundary, and choose $S^{n-k-1}(\eta)$ to be a standard sphere of radius η, so small that $D^k \times S^{n-k-1}(\eta)$ has positive scalar curvature (note $n-k-1 \geq 2$). Since both η, δ are arbitrarily small, we may take $\delta = \eta$ and complete the surgery. Corollary 7 follows by a similar application of Theorem 3.

Appendix.

In this appendix we sketch a proof of the existence of a positive function u satisfying (2.1). We are assuming that N is a compact embedded k-dimensional submanifold of a compact n-dimensional manifold M having scalar curvature $R > 0$. We are also assuming $k < n - 2$. Let $G(P,Q)$ be the Green's function on M for the operator $L = \Delta - \frac{n-2}{4(n-1)} R$. Since $R > 0$, $G(P,Q)$ exists, and by the maximum principle $G(P,Q)$ does not change sign. We take $G(P,Q)$ to be positive. (See [12, p. 136] for the construction of $G(P,Q)$.) Green's formula then says that for any function f on M, the function ϕ defined by

$$\phi(P) = - \int_M G(P,Q)f(Q)dQ$$

satisfies the equation $L\phi = f$. We let $\rho(P,Q)$ denote the intrinsic distance from P to Q, and we recall that for $n \geq 3$,

$$G(P,Q) = O(\rho(P,Q)^{2-n}) \quad . \tag{A.1}$$

We let $r(P)$ be the distance from P to N, and note that r is a Lipschitz function and r^2 is smooth in a neighborhood of N. Let ϕ_o be a smooth function on $M \sim N$ satisfying $\phi_o = r^{2+k-n}$ in a deleted neighborhood of N. It is straightforward to check that $f = L\phi_o$ satisfies

$$f = O(r^{1+k-n}) \quad \text{near } N \ . \tag{A.2}$$

Let $\{f_n\}$ be a sequence of bounded functions on N which converge uniformly to f on compact subsets of $M \sim N$. Set $\phi_n(P) = -\int_M G(P,Q)f_n(Q)dQ$. By (A.2) we see that f is an L^1 function on M, so the bounded convergence theorem implies that for any $P \notin N$ we have $\phi_n(P) \to \phi(P)$ where

$$\text{(A.3)} \qquad \phi(P) = -\int_M G(P,Q)f(Q)dQ \quad .$$

Standard elliptic theory implies that ϕ is smooth on $M \sim N$ and satisfies $L\phi = f$ on $M \sim N$. It follows that the function $u = \phi_o - \phi$ satisfies $Lu = 0$ on $M \sim N$. We now must study the asymptotic behavior of ϕ near N. We will be finished if we can show

$$\text{(A.4)} \qquad \begin{aligned} \phi &= 0(r^{3+k-n}) && \text{if } k < n-3 \\ \phi &= 0(\log r^{-1}) && \text{if } k = n-3 \quad . \end{aligned}$$

It will then follow from the maximum principle that $u > 0$ on $M \sim N$. To check (A.4) we take a point $P \notin N$ with $r(P)$ small, and let 0 be the nearest point of N to P, so that $r(P) = \rho(0,P)$. We choose a normal coordinate system centered at 0, call it $x^1,\ldots,x^n$, and suppose $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^k}$ span the tangent space to N at 0. Let δ be the radius of our normal coordinate ball which we denote $B_\delta(0)$. Since we only care about P close to N, we assume $r(P) < \delta/2$. Thus it follows from (A.1) that G(P,Q) is bounded for $Q \notin B_\delta(0)$, so that by (A.2) we see

$$\text{(A.5)} \qquad \int_{M\sim B_\delta(0)} G(P,Q)f(Q)dQ = 0(1) \quad .$$

To show (A.4), we are left with showing

$$\text{(A.6)} \quad \int_{B_\delta(0)} G(P,Q)f(Q)dQ = \begin{cases} 0(r^{3+k-n}) & \text{if } k < n-3 \\ 0(\log r^{-1}) & \text{if } k = n-3 \end{cases} \quad .$$

In our normal coordinate system, we let $x_1 = (x^1,\ldots,x^k)$ $x_2 = (x^{k+1},\ldots,x^n)$ so that $x = (x_1,x_2)$. Using (A.1), (A.2), and our choice of normal coordinates, we see that to prove (A.6) it suffices to show

$$\int_{B_\delta(0)} |x-y|^{2-n} \, |y_2|^{1+k-n} \, dy = \begin{cases} 0(|x_2|^{3+k-n}) & \text{if } k < n-3 \\ 0(\log |x_2|^{-1}) & \text{if } k = n-3 \end{cases} \quad .$$

where the integral is now a Euclidean integral in R^n. This may be seen by estimating the equivalent integral

$$\int_{B_\delta^k(0)\times B_\delta^{n-k}(0)} |x-y|^{2-n} \, |y_2|^{1+k-n} \, dy_1 dy_2 \quad .$$

Evaluating this as an iterated integral, one can first show

$$\int_{B_\delta^k(0)} |x-y|^{2-n} \, dy_1 = 0(|x_2-y_2|^{2+k-n}) \quad .$$

This reduces the problem to showing

$$\int_{B_\delta^{n-k}(0)} |x_2-y_2|^{2+k-n}\,|y_2|^{1+k-n}\,dy_2 = \begin{cases} 0(|x_2|^{3+k-n}) & \text{if } k < n-3 \\ 0(\log|x_2|^{-1}) & \text{if } k = n-3 \end{cases}$$

Both of these are elementary to check and we omit the proofs. This completes the proof of existence of u satisfying (2.1).

REFERENCES

[1] A. Lichnerowicz, Spineurs harmoniques, C. R. Acad. Sci., Paris sér A-B 257 (1963) 7-9

[2] N. Hitchin, Harmonic spinors, Advances in Math. 14 (1974) 1-55

[3] J. Milnor, Remarks concerning spin manifolds, Differential and Combinatorial Topology, a Symposium in Honor of Marston Morse, Princeton Univ. Press, 1965, 55-62

[4] R. Schoen and S. T. Yau, Incompressible minimal surfaces, three dimensional manifolds with nonnegative scalar curvature, and the positive mass conjecture in general relativity, Proc. Natl. Acad. Sci. 75, (6), p. 2567, 1978

[5] R. Schoen and S. T. Yau, Existence of incompressible minimal surfaces and the toplogy of three dimensional manifolds with non-negative scalar curvature, to appear in Annals of Math.

[6] R. Schoen and S. T. Yau, On the proof of the positive mass conjecture in general relativity, to appear in Comm. Math. Phys.

[7] J. Hempel, <u>3-manifolds</u>, Annals of Math. Studies <u>86</u>, Princeton Univ. Press 1976

[8] S. T. Yau, Remarks on the group of isometries of a Riemannian manifold, Topology <u>16</u> (1977), 239-247

[9] J. Kazdan and F. Warner, Prescribing curvatures, <u>Proc</u>. <u>Symp</u>. <u>in</u> <u>Pure</u> <u>Math</u>. <u>27</u> (1975) 309-319

[10] H. B. Lawson Jr., <u>Minimal varieties in real and complex geometry</u>, Univ. of Montreal, 1974 (lecture notes)

[11] S. S. Chern, <u>Minimal submanifolds in a Riemannian manifold</u>, Univ. of Kansas, 1968 (lecture notes)

[12] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Diff. Geom. <u>6</u>(1), 1971

[13] G. de Rham, <u>Variétés Différentiables</u>, Paris, Hermann, 1955

Richard Schoen
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, N. Y. 10012

Shing-Tung Yau
Department of Mathematics
Stanford Univeriity
Stanford, California 94305

(Received January 31, 1979)