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ON THE KORTEWEG-DE VRIES EQUATION

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Dedicated to Hans Lewy and Charles B. Morrey Jr.

Existence, uniqueness, and continuous dependence on the initial data are proved for the local (in time) solution of the (generalized) Korteweg-de Vries equation on the real line, with the initial function ϕ in the Sobolev space of order $s > 3/2$ and the solution $u(t)$ staying in the same space, $s = \infty$ being included. For the proper KdV equation, existence of global solutions follows if $s \geq 2$. The proof is based on the theory of abstract quasilinear evolution equations developed elsewhere.

1. Introduction

The purpose of this paper is to strengthen the results given in the previous paper [1] on the Cauchy problem for the (slightly generalized) KdV equation

(1.1) $$u_t + u_{xxx} + a(u)u_x = 0, \quad t \geq 0, \quad -\infty < x < \infty,$$

(1.2) $$u(0,x) = \phi(x).$$

Here all functions are real-valued, and a is assumed to be C^∞ (though we do not always need it).

In [1] a general theory of quasilinear equations of evolution was applied to (1.1-2), to deduce existence,

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uniqueness, and continuous dependence on the initial data for the solution, with the assumption that $\phi \in H^s$ for some $s \geq 3$ and with the solution u obtained in the class

$$(1.3) \qquad u \in C([0,T);H^s) \cap C^1([0,T);H^{s-3}),$$

where T depends on $\|\phi\|_s$. (Here $H^s = H^s(-\infty,\infty)$ is the Sobolev space of order s of L^2-type, with the norm denoted by $\| \ \|_s$.)

Similar results have been obtained, independently, by Bona and others [2,3] for any $s \geq 2$ and any $T > 0$ for the proper KdV equation (in which $a(u) = u$), and by Saut and Temam [4] for $s > 3/2$ and small T except for the continuous dependence in H^s. Since the methods used by these authors are quite different from ours, it would be worth while to show that the method used in [1] is also applicable to s smaller than 3. We shall indeed show that $s > 3/2$ suffices for the existence, uniqueness, and continuous dependence in H^s on the initial data for the local solution. We shall also show that global solutions are obtained if there exists a certain global estimate, which is the case with the proper KdV equation. (We note that recently Cohen Murray [5] proved the existence of global solutions for certain discontinuous initial data.)

More precisely, our theorems read.

THEOREM I. (a) *Let* $s > 3/2$ (*not necessarily an integer*). *For any* $\phi \in H^s$, *there is a unique solution* u *to* (1.1-2) *in the class* (1.3), *with* T *having a lower bound depending only on* $\|\phi\|_s$.

(b) *The map* $\phi \mapsto u(t)$ *is continuous in the* H^s-*norm*. *More precisely, if* $\phi_n \in H^s$, $n = 1,2,\ldots$, *with* $\|\phi_n-\phi\|_s \to 0$ *and* $T' < T$, *the solution* u_n *for* $u_n(0) = \phi_n$ *exists on* $[0,T']$ *for sufficiently large* n *and* $\|u_n(t)-u(t)\|_s \to 0$ *uniformly in* $t \in [0,T']$.

(c) <u>T may be chosen independent of</u> s <u>in the following sense. If</u> u <u>is a solution satisfying</u> (1.3) <u>and if</u> $u(0) = \phi \in H^{s'}$ <u>for some</u> $s' \neq s$, $s' > 3/2$, <u>then</u> (1.3) <u>is true also with</u> s <u>replaced by</u> s' <u>and with the same</u> T. <u>In particular,</u> $\phi \in H^{\infty} = \bigcap_s H^s$ <u>implies</u> (1.3) <u>with</u> $s = \infty$.

In order to obtain global solutions, we introduce

<u>CONDITION (G)</u>. <u>There are real numbers</u> $s_1 \geq s_0 > 3/2$ <u>and a monotone increasing function</u> q <u>on</u> $[0,\infty)$ <u>to itself, such that for any</u> $T > 0$ <u>and any function</u> $u \in C([0,T);H^{s_1})$ <u>satisfying</u> (1.1) <u>one has</u>

$$(1.4) \qquad \|u(t)\|_{s_0} \leq q(\|u(0)\|_{s_0}), \qquad t \in [0,T).$$

As is well known, Condition (G) is satisfied by the proper KdV equation with $s_0 = 2$. Here s_1 may be chosen sufficiently large that the <u>formal derivation</u> of the inequality is justified (say $s_1 = 4$). We need no further estimates to deduce global existence for all $s \geq s_0$. We have namely

<u>THEOREM II.</u> <u>Assume Condition</u> (G). <u>If</u> $s \geq s_0$, <u>Theorem</u> I <u>is true with</u> $T = \infty$.

For the proper KdV equation, Theorem II shows that a global solution exists whenever $\phi \in H^s$ with $s \geq 2$, and stays in H^s. Theorem I shows that a local solution exists if $\phi \in H^s$ with $3/2 < s < 2$, but we do not know whether or not it can be extended to a global solution.

<u>2. Local solutions</u>

We shall first prove parts (a), (b) of Theorem I by applying the general theory of [1]. Before doing so, we make a preliminary transformation of the unknown by

$$(2.1) \qquad u(t) = P(t)v(t), \quad P(t) = e^{-tD^3}, \quad D = d/dx,$$

where the space variable x is suppressed in $u(t) = u(t,x)$, etc. Note that D^3 is skew-adjoint in each H^s so that $\{P(t)\}$ forms a strongly continuous unitary group on H^s.

Substitution of (2.1) into (1.1-2) yilds a quasilinear equation of evolution for the unknown $v = v(t)$:

$$(2.2) \qquad dv/dt + A(t,v)v = 0, \qquad v(0) = \phi,$$

where $A(t,y)$ is a linear operator depending on t and $y \in H^s$:

$$(2.3) \qquad A(t,y) = P(-t)a(P(t)y)DP(t).$$

Here $a(P(t)y)$ is the operator of multiplication by the function $x \mapsto a((P(t)y)(x))$.

(2.2) belongs to the class of quasilinear equations of evolution for which the theory of [1] is applicable; it is simpler than (1.1) in that $A(t,v)$ is an operator of "first order," although it is not a differential operator or a pseudo-differential operator in familiar classes.

To apply the algorithm of [1] to (2.2), we choose the basic spaces

$$(2.4) \qquad X = H^0, \qquad Y = H^s.$$

Let W be the open ball $\|y\|_s < R$ in Y. According to [1], the parts (a), (b) of Theorem I will follow if we verify the following lemmas. (In these lemmas it is assumed that $t \geq 0$, $y, z \in W$; note that $W \subset C^1 \cap L^\infty$ due to $s > 3/2$.)

<u>LEMMA 2.1.</u> $A(t,y) \in G(X,1,\beta)$ <u>with</u> β <u>depending only on</u> R. <u>In other words,</u> $-A(t,y)$ <u>is the generator of a</u> C_0-<u>semigroup on</u> X <u>satisfying</u>

$$(2.5) \qquad (A(t,y)w,w)_X \geq -\beta\|w\|_X^2, \qquad w \in C_0^\infty.$$

PROOF. Since $P(t)$ is unitary on X, it suffices to note that (2.5) holds for the operator $a(P(t)y)D$ (cf. [1,Section 8]); one can take

$$2\beta = \sup_{y\in W}\|a(P(t)y)_x\|_{L^\infty} = \sup_{y\in W}\|a(y)_x\|_{L^\infty} \le M\sup_{y\in W}\|a(y)_x\|_{s-1} < \infty$$

because $s-1 > 1/2$. (M is a constant.)

LEMMA 2.2. *Let* $S = \Lambda^s$, *where* $\Lambda = (1-D^2)^{1/2}$. S *is an isomorphism of* Y *onto* X, *and*

$$SA(t,y)S^{-1} = A(t,y) + B(t,y), \tag{2.6}$$

$$\|B(t,y)\|_X \le \lambda = \lambda(R), \tag{2.7}$$

$$\|B(t,y) - B(t,z)\|_X \le \mu(R)\|y-z\|_Y, \tag{2.8}$$

where $\lambda(R)$, $\mu(R)$ *are constants depending only on* R *and* a.

PROOF. To prove (2.6), it suffices to verify that

$$[S,A(t,y)]S^{-1} = P(-t)[S,a(P(t)y)]DS^{-1}P(t) \subset B(t,y) \tag{2.9}$$

is a bounded operator in X. ($[\ ,\ ]$ denotes the commutator.) This is true because $[S,a(P(t)y)]DS^{-1}$ is bounded, as is seen from Lemma 2.6 given below, together with the estimate (2.7). (2.8) can be proved in the same way.

LEMMA 2.3. *We have, with another constant* $\mu_1(R)$,

$$\|A(t,y) - A(t,z)\|_{Y,X} \le \mu_1(R)\|y-z\|_X . \tag{2.10}$$

PROOF. Since $P(t)$ is unitary on X and Y, (2.10) follows from $\|a(P(t)y)D - a(P(t)z)D\|_{Y,X} \le \text{const}\|y-z\|_X$ $= \text{const}\|P(t)y-P(t)z\|_X$, which is proved in [1,Section 8].

LEMMA 2.4. $t \mapsto A(t,y) \in B(Y,X)$ is strongly continuous.

PROOF. Since D is bounded on Y to X and since $P(t)$ is a strongly continuous unitary group on X, it suffices to note that the multiplication operator $a(P(t)y) \in B(X)$ is strongly continuous in t. This is true even in the norm-topology, since $P(t)y \in H^s$ is continuous in t.

REMARK 2.5. To apply the general theory of [1], we would need norm-continuity in Lemma 2.4 rather than strong continuity. Fortunately, it has been shown that strong continuity is sufficient. This fact, essentially a problem for linear evolution equations, follows from the results of Darmois [6]. Darmois's thesis has not been published, but similar results are contained in a recent paper by Da Prato and Iannelli [7]. Another proof was recently given by Kobayashi [8].

With this remark, the theory of [1] can be applied to (2.2), yielding a unique solution $v \in C([0,T);H^s)$. Then the transformation (2.1) gives a solution u of (1.1-2) in the class (1.3). This proves the parts (a), (b).

We conclude this section with a technical lemma, which was used above and will be used later too.

LEMMA 2.6. Let $f \in H^r(R^m)$ for some $r > m/2 + 1$. Then

$$\|\Lambda^{-s}[\Lambda^{s+t+1}, M_f]\Lambda^{-t}\| \leq c\|\text{grad } f\|_{r-1}, \qquad |s|,|t| \leq r-1,$$

where $\Lambda = (1-\Delta)^{1/2}$, M_f is the operator of multiplication by f, and the norm $\| \ \|$ on the left is the operator norm in $L^2(R^m)$.

The proof is identical with that for Lemma A2 in [1] (which is a special case) and may be omitted.

3. The interval of existence

We now prove part (c) of Theorem I. Suppose that u is a solution of (1.1-2) satisfying (1.3). It suffices to consider the case $s' > s$, since the case $s' < s$ is obvious from uniqueness. As is seen from the following proof, this is essentially a linear problem.

We return to equation (2.2) for v and apply the operator D^2, obtaining the following linear evolution equation for $w(t) = D^2v(t)$:

$$(3.1) \qquad dw/dt + A(t)w + B(t)w = f(t),$$

$$(3.2) \qquad A(t) = DP(-t)a(u(t))P(t),$$

$$(3.3) \qquad B(t) = 2P(-t)a'(u(t))u_x(t)P(t),$$

$$(3.4) \qquad f(t) = -P(-t)a''(u(t))u_x(t)^3,$$

where

$$(3.5) \qquad u \in C([0,T);H^s)$$

is regarded as a known function. (Subscript x in (3.3) and (3.4) denotes differentiation in x, and a' and a'' denote the derivatives of the function a.)

We already know that $w \in C([0,T);H^{s-2})$ because $v \in C([0,T);H^s)$. Furthermore, $w(0) = \phi_{xx} \in H^{s'-2}$ because $\phi \in H^{s'}$. It is our purpose to deduce $w \in C([0,T);H^{s'-2})$, which will imply $v \in C([0,T);H^{s'})$, hence the same for u, completing the proof of part (c). To this end, we have to study the linear evolution equation (3.1) in more detail. In particular, we are concerned with the evolution operator $\{U(t,\tau)\}$ associated with the family $\{A(t)\}$.

<u>LEMMA 3.1.</u> <u>The family $\{A(t)\}$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$, $Y = H^k$ in the sense of [1], where</u>

(3.6) $-s \leq h \leq s-2, \quad 1-s \leq k \leq s-1, \quad \text{and} \quad k \geq h+1.$

<u>In particular</u>, $U(t,\tau)$ <u>maps</u> H^r <u>into itself for</u> $-s \leq r \leq s-1$.

<u>PROOF.</u> The proof parallels that of Theorem I, consisting of verifying three conditions. First we show that $A(t) \in G(X,1,\beta)$, which is less trivial than before since h is not zero in general. We have to show that $(A(t)z,z)_h \geq -\beta\|z\|_h^2$, where $(\ ,\ )_h$ denotes the inner product in H^h. In view of the unitarity of $P(t)$, this is equivalent to

$$(3.7) \qquad -(\Lambda^h a(u(t))z, D\Lambda^h z) \geq -\beta\|\Lambda^h z\|^2,$$

where $(\ ,\ ) = (\ ,\ )_0$ and $\|\ \| = \|\ \|_0$.

If we write

$$(3.8) \qquad \Lambda^h a(u(t)) = a(u(t))\Lambda^h + [\Lambda^h, a(u(t))],$$

the contribution of the first term to (3.7) can be dealt with by the standard technique of integration by parts, yielding an estimate of the desired form. The contribution from the commutator in (3.8) can be estimated by Lemma 2.6, to yield the same estimate.

Second, we have to take the isomorphism $S = \Lambda^{k-h}$ of Y onto X and verify that $B(t) \supset [S,A(t)]S^{-1} = DP(-t)[S,a(u(t))]S^{-1}P(t)$ is uniformly bounded on X. This is equivalent to that $\Lambda^h D[\Lambda^{k-h}, a(u(t))]\Lambda^{-k}$ is (uniformly) L^2-bounded, which again follows from Lemma 2.6.

Finally, it is easy to see that $A(t) \in B(Y,X)$ is strongly continuous in t. For the sufficiency of strong continuity for our purpose, see Remark 2.5.

This proves the existence and uniqueness of the evolution operator $\{U(t,\tau)\}$ for the family $\{A(t)\}$. A priori, $\{U(t,\tau)\}$ may depend on the choice of X and Y. Actually we have a unique one for all $X = H^h$ and $Y = H^k$ with h, k satisfying (3.6), due to the uniqueness theorem for the evolution operator (see [9]).

LEMMA 3.2. We have

$$(3.9)\qquad w(t) = U(t,0)w(0) + \int_0^t U(t,\tau)[-B(\tau)w(\tau)+f(\tau)]d\tau .$$

PROOF. In Lemma 3.1, choose $h = s-3$ and $k = s-2$ (which satisfy (3.6)). Since $w \in C([0,T);Y) \cap C^1([0,T);X)$, as is easily verified by (3.5), we can carry out the standard computation

$$dU(t,\tau)w(\tau)/d\tau = U(t,\tau)[dw(\tau)/d\tau + A(\tau)w(\tau)]$$

$$= U(t,\tau)[-B(\tau)w(\tau) + f(\tau)]$$

and obtain (3.9) on integration in $\tau \in [0,t]$ (see e.g. [9]). This is essentially a uniqueness proof for the solution w of (3.1) in the class $C([0,T);H^{s-2})$.

LEMMA 3.3. $w \in C([0,T);H^{s'-2})$.

PROOF. We have $w(0) \in H^{s'-2}$ as mentioned above. Also (3.4-5) imply that $f \in C([0,T);H^{s-1}) \subset C([0,T);H^{s'-2})$ if $s' \leq s+1$, since u_x is in the same class and $s-1 > 1/2$. For the same reason, (3.3) shows that $B(t) \in B(H^{s'-2})$ is strongly continuous in $t \in [0,T)$ if $s' \leq s+1$; note that $H^{s-1}\cdot H^{s'-2} \subset H^{s'-2}$ by $s-1 > 1/2$. Since the family $\{U(t,\tau)\}$ is strongly continuous on $H^{s'-2}$ to itself (see Lemma 3.1), the required result follows from (3.9); we have only to regard (3.9) as an integral equation of Volterra type, which can be solved for w by successive approximation.

This proves the lemma under the additional condition $s' \leq s+1$. If $s' > s+1$, we obtain the result by repeated application of the above argument.

4. Global solutions

We now prove Theorem II.

LEMMA 4.1. (1.4) *is true for any solution* u *in* $C([0,T);H^{s_0})$, *not necessarily in* $C([0,T);H^{s_1})$.

PROOF. Let u be such a solution with $u(0) = \phi \in H^{s_0}$. Let $\phi_n \in H^{s_1}$, $n = 1,2,\ldots$, be a sequence such that $\phi_n \to \phi$ in H^{s_0}, and let u_n be the solution of (1.1) with $u_n(0) = \phi_n$. Given any $T' < T$, we have $u_n \in C([0,T');H^{s_0})$ for sufficiently large n, with $u_n(t) \to u(t)$ in H^{s_0}, by part (b) of Theorem I. On the other hand, part (c) shows that $u_n \in C([0,T');H^{s_1})$. Thus (1.4) holds for u replaced by u_n. Since $u_n(t) \to u(t)$ in H^{s_0}, (1.4) must hold also for u.

Once Lemma 4.1 is proved, the standard argument can be used to continue any solution $u \in C([0,T);H^{s_0})$ to all time $t > 0$ in a finite number of steps, since the local solution exists for a time interval depending only on $\|\phi\|_{s_0}$. The same is true for any $s > s_0$, due to part (c) of Theorem I. The continuous dependence, so far proved only locally, can be extended as usual to all $t > 0$ step by step. This completes the proof of Theorem II.

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