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Nets of C^*-Algebras and Classification of States*

R. HAAG
II. Institut für Theoretische Physik der Universität Hamburg

R. V. KADISON
Department of Mathematics, University of Pennsylvania, Philadelphia

D. KASTLER
Centre Universitaire de Marseille-Luminy

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Abstract. The concept of locality in quantum physics leads to mathematical structures in which the basic object is an operator algebra with a net of distinguished subalgebras (the "local" subalgebras). Such nets provide a classification of the states of this algebra in equivalence classes determined by local or asymptotic properties. The corresponding equivalence relations are natural generalizations of the (more stringent) standard quasi-equivalence relation (they are also useful for classifying states by their properties with respect to automorphism groups). After discussing general nets from this point of view we investigate in the last section more specialized nets (funnels of von Neumann algebras) with special emphasis on their locally normal states.

Introduction

In the algebraic approach to Quantum Field Theory or Statistical Mechanics one deals with a C^*-algebra $\mathfrak{A}$ with a distinguished collection of subalgebras $\mathfrak{A}_\alpha$. The physical significance of the index α is usually to specify a region in Minkowski space (resp. Euclidean space). Then $\mathfrak{A}_\alpha$ is the algebra generated by the physical operations (or observables) which can be performed in the specified region. The collection $\{\mathfrak{A}_\alpha\}$ provides a "net" for $\mathfrak{A}$ in the sense of Definition 2 below and for many purposes we may assume that it is a "funnel" (see Definition 7).

Parallel to observables and operations we have to consider the physical states. In the mathematical frame they are given by positive linear forms (expectation functionals) over the algebra. The set of these forms is denoted by $\mathfrak{A}^{*+}$. One may take the attitude that each $\omega \in \mathfrak{A}^{*+}$ corresponds to a physical state, but that no actual experimental arrangement can prepare a state precisely. Rather an experiment specifies a weak neighborhood in the space of positive linear forms. This is the point

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of view adopted in [1]. It results if one assumes that $\mathfrak{A}$ contains all observables and that no further knowledge about a state is available beyond the information obtained from the measurement of a finite number of observables. A consequence of this point of view is that one can treat all experimentally relevant questions by using a very restricted set of states, namely the vector states of one faithful representation of $\mathfrak{A}$ because such a set is weakly dense in $\mathfrak{A}^{*+}$. Moreover, the choice of this representation is arbitrary. Another possible point of view [2] is to assume that the "physical states" are a distinguished subset $\mathscr{S} \subset \mathfrak{A}^{*+}$ so that the theory specifies $(\mathfrak{A}, \mathscr{S})$ rather than $\mathfrak{A}$ alone. The two points of view are reconciled if one acknowledges that in each case an idealization is made and that the idealizations have to be judged by their convenience rather than by fundamental principles. In the first point of view the idealization consists in the choice of the algebra $\mathfrak{A}$. This specifies precisely what is supposed to be observable. In practice the precise choice of the local subalgebras $\mathfrak{A}_\alpha$ may be a matter of dispute. For example, in the non-relativistic many body problem of a single type of Fermi particle, interpreting the index α to specify a finite region $\mathbf{V}$ in 3-dimensional Euclidean space at a time t and writing $\mathfrak{A}(\mathbf{V}, t)$ instead of $\mathfrak{A}_\alpha$, the simplest choice for $\mathfrak{A}(\mathbf{V}, t)$ would be to take the smallest C^*-algebra containing all creation- and destruction operators for a particle whose wave function at time t has support in $\mathbf{V}$. This should first be considered as defining a collection of concrete operator algebras, denoted by $\mathbf{C}(\mathbf{V}, t)$, in Fock space. Each $\mathbf{C}(\mathbf{V}, t)$, as a C^*-algebra, is isomorphic to the Clifford algebra over a separable Hilbert space. If we regard the collection of all these $\mathbf{C}(\mathbf{V}, t)$ as a net of C^*-algebras (keeping their mutual relations in so far as they are independent of the realization in Fock space) then we encounter the following two problems.

First, this algebraic structure allows many states which are not physically realizable in realistic systems, namely states which describe an actual infinity of particles within a finite volume $\mathbf{V}$. A "physical state" should be "locally normal" [3] with respect to the Fock representation, i.e. the restrictions of all physical states to one local subalgebra lie in one quasi-equivalence class. This allows us to choose as the local algebra $\mathfrak{A}(\mathbf{V}, t)$ instead of $\mathbf{C}(\mathbf{V}, t)$ its weak closure $\mathscr{M}(\mathbf{V}, t)$ in the Fock representation or any other C^*-algebra in Fock space whose weak closure coincides with $\mathscr{M}(\mathbf{V}, t)$. We can use this freedom to return to the first mentioned point of view by building up the local algebra $\mathfrak{A}(\mathbf{V}, t)$ from the relatively compact operators in $\mathscr{M}(\mathbf{V}', t)$ with $\mathbf{V}' \subset \mathbf{V}$.

Secondly, there is the problem of the relation between the algebras associated with different times. The dynamical law, formulated for instance by the Heisenberg equations of motion for the creation operators, means that a local observable at time t should be expressible also in terms of the observables at an arbitrary other time. One may therefore hope that one can (by a proper choice of the local algebras $\mathfrak{A}(\mathbf{V}, t)$) define a "kinematical algebra"

$$\mathfrak{A}(t) = \overline{\bigcup_{\mathbf{V}} \mathfrak{A}(\mathbf{V}, t)} \tag{1}$$

in such a way that

$$\mathfrak{A}(t) = \mathfrak{A}(t') = \mathfrak{A}\,, \tag{2}$$

i.e. that the kinematical algebra at an arbitrary time is already the full algebra mentioned at the beginning of this section. In that case the group of time translations will be a subgroup of the automorphisms of $\mathfrak{A}$ (i.e. time translations will be represented by automorphisms of $\mathfrak{A}$). If we choose $\mathfrak{A}(\mathbf{V}, t) = \mathbf{C}(\mathbf{V}, t)$ then for a system of non-interacting particles and also for sufficiently "mild interactions" Eqs. (1), (2) will be true. They cease to be true, however, in the case of most practical interest, where the interaction is given by a velocity independent 2-particle potential. One might be inclined to put $\mathfrak{A}(\mathbf{V}, t) = \mathscr{M}(\mathbf{V}, t)$ in that case. It has been pointed out to us by Araki that such a choice can only make matters worse. A very simple argument shows that with this choice of $\mathfrak{A}(\mathbf{V}, t)$ Eq. (2) does not even hold for the non-interacting particle system. It is at present unknown whether a suitable choice for $\mathfrak{A}(\mathbf{V}, t)$ can be found for which Eqs. (1), (2) hold in a sufficiently general class of interactions. This question will, however, not be studied here.

The objective of the present paper is to study different ways of classifying states of $\mathfrak{A}$ into equivalence classes. We shall assume that we are dealing with a distinguished subset $\mathscr{S} \subset \mathfrak{A}^{*+}$ of states. The various definitions of "equivalence" will be related to various properties with respect to a net of subalgebras $\{\mathfrak{A}_\alpha\}$.

The last section of the paper, dealing with "funnels" of von Neumann algebras, can be read independently of the earlier sections (except for the definition of nets of subalgebras). In particular it makes no use of the concepts of relative equivalence and containment.

§ 1. Relative Equivalence and Relative Containment of Representations of a C^*-Algebra

In this section we describe a generalization of the notions of quasi-equivalence and quasi-containment of representations (or states) of a C^*-algebra[1]. This will provide a unified language for dealing with the type of situations mentioned in the Introduction and discussed in Section 2 below. After a statement of definitions and notation, we list

[1] We refer to [4] the reader interested in the corresponding relative central decompositions of states.

results without giving the proofs which can easily be adapted from known arguments ([5], Chapter I, [6], Chapter 5).

Let $\mathfrak{A}$ be a C^*-algebra with double dual (von Neumann enveloping algebra) $\mathfrak{A}^{**}$. We first remind the reader that one can define quasi-equivalence (quasi-containment) of representations of $\mathfrak{A}$ by using as follows the center $\mathscr{Z}$ of $\mathfrak{A}^{**}$: if, with π a representation of $\mathfrak{A}$, we denote by $\bar{\pi}$ its ultraweakly continuous extension to $\mathfrak{A}^{**}$, by K_π^0 the generating projector of the kernel of $\bar{\pi}$; and if we set $S_\pi^0 = I - K_\pi^0$ (so that $K_\pi^0, S_\pi^0 \in \mathscr{Z}$), π_1 is quasi-equivalent to (quasi-contained in) π_2 if and only if $S_{\pi_1}^0 = S_{\pi_2}^0$ ($S_{\pi_1}^0 \leqq S_{\pi_2}^0$). In this way quasi-equivalence classes of representations of $\mathfrak{A}$ are set into a one-to-one correspondance with projectors of $\mathscr{Z}$. Furthermore, with M an arbitrary set of representations of $\mathfrak{A}$ and $\varrho = \bigoplus_{\pi \in M} \pi$, one has $S_\varrho^0 = \bigvee_{\pi \in M} S_\pi^0$ [2]; and the minimal projectors of $\mathscr{Z}$ correspond to the quasi-equivalence classes of primary (factor) representations of $\mathfrak{A}$. Our generalization will now consist on the one hand in replacing $\mathfrak{A}^{**}$ by the closure $[\mathscr{S}]^*$ of $\mathfrak{A}$ in the weak topology of operators determined by some faithful representation (having $\mathscr{S}$ as its set of normal positive linear forms); and on the other in taking instead of $\mathscr{Z}$ an arbitrary von Neumann subalgebra $\mathscr{Z}_1$ of the center of $[\mathscr{S}]^*$. We first introduce some terminology and notation.

Let $\mathfrak{A}^{*+}$ be the set of positive linear forms over $\mathfrak{A}$. A subset of $\mathfrak{A}^{*+}$ will be called a *folium* [3] whenever $\mathscr{S}$ is (i) closed under convex combinations (ii) closed in the norm-topology of linear forms (iii) such that for each $\varphi \in \mathscr{S}$ and $A \in \mathfrak{A}$, $\varphi_A = \varphi(A^* \cdot A) \in \mathscr{S}$. The normal positive forms of any representation π of $\mathfrak{A}$ form a folium which we denote by $\mathscr{S}_\pi$. Conversely any folium $\mathscr{S}$ is of the form $\mathscr{S}_\pi$ for some representation π which we can take to be the direct sum $\pi = \bigoplus_{\varphi \in \mathscr{S}} \pi_\varphi$ of all cyclic representations π_φ determined by the elements $\varphi \in \mathscr{S}$. Given two representations π_1 and π_2 of $\mathfrak{A}$ π_1 is quasi-equivalent to (quasi-contained in) π_2 iff $\mathscr{S}_{\pi_1} = \mathscr{S}_{\pi_2}$ ($\mathscr{S}_{\pi_1} \subseteqq \mathscr{S}_{\pi_2}$). This establishes a one-to-one order-preserving correspondence between the folia of $\mathfrak{A}^{*+}$ and the quasi-equivalence classes of representations of $\mathfrak{A}$. (With φ a state of $\mathfrak{A}$ we note that the normal folium $\mathscr{S}_{\pi_\varphi}$ of the representation π_φ generated by φ is the smallest folium containing φ.)

Let now $\mathscr{S}$ be a folium in $\mathfrak{A}^{*+}$, with $\tau_{\mathscr{S}} = \bigoplus_{\varphi \in \mathscr{S}} \pi_\varphi$, so that $\mathscr{S} = \mathscr{S}_{\tau_{\mathscr{S}}}$. We assume that $\mathscr{S}$ is w^*-dense in $\mathfrak{A}^{*+}$, so that $\tau_{\mathscr{S}}$ is a faithful representation [4]; and identify $A \in \mathfrak{A}$ with $\tau_{\mathscr{S}}(A)$ acting on the representation space

[2] $\bigvee$ denotes the union of projectors.

[3] Our folia correspond to the norm-closed invariant faces of [7, 8].

[4] By Theorem (2.2) of [2] this is equivalent to the assumption that $\mathscr{S}$ is a *full folium*, i.e. that $A \geqq 0$ if $\varphi(A) \geqq 0$ for all φ in $\mathscr{S}$.

$\mathscr{H}_{\tau_{\mathscr{S}}}$ of $\tau_{\mathscr{S}}$, thus considering $\mathfrak{A}$ as a concrete C^*-algebra acting on $\mathscr{H}_{\tau_{\mathscr{S}}}$. The linear hull $[\mathscr{S}]$ of $\mathscr{S}$ is a Banach subspace of the strong dual $\mathfrak{A}^*$ of $\mathfrak{A}$, consisting of all the linear forms of $\mathfrak{A}$ continuous in the ultraweak topology of operators on $\mathscr{H}_{\tau_{\mathscr{S}}}$ (or the weak topology of $\mathfrak{A}$ with respect to $[\mathscr{S}]$). The norm dual $[\mathscr{S}]^*$ of the Banach space $[\mathscr{S}]$, then coïncides with the weak closure, $\tau_{\mathscr{S}}(\mathfrak{A})''$, of $\mathfrak{A}$ acting on $\mathscr{H}_{\tau_{\mathscr{S}}}$, since $[\mathscr{S}]$ is the set of ultraweakly continuous linear forms (the predual) of $\tau_{\mathscr{S}}(\mathfrak{A})''$. Therefore $[\mathscr{S}]^* = \tau_{\mathscr{S}}(\mathfrak{A})''$ is a von Neumann algebra, the strong and w^*-topologies of $[\mathscr{S}]^*$ as a dual space coinciding respectively with the norm and ultraweak topology of operators. (In the particular case when $\mathscr{S}$ consists of all positive forms of $\mathfrak{A}$, $\mathscr{S} = \mathfrak{A}^{*+}$, $\tau_{\mathscr{S}}$ is the universal representation τ of $\mathfrak{A}$ and $[\mathscr{S}]^* = \mathfrak{A}^{**}$, the von Neumann enveloping algebra of $\mathfrak{A}$.) Let $\mathscr{R}_{\mathscr{S}}$ be the set of representations π of $\mathfrak{A}$ quasi-contained in $\tau_{\mathscr{S}}$ (i.e. such that $\mathscr{S}_\pi \subseteqq \mathscr{S}$). As a generalization of the universal property of $\mathfrak{A}^{**}$, we have that each representation $\pi \in \mathscr{R}_{\mathscr{S}}$ (each state $\varphi \in \mathscr{S}$) uniquely extends to a normal representation $\bar{\pi}$ (state $\bar{\varphi}$) of $[\mathscr{S}]^*$; moreover $\pi([\mathscr{S}]^*)$ is the weak closure of the concrete C^*-algebra $\pi(\mathfrak{A})$; and $\bar{\pi}_\varphi = \pi_{\bar{\varphi}}$. Denoting by $\mathscr{Z}_{\mathscr{S}}$ the center of the von Neumann algebra $[\mathscr{S}]^*$, we are now in a position to give

Definition 1. *Let $\mathscr{Z}_1$ be an arbitrary von Neumann subalgebra of $\mathscr{Z}_{\mathscr{S}}$. With $\pi \in \mathscr{R}_{\mathscr{S}}$, we denote by K_π the generating projector in $\mathscr{Z}_1$ of the kernel of $\bar{\pi} | \mathscr{Z}_1$, the restriction of $\bar{\pi}$ to $\mathscr{Z}_1$; and by $S_\pi = I - K_\pi$ the complementary projector of K_π in $\mathscr{Z}_1$ (called the $\mathscr{Z}_1$-carrier of π)* [5]. *With π_1, π_2 in $\mathscr{R}_{\mathscr{S}}$, we say that*

(i) *π_1 is $\mathscr{Z}_1$-contained in π_2 ($\pi_1 \underset{1}{\propto} \pi_2$) whenever $S_{\pi_1} \leqq S_{\pi_2}$ (or $K_{\pi_1} \geqq K_{\pi_2}$),*

(ii) *π_1 and π_2 are $\mathscr{Z}_1$-equivalent ($\pi_1 \underset{1}{\sim} \pi_2$) whenever $S_{\pi_1} = S_{\pi_2}$ (or $K_{\pi_1} = K_{\pi_2}$),*

(iii) *π_1 and π_2 are $\mathscr{Z}_1$-disjoint ($\pi_1 \underset{1}{\circ} \pi_2$) whenever $S_{\pi_1} S_{\pi_2} = 0$ (or $K_{\pi_1} \vee K_{\pi_2} = I$).*

Further $\pi \in \mathscr{R}_{\mathscr{S}}$ is called $\mathscr{Z}_1$-primary whenever $\pi_1 \underset{1}{\propto} \pi$ implies $\pi_1 \underset{1}{\sim} \pi$ for all $\pi_1 \in \mathscr{R}_{\mathscr{S}}$ (or else if S_π is a minimal projector of $\mathscr{Z}_1$).

For states of $\mathfrak{A}$, $\mathscr{Z}_1$-containment (-equivalence, -disjointness, -primariness) are defined as the same circumstances for the associated representations.

We have thus defined an ordering $\underset{1}{\propto}$ of $\mathscr{R}_{\mathscr{S}}$, and associated equivalence $\underset{1}{\sim}$, which reduce to quasi-containment and quasi-equivalence in the case $\mathscr{S} = \mathfrak{A}^{*+}$ and $\mathscr{Z}_1 = \mathscr{Z}$ [6]. We note that the above definitions are invariant under addition of a null representation as a direct summand

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[5] With K_π^0 the generating projector of the kernel of $\bar{\pi}$ in $[\mathscr{S}]^*$ and $S_\pi^0 = I - K_\pi^0$ we note that K_π is the greatest projector of $\mathscr{Z}_1$ smaller than K_π^0, and S_π the smallest projector of $\mathscr{Z}_1$ greater than S_π^0.

[6] Or in the case of a general $\mathscr{S}$ and $\mathscr{Z}_1 = \mathscr{Z}_{\mathscr{S}}$, to the usual notion of quasi-containment and equivalence restricted to representations in $\mathscr{R}_{\mathscr{S}}$.

to the representations under consideration. In this respect our generalized disjointness and primariness notions differ slightly from the usual ones in the case of degenerate representations: for $\mathscr{S} = \mathfrak{A}^{*+}$ and $\mathscr{Z}_1 = \mathscr{Z}$, $\mathscr{Z}_1$-disjointness of π_1 and π_2 in our sense means that the essential parts[7] of π_1 and π_2 are disjoint in the usual sense; and analogously π will be $\mathscr{Z}$-primary in our sense if and only if it is the direct sum of a null representation and a primary representation defined as usual. For essential representations all our generalized notions merge exactly into the usual ones in the case $\mathscr{S} = \mathfrak{A}^{*+}$ and $\mathscr{Z}_1 = \mathscr{Z}$. Obviously, two $\mathscr{Z}_1$-primary representations are either $\mathscr{Z}_1$-equivalent or $\mathscr{Z}_1$-disjoint. The following statements are listed without proofs, since most of them are straightforward generalizations of known facts in the theory of quasi-equivalence of representations.

(a) *Let* $\pi = \bigoplus_{\nu \in I} \pi_\nu$ *with* $\pi_\nu \in \mathscr{R}_{\mathscr{S}}$, $\nu \in I$: *then* $\pi \in \mathscr{R}_{\mathscr{S}}$ *and* $\bar{\pi} = \bigoplus_{\nu \in I} \overline{\pi_\nu}$. *Furthermore* $K_\pi = \bigcap_{\nu \in I} K_{\pi_\nu}$ *and* $S_\pi = \bigvee_{\nu \in I} S_{\pi_\nu}$. *In particular for* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$, $K_{\pi_1 \oplus \pi_2} = K_{\pi_1} K_{\pi_2}$ *and* $S_{\pi_1 \oplus \pi_2} = S_{\pi_1} + S_{\pi_2} - S_{\pi_1} S_{\pi_2}$.

Combining this with the preceding definition we have that:

(b) *There is a one-to-one correspondence between* $\mathscr{Z}_1$*-equivalence classes of representations in* $\mathscr{R}_{\mathscr{S}}$ *and projectors in* $\mathscr{Z}_1$*; whereby* $\mathscr{Z}_1$*-containment,* $\mathscr{Z}_1$*-disjointness,* $\mathscr{Z}_1$*-primariness and direct sums of (classes of) representations go over respectively into containment, orthogonality, minimality and union of projectors. The* $\mathscr{Z}_1$*-equivalence classes of representations thus build a Boolean algebra.*

We next state properties of the $\mathscr{Z}_1$-carriers connected with the representations themselves:

(c) *Let* $\pi \in \mathscr{R}_{\mathscr{S}}$. *Then* $\bar{\pi}(S_\pi)$ *is the essential projector of* π *(the smallest projector* E *acting on the Hilbert space of* π *such that* $E\pi(A) = \pi(A)$, $A \in \mathfrak{A}$*).*

(d) *Let* π_1 *be a subrepresentation of* $\pi \in \mathscr{R}_{\mathscr{S}}$. *Then* $\pi_1 \in \mathscr{R}_{\mathscr{S}}$ *and* $\pi_1 \underset{1}{\subset} \pi$. *Furthermore* $\bar{\pi}(S_{\pi_1})$ *is the smallest projector in* $\bar{\pi}(\mathscr{Z}_1)$ *containing the essential projector of* π_1.

(e) *Let* $\pi \in \mathscr{R}_{\mathscr{S}}$ *and let* S *be a projector in* $\mathscr{Z}_1$. *Defining* $\varrho(A) = \bar{\pi}(SA)$, $A \in \mathfrak{A}$, *one obtains a subrepresentation* ϱ *of* π *with* $S_\varrho = SS_\pi$ *(in particular* $S_\varrho = S$ *if* $S \leqq S_\pi$*). Moreover* $\bar{\pi}(SS_\pi)$ *and* $\bar{\pi}(S_\pi - SS_\pi)$ *are the greatest*

[7] With π a representation of $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$ we recall that a *subrepresentation of* π is a representation π_1 of $\mathfrak{A}$ on $\mathscr{H}$ of the form $\pi_1(A) = P\pi(A)$, $A \in \mathfrak{A}$, with P a projector acting on $\mathscr{H}$ which commutes with $\pi(\mathfrak{A})$. A representation π is called *essential* (non degenerate) if it has no null subrepresentation. The *essential part of* π is the subrepresentation π_1 of the foregoing form with P the greatest projector such that π_1 is essential (P is then called the *essential projector* and $P\mathscr{H}$ the *essential subspace* of π).

projectors determining essential subrepresentations of π *respectively* $\mathscr{Z}_1$*-contained in and* $\mathscr{Z}_1$*-disjoint from the* $\mathscr{Z}_1$*-equivalence class of representations determined by S.*

As immediate corollaries we have that

(f) *With* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *one has* $\pi_1 \underset{1}{\propto} \pi_2$ *if and only if* π_1 *is* $\mathscr{Z}_1$*-equivalent to a subrepresentation of* π_2.

(g) *With* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *and assuming* π_1 *essential,* π_1 *is uniquely decomposed into a direct sum* $\pi_1' \oplus \pi_1''$ *of a subrepresentation* π_1' $\mathscr{Z}_1$*-contained in* π_2 *(determined by the central projector* $\bar{\pi}_1(S_{\pi_2})$*) and a subrepresentation* π_1'' *disjoint from* π_2.

The next two properties are relative to mutual relationships of subrepresentations of the same representation:

(h) *Let* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *and consider* $\pi = \pi_1 \oplus \pi_2$. *Then* π_1 *and* π_2 *are* $\mathscr{Z}_1$*-disjoint if and only if the essential projector of* π_1 *belongs to* $\bar{\pi}(\mathscr{Z}_1)$.

(k) *Let* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *be subrepresentations of* $\pi \in \mathscr{R}_{\mathscr{S}}$*; and let* F_1, F_2 *be the smallest projectors of* $\bar{\pi}(\mathscr{Z}_1)$ *dominating the respective essential projectors of* π_1 *and* π_2 *(so that* $F_1 = \bar{\pi}(S_{\pi_1})$ *and* $F_2 = \bar{\pi}(S_{\pi_2})$ *by* (d)*). We have the following equivalences:* $\pi_1 \underset{1}{\propto} \pi_2 \Leftrightarrow F_1 \leqq F_2$*;* $\pi_1 \underset{1}{\sim} \pi_2 \Leftrightarrow F_1 = F_2$*;* $\pi_1 \underset{1}{\between} \pi_2 \Leftrightarrow F_1 F_2 = 0$.

We next characterize $\mathscr{Z}_1$-primariness of a representation or a state:

(l) $\pi \in \mathscr{R}_{\mathscr{S}}$ *is* $\mathscr{Z}_1$*-primary if and only if* $\bar{\pi}(\mathscr{Z}_1)$ *reduces to the multiples of the unit operator on the essential subspace of* π *(iff* $\bar{\pi}(\mathscr{Z}_1)$ *is one-dimensional).*

(m) *Let* $\varphi \in \mathscr{S}$ *with* π_φ *the corresponding representation of* $\mathfrak{A}$ *acting on the Hilbert space* $\mathscr{H}$ *with cyclic vector* Ω *such that* $\varphi(A) = (\Omega, \pi_\varphi(A)\Omega)$, $A \in \mathfrak{A}$.

The following are equivalent:

(i) φ *is* $\mathscr{Z}_1$*-primary;*

(ii) *for each* $A \in \mathfrak{A}$ *and* $Z \in \mathscr{Z}_1$ $\bar{\varphi}(AZ) = \varphi(A)\,\bar{\varphi}(Z)$*;*

(iii) *for each* $A \in [\mathscr{S}]^*$ *and* $Z \in \mathscr{Z}_1$ $\bar{\varphi}(AZ) = \bar{\varphi}(A)\,\bar{\varphi}(Z)$*;*

(iv) $\bar{\pi}_\varphi(\mathscr{Z}_1)$ *reduces to the multiples of the identity operator I of* $\mathscr{H}$*;*

(v) $\bar{\pi}_\varphi(Z) = \bar{\varphi}(Z) \circ I$ *for all* $Z \in \mathscr{Z}_1$*;*

(vi) $\bar{\pi}_\varphi(\mathscr{Z}_1)$ *is one-dimensional;*

(vii) *the restriction of* $\bar{\varphi}$ *to* $\mathscr{Z}_1$ *is pure.*

The next three properties state relationships between our $\mathscr{Z}_1$-primariness and $\mathscr{Z}_1$-disjointness notions and the usual ones:

(n) *If* $\pi_1 \in \mathscr{R}_{\mathscr{S}}$ *is quasi-contained in* $\pi_2 \in \mathscr{R}_{\mathscr{S}}$, π_1 *is* $\mathscr{Z}_1$*-contained in* π_2. *In particular* $\pi_1 \sim \pi_2$ *implies* $\pi_1 \underset{1}{\sim} \pi_2$.

(o) *If* $\pi \in \mathscr{R}_{\mathscr{S}}$ $(\varphi \in \mathscr{S})$ *is primary, it is* $\mathscr{Z}_1$*-primary.*

(p) *Two essential representations* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *(two states* $\varphi_1, \varphi_2 \in \mathscr{S}$*) which are* $\mathscr{Z}_1$*-disjoint are disjoint*[8].

We finally give an interpretation of the foregoing notions in terms of the "$\mathscr{Z}_1$-folia of representations":

(q) *Let* $\pi \in \mathscr{R}_{\mathscr{S}}$. *The sets*

$$M_1 = \{\psi \in \mathscr{S};\ \text{there is a } \varphi \in \mathscr{S} \text{ with } \psi(A) = \bar{\varphi}(S_\pi A), A \in \mathfrak{A}\},$$

$$M_2 = \{\psi \in \mathscr{S}; \bar{\psi}(S_\pi A) = \psi(A), A \in \mathfrak{A}\},$$

$$M_3 = \bigcup_{\substack{\varrho \in \mathscr{R}_{\mathscr{S}} \\ \varrho \underset{1}{\sim} \pi}} \mathscr{S}_\varrho ,$$

$$M_4 = \bigcup_{\substack{\varrho \in \mathscr{R}_{\mathscr{S}} \\ \varrho \underset{1}{\propto} \pi}} \mathscr{S}_\varrho ,$$

define the same subset Σ_π *of* $\mathscr{S}$ *called the* $\mathscr{Z}_1$*-folium of* π. *One has* $\varphi \in \Sigma_\pi \Leftrightarrow \pi_\varphi \underset{1}{\propto} \pi$. *For* $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ *we have the equivalences* $\pi_1 \underset{1}{\sim} \pi_2 \Leftrightarrow \Sigma_{\pi_1} = \Sigma_{\pi_2}$; $\pi_1 \underset{1}{\propto} \pi_2 \Leftrightarrow \Sigma_{\pi_1} \subseteqq \Sigma_{\pi_2}$; $\pi_1 \underset{1}{\between} \pi_2 \Leftrightarrow \Sigma_{\pi_1} \cap \Sigma_{\pi_2} = \{0\}$. *With* $\pi_\nu \in \mathscr{R}_{\mathscr{S}}, \nu \in I$, $\Sigma \bigoplus_{\nu \in I} \pi_\nu$ *is the norm-closed convex hull of the union of all* $\Sigma_{\pi_\nu}, \nu \in I$.

§ 2. Applications to Nets of Subalgebras and Groups of Automorphisms

A. Local and Asymptotic Properties Relative to a Net of Subalgebras

The general context of this sub-sections is that of a "net of local subalgebras" defined in the following way:

Definition 2. *Let* $\mathfrak{A}$ *be a* C^**-algebra. A net of* $\mathfrak{A}$ *is a collection* $\mathscr{F} = \{\mathfrak{A}_\alpha\}$ *of* C^**-subalgebras of* $\mathfrak{A}$ *(the "local subalgebras") with the following properties*

(i) *to all pairs* $\mathfrak{A}_\alpha, \mathfrak{A}_\beta \in \mathscr{F}$ *there is* $\mathfrak{A}_\gamma \in \mathscr{F}$ *with* $\mathfrak{A}_\alpha \cup \mathfrak{A}_\beta \subseteqq \mathfrak{A}_\gamma$;

(ii) *the unit of* $\mathfrak{A}$, *if it exists, is contained in all* $\mathfrak{A}_\alpha \in \mathscr{F}$; *if* $\mathfrak{A}$ *has no unit, every approximate unit of each* $\mathfrak{A}_\alpha \in \mathscr{F}$ *is an approximate unit of* $\mathfrak{A}$[9];

(iii) *the union* $\bigcup_{\mathfrak{A}_\alpha \in \mathscr{F}} \mathfrak{A}_\alpha$ *of all* $\mathfrak{A}_\alpha$ *is norm-dense in* $\mathfrak{A}$[10].

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[8] More generally if $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are von Neumann subalgebras of $\mathscr{Z}_{\mathscr{S}}$ with $\mathscr{Z}_1 \leqq \mathscr{Z}_2$, we have for $\pi_1, \pi_2, \pi \in \mathscr{R}_{\mathscr{S}}$ and $\varphi \in \mathscr{S}$, with evident notation:

(n′) $\pi_1 \underset{2}{\propto} \pi_2 \Rightarrow \pi_1 \underset{1}{\propto} \pi_2$.

(o′) If $\pi(\varphi)$ is $\mathscr{Z}_2$-primary, it is $\mathscr{Z}_1$-primary.

(p′) $\pi_1 \underset{1}{\between} \pi_2 \Rightarrow \pi_1 \underset{2}{\between} \pi_2$.

[9] We note that this property entails that each essential representation of $\mathfrak{A}$ is essential in restriction to all $\mathfrak{A}_\alpha \in \mathscr{F}$: and that a state of $\mathfrak{A}$ furnishes by restriction a state of $\mathfrak{A}_\alpha$ for all $\mathfrak{A}_\alpha \in \mathscr{F}$.

[10] Given two nets $\mathscr{F} = \{\mathfrak{A}_\alpha\}$ and $\mathscr{G} = \{\mathfrak{A}_\beta\}$ of $\mathfrak{A}$, we write $\mathscr{F} \leqq \mathscr{G}$ whenever, to each $\mathfrak{A}_\alpha \in \mathscr{F}$, there is a $\mathfrak{A}_\beta \in \mathscr{G}$ with $\mathfrak{A}_\alpha \subseteqq \mathfrak{A}_\beta$.

Given a C^*-algebra $\mathfrak{A}$ with a full folium $\mathscr{S}$ of positive forms and a net $\mathscr{F}$, we next define two von Neumann algebras $\mathscr{Z}_l$ and $\mathscr{Z}_c$ of the center of $[\mathscr{S}]^*$, each of which determines as above a notion of equivalence and containment of representations of $\mathfrak{A}$. The indexes l and c stand for "local" and "commutant", as the so obtained notions are respectively related to properties holding "within" the local subalgebras and on their relative commutants.

Definition 3. *Let $\mathfrak{A}$ be a C^*-algebra equipped with a full folium $\mathscr{S}$ of positive forms and with a net $\mathscr{F}$. Retaining the identification $\mathfrak{A} = \tau_{\mathscr{S}}(\mathfrak{A})$ and taking commutants and bicommutants in the representation $\tau_{\mathscr{S}}$, we denote*

– by $\mathscr{Z}_l$ the von Neumann algebra generated by the central supports relative to $\mathfrak{A}''$ of all elements of the centers $\mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha$ of all $\mathfrak{A}''_\alpha$;

– by $\mathscr{Z}_c$ the intersection of all $\mathfrak{A}^{c''}_\alpha$, $\mathfrak{A}_\alpha \in \mathscr{F}$, where $\mathfrak{A}^c_\alpha = \mathfrak{A}'_\alpha \cap \mathfrak{A}$ is the relative commutant of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$.

We denote accordingly by $\underset{l}{\sim}, \underset{l}{\propto}, \underset{l}{\diamond}$; and by $\underset{c}{\sim}, \underset{c}{\propto}, \underset{c}{\diamond}$ the respective relative equivalence, relative containment and relative disjointness of representations given in Definition 1 where one takes $\mathscr{Z}_1 = \mathscr{Z}_l$; and $\mathscr{Z}_1 = \mathscr{Z}_c$[11].

We note that for each $\mathfrak{A}_\alpha \in \mathscr{F}$, $\mathfrak{A}_\alpha \subseteqq \mathfrak{A}^{c'}_\alpha$ and $\mathfrak{A}^c_\alpha \subseteqq \mathfrak{A}$, so that $\mathfrak{A}^{c''}_\alpha \subseteqq \mathfrak{A}'_\alpha$ (whence $\mathfrak{A}^{c''}_\alpha \subseteqq \mathfrak{A}'$ by (iii) of Definition 2) and $\mathfrak{A}^{c''}_\alpha \subseteqq \mathfrak{A}'' = [\mathscr{S}]^*$. Thus $\mathscr{Z}_c$ belongs as implied above to the center $\mathscr{Z}_{\mathscr{S}}$ of $[\mathscr{S}]^*$.

The local nature of $\mathscr{Z}_l$ is shown by

Proposition 1. *Let $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ and assume $\pi_1 \underset{l}{\propto} \pi_2$. Then for each $\mathfrak{A}_\alpha \in \mathscr{F}$ the restriction $\pi_1 \mid \mathfrak{A}_\alpha$ of π_1 to $\mathfrak{A}_\alpha$ is quasi-contained in $\pi_2 \mid \mathfrak{A}_\alpha$. In particular, if $\pi_1 \underset{l}{\sim} \pi_2$, $\pi_1 \mid \mathfrak{A}_\alpha$ and $\pi_2 \mid \mathfrak{A}_\alpha$ are quasi-equivalent for all $\mathfrak{A}_\alpha \in \mathscr{F}$.*

Proof. Consider $\pi \in \mathscr{R}_{\mathscr{S}}$ with ultraweakly continuous extension $\bar{\pi}$ to $[\mathscr{S}]^*$ and denote by K^0_π and K_π the respective generating projectors of $\operatorname{Ker}\bar{\pi}$ and $\operatorname{Ker}\bar{\pi} \mid \mathscr{Z}_l$. For $\mathfrak{A}_\alpha \in \mathscr{F}$ and any $L \in \mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha$ with central support C_L we have (since $C_L \in \mathscr{Z}_l$ and $K^0_\pi \in \mathscr{Z}_{\mathscr{S}}$) the equivalences:

$$K_\pi C_L = C_L \Leftrightarrow \bar{\pi}(C_L) = 0 \Leftrightarrow K^0_\pi C_L = C_L \Leftrightarrow K^0_\pi L = L \Leftrightarrow \bar{\pi}(L) = 0\,.$$

Therefore if $\pi_1, \pi_2 \in \mathscr{R}_{\mathscr{S}}$ are such that $\pi_1 \underset{l}{\propto} \pi_2$, whence the implication $K_{\pi_2} C_L = C_L \Rightarrow K_{\pi_1} C_L = C_L$ for $L \in \mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha$, we have also the implication $\bar{\pi}_2(L) = 0 \Rightarrow \bar{\pi}_1(L) = 0$ for $L \in \mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha$ i.e. the inclusion $\operatorname{Ker}\bar{\pi}_1 \mid \mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha \supseteqq \operatorname{Ker}\bar{\pi}_2 \mid \mathfrak{A}''_\alpha \cap \mathfrak{A}'_\alpha$, or equivalently $\operatorname{Ker}\bar{\pi}_1 \mid \mathfrak{A}''_\alpha \supseteqq \operatorname{Ker}\bar{\pi}_2 \mid \mathfrak{A}''_\alpha$. In order to complete our proof we will now pass from there to the inclusion $\operatorname{Ker}\overline{\pi_1 \mid \mathfrak{A}_\alpha} \supseteqq \operatorname{Ker}\overline{\pi_2 \mid \mathfrak{A}_\alpha}$, where $\overline{\pi_1 \mid \mathfrak{A}_\alpha}$ and $\overline{\pi_2 \mid \mathfrak{A}_\alpha}$ are the respective ultraweakly continuous extensions of $\pi_1 \mid \mathfrak{A}_\alpha$ and $\pi_2 \mid \mathfrak{A}_\alpha$ to the von Neumann algebra $[\mathscr{S} \mid \mathfrak{A}_\alpha]^*$, $\mathscr{S} \mid \mathfrak{A}_\alpha$ denoting the set of restrictions of the states of $\mathscr{S}$ to

[11] The symbols $\sim$, $\propto$, $\diamond$ will be reserved for the usual quasi-equivalence, quasi-containment and disjointness of representations of $\mathfrak{A}$.

$\mathfrak{A}_\alpha$ (this will imply our result – see footnote after Definition 1). Now the restriction $\tau_{\mathscr{S}} \mid \mathfrak{A}_\alpha$ of $\tau_{\mathscr{S}}$ to $\mathfrak{A}_\alpha$ extends uniquely to a ultraweakly continuous representation of $[\mathscr{S} \mid \mathfrak{A}_\alpha]^*$ whose image is the weak closure $\mathfrak{A}_\alpha''$ of $\tau_{\mathscr{S}}(\mathfrak{A}_\alpha)$. Furthermore, since the normal states of $\tau_{\mathscr{S}} \mid \mathfrak{A}_\alpha$ comprise $\mathscr{S} \mid \mathfrak{A}_\alpha$, this extension is faithful, so that we may identify $\overline{[\mathscr{S} \mid \mathfrak{A}_\alpha]^*}$ with $\mathfrak{A}_\alpha''$ as a von Neumann algebra; whereby $\overline{\pi_1 \mid \mathfrak{A}_\alpha}$ and $\overline{\pi_2 \mid \mathfrak{A}_\alpha}$ are respectively identified with the restrictions of π_1 and π_2 to $\mathfrak{A}_\alpha''$; whence our result.

The next results are easy generalizations of a theorem of Powers ([9], Theorem 2.5).

Definition 4. *Let φ_1 and φ_2 be elements of the linear closure $[\mathscr{S}]$ of $\mathscr{S}$ and denote by $\overline{\varphi_1}$ and $\overline{\varphi_2}$ their respective ultraweakly continuous extensions to $[\mathscr{S}]^*$; φ_1 and φ_2 are called $\mathscr{Z}_c$-equal whenever $\overline{\varphi}_1$ and $\overline{\varphi}_2$ coincide in restriction to $\mathscr{Z}_c$.*

Lemma 1. *Let $\varphi_1, \varphi_2 \in [\mathscr{S}]$. The following are equivalent:*

(i) *φ_1 and φ_2 are $\mathscr{Z}_c$-equal;*

(ii) *to each $\varepsilon > 0$ there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ with $\|\varphi_1 \mid \mathfrak{A}_\alpha^c - \varphi_2 \mid \mathfrak{A}_\alpha^c\| < \varepsilon$.*

Proof. We prove (i)⇒(ii). Assume (i) to hold and (ii) to be false. For each $\mathfrak{A}_\alpha \in \mathscr{F}$ the weak-operator closed set

$$Q_\alpha = \{X \in \mathfrak{A}_\alpha^{c''};\ \|X\| \leqq 1,\ |\overline{\varphi}_1(X) - \overline{\varphi}_2(X)| \geqq \varepsilon\}$$

is non void. We note that $\mathfrak{A}_\alpha \supseteqq \mathfrak{A}_\beta$ implies $Q_\alpha \subseteqq Q_\beta$; thus (i) of Definition 2 entails that the family $\{Q_\alpha\}$ has the finite intersection property. The weak compactness of the unit ball of $[\mathscr{S}]^*$ then yields the existence of a $Z \in \mathscr{Z}_c$ with $|\overline{\varphi_1}(Z) - \overline{\varphi_2}(Z)| \geqq \varepsilon$, a contradiction. We now prove the converse (ii)⇒(i). Let $\varepsilon > 0$, assume (ii) and take $Z \in \mathscr{Z}_c$. Since $\mathscr{Z}_c \subseteqq \mathfrak{A}_\alpha^{c''}$, Kaplansky's density theorem yields a $B \in \mathfrak{A}_\alpha^c$ with $\|B\| \leqq \|Z\|$, $|\overline{\varphi_1}(Z) - \varphi_1(B)| \leqq \varepsilon$ and $|\overline{\varphi}_2(Z) - \varphi_2(B)| \leqq \varepsilon$; then

$$|\overline{\varphi_1}(Z) - \overline{\varphi_2}(Z)| \leqq \varepsilon(2 + \|Z\|)$$

and (i) follows.

Proposition 2. *Let $\varphi \in \mathscr{S}$. The following are equivalent:*

(i) *φ is $\mathscr{Z}_c$-primary;*

(ii) *to each $A \in \mathfrak{A}$ there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ with*

$$|\varphi(AB) - \varphi(A)\,\varphi(B)| \leqq \|B\|, \qquad B \in \mathfrak{A}_\alpha^c .$$

Proof. Define $\varphi_1, \varphi_2 \in [\mathscr{S}]$ by $\varphi_1(B) = \varphi(AB)$ and $\varphi_2(B) = \varphi(A)\,\varphi(B)$, $B \in \mathfrak{A}$. By (m) above, (i) means that φ_1 and φ_2 are $\mathscr{Z}_c$-equal. But this is equivalent to (ii) where one takes A/ε instead of A from the previous Lemma.

Proposition 3. *Let $\varphi_1, \varphi_2 \in \mathscr{S}$. Then $\varphi_1 \underset{c}{\circ} \varphi_2$ implies $\|\varphi_1 \mid \mathfrak{A}_\alpha^c - \varphi_2 \mid \mathfrak{A}_\alpha^c\| = 2$ for all $\mathfrak{A}_\alpha \in \mathscr{F}$. The converse holds if φ_1 and φ_2 are $\mathscr{Z}_c$-*

primary. Two $\mathscr{Z}_c$-primary states φ_1 and φ_2 are $\mathscr{Z}_c$-equivalent if and only if, to each $\varepsilon > 0$, there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ with $\|\varphi_1 \mid \mathfrak{A}_\alpha^c - \varphi_2 \mid \mathfrak{A}_\alpha^c\| \leqq \varepsilon$.

Proof. Consider $\pi = \pi_{\varphi_1} \oplus \pi_{\varphi_2}$ acting on $\mathscr{H} = \mathscr{H}_{\pi_{\varphi_1}} \oplus \mathscr{H}_{\pi_{\varphi_2}}$ and denote by E_1, E_2 the respective projectors onto $\mathscr{H}_{\pi_{\varphi_1}}$ and $\mathscr{H}_{\pi_{\varphi_2}}$ in $\mathscr{H}$. We know by (h) above that $\varphi_1 \underset{c}{\between} \varphi_2$ entails $E_1 = \bar{\pi}(Z_1)$, $E_2 = \bar{\pi}(Z_2)$ with $Z_1, Z_2 \in \mathscr{Z}_c$. For each $\mathfrak{A}_\alpha \in \mathscr{F}$, $Z_1, Z_2 \in \mathfrak{A}_\alpha^{c''}$, Kaplansky's density theorem then asserts the existence, to each $\varepsilon > 0$, of $A \in \mathfrak{A}_\alpha^c$ with $\|A\| = 1$, $|\varphi_1(A - Z_1)| = |\varphi_1(A) - 1| \leqq \varepsilon$ and $|\varphi_2(A + Z_2)| = |\varphi_2(A) + 1| \leqq \varepsilon$; since ε is arbitrary it follows that $\|\varphi_1 \mid \mathfrak{A}_1^c - \varphi_2 \mid \mathfrak{A}_1^c\| = 2$. If φ_1 and φ_2 are $\mathscr{Z}_c$-primary we have either $\varphi_1 \underset{c}{\between} \varphi_2$ or $\varphi_1 \underset{c}{\sim} \varphi_2$. The latter entails by Lemma 1 the existence of $\mathfrak{A}_\alpha \in \mathscr{F}$ for which $\|\varphi_1 \mid \mathfrak{A}_\alpha^c - \varphi_2 \mid \mathfrak{A}_\alpha^c\| < \varepsilon$.

Combining the two last propositions with (n) and (o) of Section I we get the

Corollary. *If φ is a primary state of the C^*-algebra $\mathfrak{A}$ and $\mathscr{F} (= \{\mathfrak{A}_\alpha\})$ is a net for $\mathfrak{A}$, then, given A in $\mathfrak{A}$, there is an $\mathfrak{A}_\alpha$ in $\mathscr{F}$ such that*

$$|\varphi(AB) - \varphi(A)\,\varphi(B)| \leqq \|B\|, \qquad B \in \mathfrak{A}_\alpha^c .$$

Moreover, with φ_1 and φ_2 quasi-equivalent, primary states of $\mathfrak{A}$, for each positive ε there is an $\mathfrak{A}_\alpha$ in $\mathscr{F}$ such that $\|\varphi_1 \mid \mathfrak{A}_\alpha^c - \varphi_2 \mid \mathfrak{A}_\alpha^c\| \leqq \varepsilon$.

Remark. We note that Lemma 1 and Propositions 2 and 3 hold more generally replacing $\mathfrak{A}_\alpha^c$ by $\mathscr{L}_\alpha$ and $\mathscr{Z}_c$ by $\bigcap_\alpha \mathscr{L}_\alpha''$, with $\{\mathscr{L}_\alpha\}$ a family of subalgebras of $\mathfrak{A}$ such that (i) to each $A \in \mathfrak{A}$ and $\varepsilon > 0$ there is a $\mathscr{L}_\alpha$ with $\|[A, B]\| \leqq \varepsilon$ for each B in the unit ball of $\mathscr{L}_\alpha$ (ii) to each pair $\mathscr{L}_\alpha, \mathscr{L}_\beta$ there is $\mathscr{L}_\gamma$ contained in $\mathscr{L}_\alpha \cap \mathscr{L}_\beta$. The algebra $\mathscr{B}_\pi$ (cf. [10], § 2) of Lanford and Ruelle, which is more genuinely asymptotic in nature than our $\mathscr{Z}_c$ in that it excludes superselection operators, comes under the scope of this remark.

B. Central Subalgebras Related to a Group of Automorphisms

Another structure to which the general notions of Section 1 apply is that of a C^*-algebra $\mathfrak{A}$ together with a weakly dense folium $\mathscr{S}$ of positive forms and a homomorphism $g \in \mathbf{G} \to \alpha_g$ of a group $\mathbf{G}$ into the group of $\sigma(\mathfrak{A}, [\mathscr{S}])$-continuous automorphisms of $\mathfrak{A}$[12]. (We note that each α_g then possesses a transposed α_g^t on $[\mathscr{S}]$ and a bitransposed α_g^{tt} on $[\mathscr{S}]^*$, respectively $\sigma([\mathscr{S}], \mathfrak{A})$-continuous and ultraweakly continuous. Furthermore $g \to \alpha_{g^{-1}}^t$ and $g \to \alpha_g^{tt}$ are group homomorphisms, α_g^{tt} is an automorphism of $[\mathscr{S}]^*$; and $\bar{\varphi} \circ \alpha_g^{tt} = \overline{\alpha_g^t(\varphi)}$, $\pi_{\alpha_g^t(\varphi)} = \pi_\varphi \circ \alpha_g$, $\bar{\pi} \circ \alpha_g^{tt} = \overline{\pi \circ \alpha_g}$ for all $\varphi \in \mathscr{S}$ and $\pi \in \mathscr{R}_{\mathscr{S}}$).

[12] With V and W two vectors spaces in duality $\sigma(V, W)$ denotes the weak topology of V with respect to W. We note that the automorphisms α_g are $\sigma(\mathfrak{A}, \mathscr{S}])$-continuous if and only if their transposed in $\mathfrak{A}^*$ leave the folium $\mathscr{S}$ invariant.

Definition 5. *We denote by $\mathscr{Z}_{\mathbf{G}}$ the set of elements of $\mathscr{Z}_{\mathscr{S}}$ invariant under all α_g^{tt}, $g \in \mathbf{G}$. A representation $\pi \in \mathscr{R}_{\mathscr{S}}$ (state $\varphi \in \mathscr{S}$) is called quasi-invariant*[13] *whenever the $\mathscr{Z}_{\mathscr{S}}$-carrier S_π of π (of π_φ) belongs to $\mathscr{Z}_{\mathbf{G}}$; π and φ are called ergodic whenever they are quasi-invariant and $\mathscr{Z}_{\mathbf{G}}$-primary.*

It follows from the noted facts that $\mathscr{Z}_{\mathbf{G}}$ is a von Neumann subalgebra of $\mathscr{Z}_{\mathscr{S}}$; and that $\pi \in \mathscr{R}_{\mathscr{S}}$ is quasi-invariant iff the kernel of $\bar{\pi}$ in $[\mathscr{S}]^*$ is α_g^{tt}-invariant for all $g \in \mathbf{G}$; or iff $\pi \circ \alpha_g \sim \pi$ for all $g \in \mathbf{G}$; or iff $\mathscr{S}_\pi$ is α^t-invariant. Ergodicity of π means that the only $\mathbf{G}$-invariant elements of $\pi(\mathfrak{A})''$ are the scalar multiples of the unit operator. For $\mathfrak{A}$ and $\mathbf{G}$ separable one can decompose any quasi-invariant state into ergodic states [4].

Proposition 4. *Let $\pi \in \mathscr{R}_{\mathscr{S}}$: the $\mathscr{Z}_{\mathbf{G}}$-folium Σ_π of π is the smallest α^t-invariant folium containing $\mathscr{S}_\pi$. Let $\varphi, \varphi' \in \mathscr{S}$: φ' is $\mathscr{Z}_{\mathbf{G}}$-contained in φ iff it is a norm-limit of convex combinations of states of the form $\alpha_g^t(\varphi_A)$, $g \in \mathbf{G}$, $A \in \mathfrak{A}$; and φ_1 and φ_2 in $\mathscr{S}$ are $\mathscr{Z}_{\mathbf{G}}$-disjoint iff φ_1 and $\alpha_g^t(\varphi_2)$ are disjoint for all $g \in \mathbf{G}$.*

Proof. Let S_π and S_π^0 be the respective $\mathscr{Z}_{\mathbf{G}}$- and $\mathscr{Z}_{\mathscr{S}}$-carrier of π. The first assertion follows from the fact that S_π is the smallest α^{tt}-invariant projector of $\mathscr{Z}_{\mathscr{S}}$ larger than S_π^0, and thus equals $\bigvee_{g \in \mathbf{G}} \alpha_g^{tt}(S_\pi^0)$. Further, for $\varphi \in \mathscr{S}$, the $\mathscr{Z}_{\mathbf{G}}$-folium Σ_{π_φ} of π_φ is the smallest α^t-invariant folium containing φ, and thus coincides with the norm-closure of the convex hull of states of the form $\alpha_g^t(\varphi_A)$, $g \in \mathbf{G}$, $A \in \mathfrak{A}$; whence our second assertion using (q) of § 1. The last assertion is proved as follows: if φ_1 and $\alpha_g^t \varphi_2$ are not disjoint for some $g \in \mathbf{G}$, then

$$\mathscr{S}_{\pi_{\varphi_1}} \left(\subseteqq \Sigma_{\pi_{\varphi_1}}\right) \quad \text{and} \quad \mathscr{S}_{\pi_{\alpha_g^t(\varphi_2)}} \left(= \alpha_g^t(\mathscr{S}_{\pi_{\varphi_2}}) \subseteqq \Sigma_{\pi_{\varphi_2}}\right)$$

have a non vanishing common element; therefore φ_1 and φ_2 are not $\mathscr{Z}_{\mathbf{G}}$-disjoint. Conversely if φ_1 and $\alpha_g^t(\varphi_2)$ are disjoint for all $g \in \mathbf{G}$, so are $\alpha_g^t(\varphi_1)_A$ and $\alpha_h^t(\varphi_2)_B$ for all $g, h \in \mathbf{G}$ and $A, B \in \mathfrak{A}$. The direct sums $\bigoplus_{\substack{g \in \mathbf{G} \\ A \in \mathfrak{A}}} \pi_{\alpha_g^t(\varphi_1)_A}$ and $\bigoplus_{\substack{h \in \mathbf{G} \\ B \in \mathfrak{A}}} \pi_{\alpha_h^t(\varphi_2)_B}$, whose respective normal folia coincide with $\Sigma_{\pi_{\varphi_1}}$ and $\Sigma_{\pi_{\varphi_2}}$, are then disjoint; therefore φ_1 and φ_2 are $\mathscr{Z}_{\mathbf{G}}$-disjoint, from (*q*) of § 1.

We now examine the particularly interesting special case in which $\mathbf{G}$ is locally compact amenable and acts on $\mathfrak{A}$ with the following properties:

a) $g \to \varphi(\alpha_g(A))$ is continuous for all $\varphi \in \mathscr{S}$ and $A \in \mathfrak{A}$;

b) the system $\{\mathfrak{A}, \alpha\}$ is *asymptotically abelian* in the following sense: denoting by $\mathfrak{M}_{\mathbf{G}}$ the set of invariant means of G, we have, for all $\eta \in \mathfrak{M}_{\mathbf{G}}$, $\varphi \in \mathscr{S}$ and $A, B \in \mathfrak{A}$

$$\eta\{\varphi(A \cdot \alpha_{\hat{g}}(B)) - \varphi(A)\, \varphi(\alpha_{\hat{g}}(B))\} = 0$$

[13] According to Zeller-Meier [11].

where $\eta(f(\hat{g}))=\eta(f)$ with $f: g \to f(g)$ a bounded continuous function on $\mathbf{G}$.

Arguing as in ([12], Lemma 1), we can then define, for each $\eta \in \mathfrak{M}_{\mathbf{G}}$, unique mappings $M_\eta : \mathfrak{A} \to [\mathscr{S}]^*$ and $M_\eta^t : [\mathscr{S}] \to \mathfrak{A}^*$ by the properties

$$\eta\{\varphi(\alpha_{\hat{g}}(A))\} = \overline{\varphi}(M_\eta A) = (M_\eta^t \varphi)(A), \qquad A \in \mathfrak{A},\ \varphi \in [\mathscr{S}]. \tag{1}$$

These mappings are linear, of norm not exceeding 1, and such that

$$\begin{cases} \alpha_g^t(M_\eta^t \varphi) = M_\eta^t(\alpha_g^t(\varphi)) = M_\eta^t \varphi, & \varphi \in \mathscr{S}, \\ \alpha_g^{tt}(M_\eta A) = M_\eta(\alpha_g(A)) = M_\eta A & A \in \mathfrak{A} \end{cases} \quad g \in G. \tag{2}$$

Further, M_η has its range in $\mathscr{Z}_{\mathscr{S}}$, and therefore in $\mathscr{Z}_{\mathbf{G}}$.

Assuming the foregoing structure and notation, we make:

Definition 6. *We denote by $\mathscr{Z}_M$ the von Neumann subalgebra of $[\mathscr{S}]^*$ generated by all $M_\eta A$, $A \in \mathfrak{A}$, $\eta \in \mathfrak{M}_{\mathbf{G}}$. A state $\varphi \in \mathscr{S}$ is called η-clustering whenever*

$$\eta\{\varphi(A \cdot \alpha_{\hat{g}}(B)) - \varphi(A)\,\varphi(\alpha_{\hat{g}}(B))\} = 0 \tag{3}$$

or equivalently

$$\overline{\varphi}(A \cdot M_\eta B) = \varphi(A)\,\overline{\varphi}(M_\eta B) \tag{3a}$$

for all $A, B \in \mathfrak{A}$.

Proposition 5. *With φ in $\mathscr{S}$, the following are equivalent*

(i) *φ is $\mathscr{Z}_M$-primary,*

(ii) *φ is η-clustering for all $\eta \in \mathfrak{M}_{\mathbf{G}}$.*

In particular $\mathscr{Z}_{\mathbf{G}}$-primary (and especially ergodic) states are η-clustering for all $\eta \in \mathfrak{M}_{\mathbf{G}}$. If $\varphi_1, \varphi_2 \in \mathscr{S}$ are $\mathscr{Z}_{\mathbf{G}}$-primary, they are $\mathscr{Z}_M$-equivalent if and only if $M_\eta^t \varphi_1 = M_\eta^t \varphi_2$ for all $\eta \in \mathfrak{M}_{\mathbf{G}}$.

Proof. According to (m) of § 1 φ is $\mathscr{Z}_M$-primary iff

$$\overline{\varphi}(AZ) = \overline{\varphi}(A)\,\overline{\varphi}(Z), \qquad A \in [\mathscr{S}]^*,\ Z \in \mathscr{Z}_M. \tag{4}$$

Now (i)$\Rightarrow$(ii) since (4)$\Rightarrow$(3a). Conversely, assume (3a) for all $A, B \in \mathfrak{A}$; since $\overline{\varphi}$ is ultraweakly continuous, we have (4) for all $A \in [\mathscr{S}]^*$ and Z of the form $M_\eta B$, $\eta \in \mathfrak{M}_{\mathbf{G}}$, $B \in \mathfrak{A}$. Repeated application of this property yields, for $A, B_i \in \mathfrak{A}$, $\eta_i \in \mathfrak{M}_{\mathbf{G}}$, $i = 1, 2, \dots n$

$$\begin{aligned} \varphi\left(A \prod_{i=1}^{n} M_{\eta_i} B_i\right) &= \varphi\left(A \prod_{i=1}^{n-1} M_{\eta_i} B_i\right) \varphi(M_{\eta_n} B_n) \\ &= \cdots = \varphi(A)\,\varphi\left(\prod_{i=1}^{n} M_{\eta_i} B_i\right). \end{aligned}$$

By linearity we then have (4) with Z in an ultraweakly dense set of $\mathscr{Z}_M$, and thus everywhere in $\mathscr{Z}_M$ by density. For $\mathscr{Z}_M$-primary states $\varphi_1, \varphi_2 \in \mathscr{S}$ $\mathscr{Z}_M$-equivalence means $\mathscr{Z}_M$-equality which is the same as equality on all $M_\eta A$, $A \in \mathfrak{A}$, $\eta \in \mathfrak{M}_{\mathbf{G}}$, by multiplicativity and ultraweak continuity.

§ 3. Funnels and Their Locally Normal States

In this section we consider special types of nets, the *funnels* of Definition 7 below. These are encountered e.g. as the von Neumann algebras of local regions in quantum field theory and statistical mechanics. The specification of a funnel of a C^*-algebra provides a natural selection of a particular subset of states, the *locally normal states* of Definition 8 below.

Definition 7. *A bet $\mathcal{F} = \{\mathcal{M}_\alpha\}$ of a C^*-algebra (cf. Definition 2) is called a funnel whenever each local algebra $\mathcal{M}_\alpha \in \mathcal{F}$ is a factor possessing a representation on a separable Hilbert space. $\mathcal{F}$ is called a Type I-, Type I_∞-, Type III-funnel if the $\mathcal{M}_\alpha$ are factors of Type I, Type I_∞, Type III ... etc. If $\mathcal{F}$ consists of an ascending sequence $\{\mathcal{M}_n\}_{n=1,2,\ldots}$, we call it a sequential funnel.*

We will get a funnel by considering e.g. a C^*-algebra acting on a separable Hilbert space $\mathcal{H}$ with a net of concrete factors $\mathcal{M}_\alpha$ on $\mathcal{H}$. Note that as a consequence of Definition 7 *for all ordered pairs $\mathcal{M}_\alpha \subseteqq \mathcal{M}_\beta$ in $\mathcal{F}$ the embedding of $\mathcal{M}_\alpha$ in $\mathcal{M}_\beta$ is normal.* Indeed we know from [13, 14, and 15] that every representation of a factor on a separable Hilbert space is normal: any representation φ of $\mathcal{M}_\beta$ on a separable Hilbert space $\mathcal{H}$ is thus normal and, since $\mathcal{M}_\beta$ is a factor, faithful; φ is therefore faithful in restriction to $\mathcal{M}_\alpha$ and normal by the separability of $\mathcal{H}$, whence the normal character of the embedding $\mathcal{M}_\alpha \subseteqq \mathcal{M}_\beta$. Note also, that for Type I-factors, the property of being separably representable is equivalent to countable decomposability.

Definition 8. *We say that a state (a representation; a linear form) of a C^*-algebra $\mathfrak{A}$ with a net $\mathcal{F}$ of von Neumann rings is locally normal (relative to $\mathcal{F}$) whenever it is normal in restriction to all elements of $\mathcal{F}$. The set of positive linear forms of $\mathfrak{A}$ locally normal relative to $\mathcal{F}$ will be denoted by $\mathcal{S}_{\mathcal{F}}$.*

The two next propositions are valid in a somewhat wider context than that of funnels:

Proposition 6. *Let $\mathfrak{A}$ be a C^*-algebra with a net $\mathcal{F} = \{\mathfrak{A}_\alpha\}$ of von Neumann algebras such that for all ordered pairs $\mathfrak{A}_\alpha \subseteqq \mathfrak{A}_\beta$ of $\mathcal{F}$ the embedding of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_\beta$ is normal. Then the locally normal positive linear forms of $\mathfrak{A}$ form a folium*[14]*. A state of $\mathfrak{A}$ is locally normal iff it generates a locally normal representation. If φ is a hermitian locally normal linear form on $\mathfrak{A}$ with $\varphi = \varphi^+ - \varphi^-$ the (unique) decomposition of φ as the difference of two positive forms on $\mathfrak{A}$ such that $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$, then φ^+ and φ^- are locally normal.*

[14] Recall that we defined a folium in Section 1 as a convex set of positive linear forms closed for the norm of linear forms and containing $\varphi_A = \varphi(A^* \circ A)$, $A \in \mathfrak{A}$, with every φ. Note that the assumptions of Proposition 6 are realized if $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$ with the $\mathfrak{A}_\alpha$ concrete von Neumann algebras acting on $\mathcal{H}$.

Proof. We first observe the trivial facts that a linear combination of locally normal linear forms is locally normal; and that every normal state of a locally normal representation is locally normal (so that a state φ which generates a locally normal representation π_φ is locally normal). Further the real and imaginary parts of a locally normal linear form are obviously locally normal.

Let now the linear form φ of $\mathfrak{A}$ be the norm-limit of a sequence $\{\varphi_n\}$ of locally normal linear forms; its restriction $\varphi \mid \mathfrak{A}_\alpha$ to an $\mathfrak{A}_\alpha \in \mathscr{F}$ is then the norm-limit of the normal forms $\varphi_n \mid \mathfrak{A}_\alpha$ of $\mathfrak{A}_\alpha$ and thus normal, therefore φ is locally normal. We further consider a locally normal linear form φ of $\mathfrak{A}$ with $\varphi_A = \varphi(A^* \cdot A)$, $A \in \mathfrak{A}$, and wish to show that the restriction $\varphi_A \mid \mathfrak{A}_\alpha$ to any $\mathfrak{A}_\alpha \in \mathscr{F}$ is normal. By (i) and (iii) of Definition 2 to each $\varepsilon > 0$ we find a $B \in \mathfrak{A}_\beta$ with $\mathfrak{A}_\beta \in \mathscr{F}$, $\mathfrak{A}_\alpha \subseteqq \mathfrak{A}_\beta$, and $\|A - B\| \leqq \varepsilon$; and thus $\|\varphi_A - \varphi_B\| \leqq \|\varphi\| \, \|A\| \, \varepsilon(1+\varepsilon)$. This majorization holds a fortiori for the norm of $(\varphi_A - \varphi_B) \mid \mathfrak{A}_\alpha$; but, since the embedding $\mathfrak{A}_\alpha \subseteqq \mathfrak{A}_\beta$ is by assumption normal, $\varphi_B \mid \mathfrak{A}_\alpha$ is normal as the restriction to $\mathfrak{A}_\alpha$ of the normal form $\varphi_B \mid \mathfrak{A}_\beta$ of $\mathfrak{A}_\beta$: $\varphi_A \mid \mathfrak{A}_\alpha$ is therefore a norm-limit of normal forms and thus normal. Having established that $\mathscr{S}_{\mathscr{F}}$ is a folium, we now show that a locally normal state φ generates a locally normal representation π_φ: we have to check that the restriction $\pi_\varphi \mid \mathfrak{A}_\alpha$ of π_φ to each $\mathfrak{A}_\alpha \in \mathscr{F}$ is normal, i.e. that all the vector states of $\pi_\varphi \mid \mathfrak{A}_\alpha$ are normal. But the latter are norm-limits of states of the form $\varphi_A \mid \mathfrak{A}_\alpha$, $A \in \mathfrak{A}$, and we have shown in the foregoing that φ_A is locally normal for all $A \in \mathfrak{A}$.

For the proof of the last assertion, we need the following lemma. Its proof adapts Kjaergård-Pedersen's elegant argument ([16]; Propositions 1 and 2), for proving uniqueness, to our estimates.

Lemma 2. *If φ is a continuous linear form of norm* 1 *on the C^*-algebra $\mathfrak{A}$ and $\varphi = \varphi^+ - \varphi^-$ with φ^+ and φ^- positive and $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$, then, if $\|\varphi_0^+ - \varphi_0^- - \varphi\| \leqq \varepsilon \leqq 1$ and $\big|\|\varphi_0^+\| + \|\varphi_0^-\| - \|\varphi\|\big| \leqq \varepsilon$, with φ_0^+ and φ_0^- positive, it follows that $\|\varphi_0^+ - \varphi^+\| \leqq 6\varepsilon^{\frac{1}{2}}$ and $\|\varphi_0^- - \varphi^-\| \leqq 6\varepsilon^{\frac{1}{2}}$.*

Proof. Since $\varphi = \varphi^+ - \varphi^-$, φ is hermitian; and we can find a self-adjoint A in $\mathfrak{A}$ such that $-I \leqq A \leqq I$ and $1 \leqq \varphi(A) + \varepsilon$. Let B be $\frac{1}{2}(I-A)$; so that $I - B$ is $\frac{1}{2}(I+A)$. Note that $0 \leqq B \leqq I$ and:

$$\begin{aligned}\varphi^+(B) + \varphi^-(I-B) &= \tfrac{1}{2}[\varphi^+(I) + \varphi^-(I) - (\varphi^+(A) - \varphi^-(A))]\\ &= \tfrac{1}{2}[\|\varphi^+\| + \|\varphi^-\| - \varphi(A)] = \tfrac{1}{2}[\|\varphi\| - \varphi(A)] = \tfrac{1}{2}[1 - \varphi(A)] \leqq \tfrac{1}{2}\varepsilon .\end{aligned}$$

By the same token,

$$\begin{aligned}\varphi_0^+(B) + \varphi_0^-(I-B) &= \tfrac{1}{2}[\|\varphi_0^+\| + \|\varphi_0^-\| - (\varphi_0^+ - \varphi_0^-)(A)]\\ &\leqq \tfrac{1}{2}[\|\varphi\| - \varphi(A)] + \varepsilon \leqq \tfrac{3}{2}\varepsilon .\end{aligned}$$

Since $0 \leqq B \leqq I$, $\varphi_0^+(B) \leqq \frac{3}{2}\varepsilon$ and $\varphi_0^-(I-B) \leqq \frac{3}{2}\varepsilon$ (along with $\varphi^+(B) \leqq \frac{1}{2}\varepsilon$, $\varphi^-(I-B) \leqq \frac{1}{2}\varepsilon$). Thus, with C in the unit ball of $\mathfrak{A}$, since

$$\|\varphi_0^+ - \varphi_0^- - \varphi\| = \|\varphi_0^+ - \varphi^+ - (\varphi_0^- - \varphi^-)\| \leqq \varepsilon,$$

$$\begin{aligned}|(\varphi_0^+ - \varphi^+)(C)| &= |(\varphi_0^+ - \varphi^+)(BC) + (\varphi_0^+ - \varphi^+)((I-B)C)| \\ &\leqq |\varphi_0^+(BC)| + |\varphi^+(BC)| + |(\varphi_0^- - \varphi^-)((I-B)C)| + \varepsilon \\ &\leqq \varepsilon + \varphi_0^+(B)^{\frac{1}{2}}\varphi_0^+(C^*BC)^{\frac{1}{2}} + \varphi^+(B)^{\frac{1}{2}}\varphi^+(C^*BC)^{\frac{1}{2}} \\ &\quad + \varphi_0^-(I-B)^{\frac{1}{2}}\varphi_0^-(C^*(I-B)C)^{\frac{1}{2}} \\ &\quad + \varphi^-(I-B)^{\frac{1}{2}}\varphi^-(C^*(I-B)C)^{\frac{1}{2}} \\ &\leqq \varepsilon + [6\varepsilon(1+\varepsilon)]^{\frac{1}{2}} + (2\varepsilon)^{\frac{1}{2}} \leqq 6\varepsilon^{\frac{1}{2}}.\end{aligned}$$

By the same argument, $\|\varphi_0^- - \varphi^-\| \leqq 6\varepsilon^{\frac{1}{2}}$, completing our proof.

Remark. Lemma 2 establishes that $\varphi \to \varphi^+$ and $\varphi \to \varphi^-$ are norm continuous mappings. Kjaergård-Pedersen remarks to us that the same argument, with minor modifications, also proves that these mappings are w^*-continuous on restriction to spheres of constant norm – slightly more, that if $\|\varphi_n\| \to \|\varphi\|$ then φ_n^+ and φ_n^- tend to φ^+ and φ^-, respectively, in the w^*-topology. (Some condition on norms is required for w^*-convergence, for with $\{x_n\}$ an orthonormal set, ω_{x_1, x_n} tends to 0 on the algebra of all bounded operators in the w^*-topology, while $\|\omega_{x_1, x_n}\| = 1$ for all n; so that not both $\omega_{x_1, x_n}^-(I)$ $(= \|\omega_{x_1, x_n}^+\|)$ and $\omega_{x_1, x_n}^-(I)$ $(= \|\omega_{x_1, x_n}^-\|)$ tend to 0.)

Proof of end of Proposition 6. We wish to show that $\varphi^+ \mid \mathfrak{A}_\alpha$ and $\varphi^- \mid \mathfrak{A}_\alpha$ are normal. Choose A in $\mathfrak{A}$ such that $-I \leqq A \leqq I$ and $|1 - \varphi(A)| \leqq \frac{1}{2}\varepsilon$. (We may assume $\|\varphi\| = 1$.) Find $\mathfrak{A}_\beta$ containing $\mathfrak{A}_\alpha$ and B in $\mathfrak{A}_\beta$ such that $-I \leqq B \leqq I$ and $\|A - B\| \leqq \frac{1}{2}\varepsilon$. Then $|\varphi(B) - 1| \leqq \varepsilon$; so that

$$1 - \varepsilon \leqq \|\varphi \mid \mathfrak{A}_\beta\| \leqq \|\varphi^+ \mid \mathfrak{A}_\beta\| + \|\varphi^- \mid \mathfrak{A}_\beta\| \leqq \|\varphi^+\| + \|\varphi^-\| = \|\varphi\| = 1.$$

Thus

$$\big|\|\varphi^+ \mid \mathfrak{A}_\beta\| + \|\varphi^- \mid \mathfrak{A}_\beta\| - \|\varphi \mid \mathfrak{A}_\beta\|\big| \leqq \varepsilon$$

and

$$\|\varphi^+ \mid \mathfrak{A}_\beta - \varphi^- \mid \mathfrak{A}_\beta - \varphi \mid \mathfrak{A}_\beta\| = 0.$$

From Lemma 2,

$$\|\varphi^+ \mid \mathfrak{A}_\alpha - \varphi_\beta^+ \mid \mathfrak{A}_\alpha\| \leqq \|\varphi^+ \mid \mathfrak{A}_\beta - \varphi_\beta^+\| \leqq 6\varepsilon^{\frac{1}{2}},$$

where

$$\varphi \mid \mathfrak{A}_\beta = \varphi_\beta^+ - \varphi_\beta^- \quad \text{with} \quad \|\varphi \mid \mathfrak{A}_\beta\| = \|\varphi_\beta^+\| + \|\varphi_\beta^-\|.$$

From ([6], 12.3.3 and 4), φ_β^+ is normal. Since $\mathfrak{A}_\alpha$ is normally imbedded in $\mathfrak{A}_\beta$, $\varphi_\beta^+ \mid \mathfrak{A}_\alpha$ is normal. Thus $\varphi^+ \mid \mathfrak{A}_\alpha$ (and $\varphi^- \mid \mathfrak{A}_\alpha$) is a normal limit of normal forms.

In the next proposition, we introduce a topology which singles out the locally normal states as the corresponding continuous states. This allows us, in particular, to state conditions (fulfilled by the sequential funnels) which guarantee the abundance of locally normal states.

Proposition 7. *With the assumptions of Proposition 6; denoting by* $\mathfrak{A}_{\mathscr{F}}$ *the union of all* $\mathfrak{A}_\alpha \in \mathscr{F}$*; and by* $\mathscr{T}_{\mathscr{F}}$ *the locally convex topology on* $\mathfrak{A}_{\mathscr{F}}$ *which is the inductive limit of the ultraweak topologies* $\mathscr{T}_\alpha$ *on the* $\mathfrak{A}_\alpha \in \mathscr{F}$*; then the locally normal states (linear forms) of* $\mathfrak{A}$ *consists of the norm-continuous extensions of the states (linear forms) of* $\mathfrak{A}_{\mathscr{F}}$ *continuous for* $\mathscr{T}_{\mathscr{F}}$*. If in addition* $\mathfrak{A}$ *consists of an ascending sequence of von Neumann algebras,* $\mathscr{T}_{\mathscr{F}}$ *is Hausdorff, the set of locally normal states is *-weakly dense in the set of all states of* $\mathfrak{A}$*; and each normal state of any* $\mathfrak{A}_i \in \mathscr{F}$ *can be extended to a locally normal state of* $\mathfrak{A}$.

We note that in the latter case the set $\mathscr{S}_{\mathscr{F}}$ of locally normal positive linear forms of $\mathfrak{A}$ will thus be of the type of the set $\mathscr{S}$ considered in the beginning of Section 1 and can therefore be used for the constructs presented there.

Proof. We first observe that, due to (iii) of Definition 2, taking the restrictions to $\mathfrak{A}_{\mathscr{F}}$ of the states (bounded linear forms) of $\mathfrak{A}$ yields a one-to-one mapping of the latter onto the bounded states (bounded linear forms) of $\mathfrak{A}_{\mathscr{F}}$. Since the locally normal states (linear forms) of $\mathfrak{A}_{\mathscr{F}}$ [15] are automatically bounded, we can thus identify the locally normal states (linear forms) of $\mathfrak{A}$ with those of $\mathfrak{A}_{\mathscr{F}}$, from which they are obtained by norm-continuous extension.

We recall that one gets a complete system of neighbourhoods of zero for $\mathscr{T}_{\mathscr{F}}$ by taking the balanced convex sets $\mathbf{V}$ of $\mathfrak{A}_{\mathscr{F}}$ which intersect each $\mathfrak{A}_\alpha \in \mathscr{F}$ along a neighbourhood of zero for $\mathscr{T}_\alpha$ ([17], § 3, 15) (those $\mathbf{V}$ are automatically absorbing for $\mathfrak{A}_{\mathscr{F}}$). From this, it follows immediately that the $\mathscr{T}_{\mathscr{F}}$-continuous linear forms on $\mathfrak{A}_{\mathscr{F}}$ are those whose restriction to each $\mathfrak{A}_\alpha \in \mathscr{F}$ is $\mathscr{T}_\alpha$-continuous, in other terms the locally normal linear forms of $\mathfrak{A}_{\mathscr{F}}$. If we now add the assumption that the net $\mathscr{F}$ is an increasing sequence $\{\mathfrak{A}_j\}_{j=1,2,\ldots}$, we can assert ([17], § 3,16) that $\mathscr{T}_{\mathscr{F}}$ is Hausdorff (note that the assumed normal character of the embeddings $\mathfrak{A}_j \subseteqq \mathfrak{A}_{j+1}$ entails that the ultraweak topology $\mathscr{T}_j$ of $\mathfrak{A}_j$ is the restriction of $\mathscr{T}_{j+1}$ on $\mathfrak{A}_{j+1}$ ([18], Chapter I, § 4,3)). Further, since $\mathfrak{A}_j$ (in $\mathscr{F}$) is closed in $\mathfrak{A}_{j+1}$ ([18], loc. cit.), $\mathfrak{A}_j$ is a $\mathscr{T}_{\mathscr{F}}$-closed linear subspace of $\mathfrak{A}_{\mathscr{F}}$ from the Hahn-Banach theorem, each normal linear functional on $\mathfrak{A}_j$ is the restriction of one on $\mathfrak{A}$ which is locally normal.

Since the locally normal states of $\mathfrak{A}$ form a folium (Proposition 6), and the vector states of a faithful representation of a C^*-algebra have convex hull which is w^*-dense in the set of all states of that algebra

[15] Obviously defined as those normal in restriction to all $\mathfrak{A}_\alpha \in \mathscr{F}$.

([6], 3.4.4); to establish that the locally normal states of $\mathfrak{A}$ are w^*-dense in the set of all states of $\mathfrak{A}$, it will suffice to show that the direct sum τ of the representations of $\mathfrak{A}$ arising from all locally normal states of $\mathfrak{A}$ is faithful. If $\tau(A)=0$, with A in $\mathfrak{A}_j$, each locally normal state (and, from Proposition 6, each locally normal form) of $\mathfrak{A}$ vanishes on A. Since $\mathcal{T}_{\mathcal{F}}$ is Hausdorff, $A=0$. Thus τ is faithful, hence isometric, on each $\mathfrak{A}_j$. By norm-density of $\mathfrak{A}_{\mathcal{F}}$ in $\mathfrak{A}$ and norm-continuity of τ, τ is isometric, hence faithful, on $\mathfrak{A}$.

If ϱ is a normal state of $\mathfrak{A}_j$, there is a T of trace-class on H_τ (with $0 \leq T \leq I$) such that $\varrho(A) = \text{Trace}[T\tau(A)]$, for A in $\mathfrak{A}_j$. Then $B \to \text{Trace}[T\tau(B)]$ defines a locally normal state extension of ϱ to $\mathfrak{A}$.

Proposition 8. *Let $\mathfrak{A}$ be a C^*-algebra with a sequential funnel $\mathcal{F} = \{\mathcal{M}_j\}_{j=1,2,\ldots}$. Then*

(i) *a state ϱ of $\mathfrak{A}$ is locally normal (relative to $\mathcal{F}$) if and only if the representation it generates is separable;*

(ii) *each automorphism α of $\mathfrak{A}$ maps each locally normal state ϱ of $\mathfrak{A}$ into a locally normal state $\varrho \circ \alpha$.*

Proof. We know by [13, 14, and 15] that every representation of a factor algebra on a separable Hilbert space is normal.

Let ϱ be a state of $\mathfrak{A}$ generating the representation φ_ϱ acting on $\mathcal{H}$ with corresponding cyclic vector χ_0. If φ_ϱ is separable, so is its restriction to each $\mathcal{M}_j \in \mathcal{F}$ which is thus normal, hence ϱ is locally normal. Conversely if ϱ (and, hence, φ_ϱ) is locally normal, $\varphi_\varrho \mid \mathcal{M}_j$ is normal for all $\mathcal{M}_j \in \mathcal{F}$; and thus $[\varphi_\varrho(\mathcal{M}_j)\chi_0]$ is separable for all $\mathcal{M}_j \in \mathcal{F}$ [16]. Then $\mathcal{H}$ is separable as the closure of the union of the $[\varphi_\varrho(\mathcal{M}_j)\chi_0]$, $\mathcal{M}_j \in \mathcal{F}$. The proof of (i) is complete. Property (ii) then follows from (i) and the fact that the state $\varrho \circ \alpha$ generates the representation $\varphi_\varrho \circ \alpha$ acting in the same Hilbert space as φ_ϱ.

Proposition 9. *Let $\mathfrak{A}$ be a C^*-algebra with a Type I-funnel $\mathcal{F} = \{\mathcal{M}_\alpha\}$. Denote by $\mathcal{C}_\alpha$ the set of operators compact relative to $\mathcal{M}_\alpha$ [17]; and by $\mathcal{C}$ the C^*-algebra generated by all $\mathcal{C}_\alpha$. Each locally normal state of $\mathfrak{A}$ restricts to a state φ of $\mathcal{C}$ for which $\|\varphi \mid \mathcal{C}_\alpha\| = 1$ for all $\mathcal{C}_\alpha$; and each such state of $\mathcal{C}$ has a unique state extension to $\mathfrak{A}$ which is locally normal.*

Proof. Let ϱ be a locally normal state of $\mathfrak{A}$. Since $\varrho \mid \mathcal{M}_\alpha$ is normal and since $\mathcal{C}_\alpha$ is ultraweakly dense in $\mathcal{M}_\alpha$, $\varrho \mid \mathcal{M}_\alpha$ is the unique normal extension of $\varrho \mid \mathcal{C}_\alpha$ to $\mathcal{M}_\alpha$. As $\bigcup_\alpha \mathcal{M}_\alpha$ is norm dense in $\mathfrak{A}$, ϱ is the unique locally normal extension of $\varphi(=\varrho \mid \mathcal{C})$. With E_n a sequence of projections

[16] $[M]$ denotes the closed subspace generated by the set M.

[17] I.e. elements of $\mathcal{M}_\alpha$ which are compact operators in the faithful irreducible representation of $\mathcal{M}_\alpha$. Proposition 9 is in fact independent of the assumption that the factors $\mathcal{M}_\alpha$ are countably decomposable embeddings of Type I-factors being automatically normal. This remark holds also for Proposition 11, 12, and 13 below.

in $\mathscr{C}_\alpha$ tending strongly to I, $\varrho(E_n) \to 1$, since $\varrho \mid \mathscr{M}_\alpha$ is normal. Thus $\|\varrho \mid \mathscr{C}_\alpha\| \,(= \|\varphi \mid \mathscr{C}_\alpha\|) = 1$.

Firstly, let $\bar{\varphi}$ be a state extension of φ from $\mathscr{C}$ to $\mathfrak{A}$ ([6], 1.7.2, 2.1.5 (vi), 2.10.1). Then $\bar{\varphi} \mid \mathscr{M}_\alpha$ is a state extension of $\varphi \mid \mathscr{C}$. Since $\|\varphi \mid \mathscr{C}_\alpha\| = 1$, such extension is unique and normal ([6], 2.11.7). Thus $\bar{\varphi}$ is a locally normal extension of φ to $\mathfrak{A}$.

Proposition 10. *Let $\mathfrak{A}$ be a C^*-algebra with a Type I-funnel $\mathscr{F} = \{\mathscr{M}_\alpha\}$ and assume that each $\mathscr{M}_\alpha \in \mathscr{F}$ is included in a $\mathscr{M}_\beta \in \mathscr{F}$ in which it has infinite relative commutant. Then $\mathfrak{A}$ is simple.*

Proof (cf. [19], Theorem 7). If $\mathscr{I}$ is a proper two-sided ideal in $\mathfrak{A}$ the norm closure of $\mathscr{I}$ is proper (since an operator near I is invertible, so that I is not in the norm closure of $\mathscr{I}$). We assume that $\mathscr{I}$ is norm-closed. As $\mathscr{I} \cap \mathscr{M}_\alpha$, $\mathscr{M}_\alpha \in \mathscr{F}$, is a norm-closed two sided ideal of $\mathscr{M}_\alpha$ and $\mathscr{M}_\alpha$ is a countably decomposable factor of the type I_∞, $\mathscr{I} \cap \mathscr{M}_\alpha$ is either $\mathscr{M}_\alpha$, $\mathscr{C}_\alpha$, or (0), where $\mathscr{C}_\alpha$ is the ideal of operators compact relative to $\mathscr{M}_\alpha$. Since $I \notin \mathscr{I}$, $\mathscr{I} \cap \mathscr{M}_\alpha \neq \mathscr{M}_\alpha$. Suppose $\mathscr{I} \cap \mathscr{M}_\alpha = \mathscr{C}_\alpha$ and take $\mathscr{M}_\beta \in \mathscr{F}$ containing $\mathscr{M}_\alpha$ with an infinite relative commutant $\mathscr{N}_\alpha^c = \mathscr{M}_\alpha^c \cap \mathscr{M}_\beta$: $\mathscr{N}_\alpha^c$ is a type I_∞ factor and one has $\mathscr{M}_\beta = \mathscr{M}_\alpha \otimes \mathscr{N}_\alpha^c$ ([18], Chapter I, § 8, 2, Corollaire 3). With $\{E_j\}$ an infinite orthogonal family of projections in $\mathscr{N}_\alpha^c$ and C a non-zero operator in $\mathscr{C}_\alpha$; C^*C dominates some non-zero projection E in $\mathscr{M}_\alpha$ (which must, then, lie in $\mathscr{M}_\alpha$). Since E_j commutes with $\mathscr{M}_\alpha$ and is non-zero; $\{EE_j\}$ is an infinite orthogonal family of non-zero projections in $\mathscr{M}_\beta$ dominated by E. Thus E is not finite relative to $\mathscr{M}_\beta$ (recall that $\mathscr{M}_\beta$ is of type I, so that each EE_j contains some minimal projection and all minimal projections are equivalent in $\mathscr{M}_\beta$). Hence C^*C, and, therefore, C are not in $\mathscr{C}_\beta$.

Since $C \in \mathscr{I} \cap \mathscr{M}_\beta$; $\mathscr{I} \cap \mathscr{M}_\beta = \mathscr{M}_\beta$; contradicting the choice of $\mathscr{I}$ as "proper". Hence $\mathscr{I} \cap \mathscr{M}_\alpha = (0)$ for all α; and the representation φ of $\mathfrak{A}$ on $\mathfrak{A}/\mathscr{I}$ defined by $\varphi(A) = A + \mathscr{I}$ is an isomorphism on each $\mathscr{M}_\alpha$. Thus φ is isometric on each $\mathscr{M}_\alpha$ and has a unique isometric isomorphic extension φ mapping $\mathfrak{A}$ onto $\mathfrak{A}/\mathscr{I}$. Thus $\mathscr{I} = (0)$; and $\mathfrak{A}$ is simple.

Proposition 11. *Let $\mathfrak{A}^{(j)}, j = 1, 2$, be C^*-algebras with respective sequential Type I-funnels $\mathscr{F}^{(j)} = \{\mathscr{M}_n^{(j)}\}_{n=0,1,2,\ldots}$ (where $\mathscr{M}_0^{(j)} = \{\lambda I^{(j)}\}$, with $I^{(j)}$ the unit of $\mathfrak{A}^{(j)}$). Let $\mathscr{N}_n^{(j)} = \mathscr{M}_{n-1}^{(j)c} \cap \mathscr{M}_n^{(j)}$, $n = 1, 2, \ldots$. If we assume that $\mathscr{N}_n^{(1)}$ is $*$-isomorphic to $\mathscr{N}_n^{(2)}$ for all n, $\mathfrak{A}^{(1)}$ and $\mathfrak{A}^{(2)}$ are isomorphic as C^*-algebras.*

Proof. With φ_n an isomorphism of $\mathscr{N}_n^{(1)}$ onto $\mathscr{N}_n^{(2)}$, ψ_p defined on $A_1 \ldots A_p$, $A_n \in \mathscr{N}_n^{(1)}$, by $\psi_p(A_1 \ldots A_p) = \varphi_1(A_1) \ldots \varphi_p(A_p)$ extends to an isomorphism ψ_p of the von Neumann algebra generated by $\mathscr{N}_1^{(1)} \ldots \mathscr{N}_p^{(1)}$, i.e. $\mathscr{M}_p^{(1)}$, onto that generated by $\mathscr{N}_1^{(2)} \ldots \mathscr{N}_p^{(2)}$, i.e. $\mathscr{M}_p^{(2)}$. Since the ψ_p, $p = 1, 2, \ldots$ are extensions of one another and are isometric, they extend by continuity to an isometric isomorphism of $\mathfrak{A}^{(1)}$ onto $\mathfrak{A}^{(2)}$.

The following propositions (12 and 13) are generalizations of ([9], Lemma 2.4 and Theorems 2.5, 2.7). We restate Proposition 12 as a corollary using the language of Section 2. In this same style, Proposition 13 follows from Propositions 2 and 3 there. However, in order to keep this Section independent of the somewhat heavy apparatus of Section 1, we will give a self-contained proof. For establishing Proposition 12 we need:

Lemma 3. *If $\mathcal{M}$ is a factor of type I contained in a C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$ and $\mathcal{M}'\cap\mathfrak{A}$ together with $\mathcal{M}$ generate $\mathfrak{A}''$ (as a von Neumann algebra), then $\mathcal{M}'\cap\mathfrak{A}'' = (\mathcal{M}'\cap\mathfrak{A})''$.*

Proof. Since both $\mathcal{M}'$ and $\mathfrak{A}''$ are weak-operator closed, the weak-operator closure, $(\mathcal{M}'\cap\mathfrak{A})''$, of $\mathcal{M}'\cap\mathfrak{A}$ is contained in $\mathcal{M}'\cap\mathfrak{A}''$. To establish the reverse inclusion, suppose E is a minimal projection in $\mathcal{M}$. The mapping $T'\to T'E$ is an isomorphism of $\mathcal{M}'$ onto $\mathcal{M}'E$, since $\mathcal{M}$ is a factor. We show that the two subsets $(\mathcal{M}'\cap\mathfrak{A})''$ and $\mathcal{M}'\cap\mathfrak{A}''$ of $\mathcal{M}'$ have the same image, $(\mathcal{M}'\cap\mathfrak{A})''E$, under this mapping; so that $(\mathcal{M}'\cap\mathfrak{A})'' = \mathcal{M}'\cap\mathfrak{A}''$. For this, note that ETE is a scalar multiple of E for each T in $\mathcal{M}$; so that the strongly continuous mapping, $S\to ESE$, carries the algebra generated (algebraically) by $\mathcal{M}$ and $\mathcal{M}'\cap\mathfrak{A}$ onto $(\mathcal{M}'\cap\mathfrak{A})E$ – hence the strong closure, $\mathfrak{A}''$ (by assumption), of this algebra into the strong closure, $(\mathcal{M}'\cap\mathfrak{A})''E$, of $(\mathcal{M}'\cap\mathfrak{A})E$. In particular $(\mathcal{M}'\cap\mathfrak{A}'')E$ is contained in (hence, coincides with) $(\mathcal{M}'\cap\mathfrak{A})''E$, as we wished to show.

Proposition 12. *If the C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$ has a net $\mathcal{F} = \{\mathcal{M}_\alpha\}$ of (concrete) factors of Type I, then $\mathfrak{A}'\cap\mathfrak{A}'' = \bigcap_{\mathcal{M}_\alpha\in\mathcal{F}} (\mathcal{M}_\alpha'\cap\mathfrak{A})''$.*

Corollary. *If the C^*-algebra $\mathfrak{A}$ has a Type I-funnel $\mathcal{F} = \{\mathcal{M}_\alpha\}$ and if we denote by $\mathcal{S}$ the corresponding set of locally normal positive linear forms, with $[\mathcal{S}]^* = \tau_{\mathcal{S}}(\mathfrak{A})''$, $\mathcal{Z}_{\mathcal{S}}$, $\mathcal{Z}_l$ and $\mathcal{Z}_c$ as defined in Section 1, then $\mathcal{Z}_l$ reduces to the multiples of the unit, while $\mathcal{Z}_c = \mathcal{Z}_{\mathcal{S}}$.*

Proof. Note first that for each $\mathcal{M}_\alpha\in\mathcal{F}$, every $\mathcal{M}_\beta\in\mathcal{F}$ is contained in a $\mathcal{M}_\gamma \supseteqq \mathcal{M}_\alpha$. Since $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha'\cap\mathcal{M}_\gamma$ generate $\mathcal{M}_\gamma$ as a von Neumann algebra; $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha'\cap\mathfrak{A}$ generate a von Neumann algebra containing each $\mathcal{M}_\beta$, i.e. $\mathfrak{A}''$. Applying Lemma 2 we have $(\mathcal{M}_\alpha'\cap\mathfrak{A})'' = \mathcal{M}_\alpha'\cap\mathfrak{A}''$. Clearly $\mathfrak{A}''\cap\mathfrak{A}' = \bigcap_{\mathcal{M}_\alpha\in\mathcal{F}} \mathcal{M}_\alpha'\cap\mathfrak{A}''$; so that $\bigcap_{\mathcal{M}_\alpha\in\mathcal{F}} (\mathcal{M}_\alpha'\cap\mathfrak{A})'' = \mathfrak{A}''\cap\mathfrak{A}'$. To obtain the corollary, we note that since $\mathcal{S}$ consists of locally normal states of $\mathfrak{A}$, $\tau_{\mathcal{S}}$ is normal (thus faithful and isometric) in restriction to each $\mathcal{M}_\alpha$. Identifying $\mathcal{M}_\alpha (\subseteqq \mathfrak{A})$ with $\tau_{\mathcal{S}}(\mathcal{M}_\alpha)$ $(\subseteqq \tau_{\mathcal{S}}(\mathfrak{A}))$ and $\mathfrak{A}$ with $\tau_\varrho(\mathfrak{A})$, $\mathfrak{A}'\cap\mathfrak{A}''$ is identified with $\mathcal{Z}_{\mathcal{S}}$. Moreover $\mathcal{Z}_l$ consists of scalars since the center of each $\mathcal{M}_\alpha$ consists of scalars (and all its elements have central support I in $\mathfrak{A}''$). As for $\mathcal{Z}_c = \left(\bigcap_{\mathcal{M}_\alpha\in\mathcal{F}} \mathcal{M}_\alpha^{c\,\prime\prime} = \right) \bigcap_{\mathcal{M}_\alpha\in\mathcal{F}} (\mathcal{M}_\alpha'\cap\mathfrak{A})''$, we have just noted that it coincides with all of $\mathscr{Z}_{\mathscr{S}}$. From the fact that $\mathscr{Z}_c = \mathscr{Z}_{\mathscr{S}}$, combined with Propositions 2 and 3 of Section 1, we deduce:

Proposition 13. *If the C*-algebra $\mathfrak{A}$ has a Type I-funnel $\mathscr{F} = \{\mathscr{M}_\alpha\}$, then*

(i) *a locally normal state ϱ of $\mathfrak{A}$ is primary if and only if, to each $A \in \mathfrak{A}$, there is a $\mathscr{M}_\alpha \in \mathscr{F}$ with*

$$|\varrho(AB) - \varrho(A)\,\varrho(B)| \leqq \|B\| \tag{5}$$

for all $B \in \mathscr{M}_\alpha^c$;

(ii) *if two locally normal states ϱ and τ of $\mathfrak{A}$ generate disjoint representations, they are such that $\|\varrho \mid \mathscr{M}_\alpha^c - \tau \mid \mathscr{M}_\alpha^c\| = 2$ for all $\mathscr{M}_\alpha \in \mathscr{F}$. The converse holds for ϱ and τ primary. Two locally normal primary states ϱ and τ of $\mathfrak{A}$ generate quasi-equivalent representations if and only if, to each $\varepsilon > 0$, there is a $\mathscr{M}_\alpha \in \mathscr{F}$ with $\|\varrho \mid \mathscr{M}_\alpha^c - \tau \mid \mathscr{M}_\alpha^c\| \leqq \varepsilon$.*

For the direct proof of this proposition, we use:

Lemma 4. *If $\mathfrak{A}$ is a C*-algebra acting on the Hilbert space $\mathscr{H}$, $\mathscr{F} = \{\mathfrak{A}_\alpha\}$ is a net for $\mathfrak{A}$, $\mathfrak{A}''$ is a factor, and x_0 is a unit vector in $\mathscr{H}$; then to each $y \in \mathscr{H}$ and $\varepsilon > 0$, there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ such that*

$$|(y, Ax_0) - (x_0, Ax_0)\,(y, x_0)| \leqq \varepsilon \tag{6}$$

for all A of unit norm in $\mathfrak{A}_\alpha^c$ (the relative commutant of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$).

Proof. Suppose the contrary. Then there is an $\varepsilon > 0$ and a $y \in \mathscr{H}$ such that, for all $\mathfrak{A}_\alpha \in \mathscr{F}$ the set

$$Q_\alpha = \{X \in \mathfrak{A}_\alpha^{c''};\ |(y, Xx_0) - (x_0, Xx_0)\,(y, x_0)| \geqq \varepsilon,\ \|X\| \leqq 1\}$$

is non void. Due to (i) of Definition 2, the Q_α have the finite intersection property; so that, by the weak-operator compactness of the unit ball of $\mathfrak{A}''$, all Q_α have a common element A for which

$$|(y, Ax_0) - (x_0, Ax_0)\,(y, x_0)| \geqq \varepsilon\,.$$

As A is in $\mathfrak{A}_\alpha^{c''}$ for each $\mathfrak{A}_\alpha \in \mathscr{F}$, A is in $\mathfrak{A}'$ as well as $\mathfrak{A}''$. Thus, since $\mathfrak{A}''$ is a factor, $A = aI$ for some scalar a, and

$$(y, Ax_0) = a(y, x_0) = (x_0, Ax_0)\,(y, x_0)$$

contradicting the previous inequality.

Taking $y = T^* x_0$, $T \in \mathscr{B}(\mathscr{H})$[19], in the preceding lemma, we have:

Corollary. *With the same assumptions as in Lemma 3; to each $\varepsilon > 0$ and $T \in \mathscr{B}(\mathscr{H})$, there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ such that, for all A in the unit ball of $\mathfrak{A}_\alpha^c$,*

$$|\omega_{x_0}(TA) - \omega_{x_0}(T)\,\omega_{x_0}(A)| \leqq \varepsilon\,. \tag{7}$$

[19] $\mathscr{B}(\mathscr{H})$ denotes the set of bounded linear operators on $\mathscr{H}$ and ω_{x_0} the vector state defined by $x_0 \in \mathscr{H}$.

When $\mathscr{F}$ is a net of Type I-factors, Proposition 12 can be used to establish the following partial converse:

Lemma 5. *If $\mathfrak{A}$ is a C^*-algebra acting on the Hilbert space $\mathscr{H}$ with a net $\mathscr{F} = \{\mathscr{M}_\alpha\}$ of (concrete) factors of Type I; and if x_0 is a unit vector in $\mathscr{H}$ cyclic for $\mathfrak{A}$ and such that, to each $T \in \bigcup_{\mathscr{M}_\beta \in \mathscr{F}} \mathscr{M}_\beta$ and $\varepsilon > 0$, there is a $\mathscr{M}_\alpha \in \mathscr{F}$ with*

$$\|(\omega_{T^* x_0, x_0} - \omega_{x_0}(T)\omega_{x_0}) \mid \mathscr{M}_\alpha^c\| < \varepsilon, \tag{8}$$

then $\mathfrak{A}''$ is a factor.

Proof. Suppose P is a central projector in $\mathfrak{A}''$; and $0 \neq P \neq I$. Since x_0 is cyclic for $\mathfrak{A}$, it is separating for $\mathfrak{A}'$ – hence for $\mathfrak{A}'' \cap \mathfrak{A}'$. Thus

$$(x_0, Px_0) = \|Px_0\|^2 \neq 0 \neq \|(I-P)x_0\|^2 = (x_0, (I-P)x_0).$$

Setting $(x_0, Px_0)(x_0, (I-P)x_0)$ equal to 5ε, we can choose a self-adjoint T in $\bigcup_{\mathscr{M}_\beta \in \mathscr{F}} \mathscr{M}_\beta$ such that $\|(T-P)x_0\| \leqq \varepsilon$ $\left(\text{since } \left\{\bigcup_{\mathscr{M}_\beta \in \mathscr{F}} \mathscr{M}_\beta\right\}'' = \mathfrak{A}''\right)$. From Proposition 12 and the Kaplansky density theorem, given $\mathscr{M}_\alpha \in \mathscr{F}$, we can choose a self-adjoint S in the unit ball of $\mathscr{M}_\alpha^c$ such that $\|(S-(I-P))x_0\| \leqq \varepsilon$. Then

$$\begin{aligned}
\|(\omega_{T^* x_0, x_0} - \omega_{x_0}(T)\omega_{x_0}) \mid \mathscr{M}_\alpha^c\| &\geqq |\omega_{T^* x_0, x_0}(S) - \omega_{x_0}(T)\,\omega_{x_0}(S)| \\
&= |(x_0, TSx_0) - (x_0, Tx_0)(x_0, Sx_0)| \\
&= |(x_0, (T-P)Sx_0) + (x_0, P(S-(I-P))x_0) \\
&\quad + (x_0, (P-T)x_0)(x_0, Sx_0) \\
&\quad - (x_0, Px_0)(x_0, (S-(I-P))x_0) \\
&\quad - (x_0, Px_0)(x_0, (I-P)x_0)| \\
&\geqq -4\varepsilon + 5\varepsilon = \varepsilon
\end{aligned}$$

for all $\mathscr{M}_\alpha$, contradicting (8). Thus $\mathfrak{A}''$ is a factor.

Proof of Proposition 13. Let $\mathfrak{A}$ be a C^*-algebra with a net $\mathscr{F} = \{\mathfrak{A}_\alpha\}$; and let ϱ be a primary state of $\mathfrak{A}$. Applying the corollary to Lemma 4 to the concrete C^*-algebra $\varphi_\varrho(\mathfrak{A})$ (with the net $\{\varphi_\varrho(\mathfrak{A}_\alpha)\}$ and the cyclic vector x_ϱ), we get the direct statement of (i), Proposition 13 (in fact the result obtained is somewhat more general – cf. corollary to Propositions 2, 3 of Section 2). For obtaining the converse statement of (i), Proposition 13, we need the additional assumptions that $\mathscr{F}$ is a Type I-funnel and ϱ a locally normal state, in which case $\{\varphi(\mathfrak{A}_\alpha)\}$ is a Type I-funnel for $\varphi(\mathfrak{A})$ and Lemma 5 applies to give the result.

Assuming, still, that $\mathscr{F}$ is a Type I-funnel and that ϱ, τ are locally normal states, the direct sum ψ of the representations they generate maps $\mathfrak{A}$ onto a C^*-algebra $\psi(\mathfrak{A})$ acting on a Hilbert space $\mathscr{H}$, with ϱ and τ given by $\varrho = \omega_x \circ \psi$, $\tau = \omega_y \circ \psi$, $x, y \in \mathscr{H}$. In addition, $\{\psi(\mathfrak{A}_\alpha) = \mathscr{N}_\alpha\}$ is a Type I-funnel for $\psi(\mathfrak{A})$; and the representations induced by ϱ and τ are given by restriction of ψ to the cyclic projections E and F respectively generated by x and y (under $\psi(\mathfrak{A})$).

These representations are disjoint if and only if the central carriers C_E and C_F of E and F are orthogonal. In this case $C_E - C_F$ is an operator in the unit ball of the center $\mathscr{Z}$ of $\psi(\mathfrak{A})''$ (coinciding with the intersection of all $\mathscr{N}_\alpha^{c''}$, $\mathfrak{A}_\alpha \in \mathscr{F}$, as we know by Proposition 12). Now

$$\|(\omega_x - \omega_y) \mid \mathscr{Z}\| \geqq \|(\omega_x - \omega_y)(C_E - C_F)\| = 2\,;$$

while we have for each $\mathfrak{A}_\alpha \in \mathscr{F}$, using the Kaplansky density theorem,

$$\|(\omega_x - \omega_y) \mid \mathscr{N}_\alpha^{c''}\| = \|(\varrho - \tau)(\mathfrak{A}_\alpha^c)\| \geqq \|(\omega_x - \omega_y) \mid \mathscr{Z}\|\,.$$

We see, by comparison, that two disjoint ϱ and τ are such that $\|(\varrho-\tau) \mid \mathfrak{A}_\alpha^c\| = 2$ for all $\mathfrak{A}_\alpha \in \mathscr{F}$, the first assertion of (ii), Proposition 13. For the converse, with ϱ and τ primary, if they are not disjoint they are quasi-equivalent; and $\|(\varrho - \tau) \mid \mathfrak{A}_\alpha^c\| < 2$ for some α will follow from the proof of the (direct part) of the last statement of the proposition. We assume that ϱ and τ are primary and quasi-equivalent. In this case we know that $\psi(\mathfrak{A})''$ is a factor; and we can apply Lemma 4 with $x_0 = 2^{-\frac{1}{2}}(x-y)$, $y = x+y$ to conclude that to each $\varepsilon > 0$, there is a $\mathfrak{A}_\alpha \in \mathscr{F}$ with

$$\begin{aligned} &|(x+y, \psi(A)(x-y)) - \tfrac{1}{2}(x-y, \psi(A)(x-y))(x+y, x-y)| \\ &\quad = |\varrho(A) - \tau(A)| \leqq \varepsilon \end{aligned}$$

for all A in the unit ball of $\mathfrak{A}_\alpha^c$. Conversely, with $\|(\varrho - \tau) \mid \mathfrak{A}_\alpha^c\| < 2$ for some $\mathfrak{A}_\alpha$ in $\mathscr{F}$, ϱ, and τ are not disjoint (from the above). Being primary, ϱ and τ are quasi-equivalent.

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R. Haag
Institut für theoretische Physik der Universität
2000 Hamburg 50
Luruper Chaussee 149

R. V. Kadison
Department of Mathematics
University of Pennsylvania
Philadelphia, Pa., USA

D. Kastler
Centre de Physique Théorique
C.N.R.S.
31, Chemin Joseph-Aiguier
13 Marseille 9e, France