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Boson Fields with Nonlinear Selfinteraction in Two Dimensions*

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Abstract. Semiboundedness of the total Hamiltonian is proved for a selfinteracting Boson field in two dimensional space time. The interaction is given by a Wick polynomial: $P(\Phi)$:. The polynomial P is required to have even degree and its leading coefficient must be positive. A space cutoff is introduced in the interaction Hamiltonian.

§ 1. Introduction

In [10] NELSON considered the following problem. Let Φ be a neutral scalarfield of mass $\mu_0 > 0$ in two dimensional space time and let

$$H = H_0 + g \int : \Phi^4(x) : dx \, , \tag{1.1}$$

where H_0 is the free Hamiltonian for the mass μ_0 and $g > 0$. If the system is placed in a box with periodic boundary conditions then Nelson proved that H is bounded from below. H thus has a natural selfadjoint semibounded extension (the Friedrichs extension), which can (presumably) be used to solve the Schrödinger equation. In [5], Jaffe considered the related Hamiltonian

$$H = H_0 + \int : P(\Phi(x)) : h(x) \, dx \, , \tag{1.2}$$

P a polynomial, again in two dimensional space time. Jaffe showed H to be a symmetric densely defined operator; no box is needed here. In this paper we apply Nelson's method to Jaffe's Hamiltonian (1.2). Our main result is

Theorem A. *Let h be a nonnegative function in $L_1 \cap L_2$. Suppose that the polynomial P in* (1.2) *has even degree and that the leading coefficient is positive. Then H is bounded from below.*

By elementary methods we also show that the Hamiltonian

$$\varepsilon N + \int : \Phi^2(x) : h^2(x) \, dx \tag{1.3}$$

is bounded from below, where ε is any positive number and N is the number of particles operator. This bound on (1.3) permits an improve-

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ment in the results of [2]. In [1], we defined a renormalized Hamiltonian H_{ren} for Fermi and Boson fields with Yukawa coupling (and with a space cutoff). The Hamiltonian was shown to exist as a bilinear form. In [2] we showed that

$$0 \leqq 2^{-1} F_\tau \leqq H_{\text{ren}} + \delta \int : \Phi^2(x) : h^2(x)\, dx + cI\,. \tag{1.4}$$

Here F_τ is an operator defined in [2], and all that concerns us is its property:

$$N \leqq F_\tau\,. \tag{1.5}$$

Also δ is a number which may be taken as small as desired, but is bounded above, $\delta \leqq \delta_0$, and in general δ_0 is negative. Combining the bound on (1.3) with (1.4) and (1.5), we have the following theorem.

Theorem B. *The bilinear form H_{ren} of [2] is bounded from below. Furthermore for any a in $[0,2^{-1})$ and for any number δ there is a $c = c(a, \delta)$ such that*

$$0 \leqq aF_\tau \leqq H_{\text{ren}} + \delta \int : \Phi^2(x) : h^2(x)\, dx + cI\,. \tag{1.6}$$

Thus the upper bound on the δ in (1.4) is removed and any choice of δ is possible. If our model were realistic ($h \equiv 1$, four dimensions) we would then choose δ so that the spectrum of the operator on right side of (1.6) agrees with experiment. See [7] for a further discussion of this point. One would like to know how the spectrum depends on δ. The idea expressed in [1, p. 345] on the removal of the space cutoff was first advanced by GUENIN [3]. GUENIN proposed that if A is an observable associated with a bounded region and $h \equiv 1$ on a larger region then

$$e^{-itH} A\, e^{itH}$$

is (formally) independent of h for small t. See [12, Theorem 3] for a further development of this point of view.

SEGAL [12] and SYMANZIK (unpublished) have studied Hamiltonians such as (1.2). As part of a general study of Wick products, [12, 13], SEGAL has announced a new proof of Jaffe's theorem that (1.2) is densely defined. SYMANZIK observed that Nelson's methods were not limited to fourth powers in the interaction and that the periodic boundary conditions in a finite region (as used by NELSON) could be replaced by other, for example Dirichlet, conditions.

In Theorem A, H_0 could be replaced by N with only trivial changes in the proof.

§ 2. A Domain for H

We use the Fock space representation for our field Φ. The Fock Hilbert space $\mathscr{F}$ is a direct sum

$$\mathscr{F} = \sum_{n=0}^{\infty} \oplus \mathscr{F}_n$$

and $\mathscr{F}_n$ is the space of symmetric square integrable functions of n variables. Let

$$\mu(k) = (\mu_0^2 + k^2)^{1/2} .$$

Then

$$\Phi^-(x) = \int e^{ikx} a(k) \mu(k)^{-1/2} dk \tag{2.1}$$

$$\Phi^+(x) = \int e^{ikx} a^*(-k) \mu(k)^{-1/2} dk \tag{2.2}$$

and $\Phi = \Phi^- + \Phi^+$, where a and a^* are the annihilation and creation operators,

$$[a(k), a^*(k')] = \delta(k - k') . \tag{2.3}$$

By definition,

$$:\Phi^p(x): = \sum_j \binom{p}{j} \Phi^+(x)^j \Phi^-(x)^{p-j} , \tag{2.4}$$

or in other words, the Wick product differs from the ordinary product in that all the annihilators are placed to the right and the creators are placed to the left. $:\Phi^p(x):$ is not an operator, but it is a densely defined bilinear form.

We take Fourier transforms to compute

$$\begin{aligned} \int :\Phi^p(x): h(x) dx \\ &= \sum_j \binom{p}{j} \int a^*(-k_1) \ldots a^*(-k_j) a(k_{j+1}) \ldots a(k_p) \\ &\times \hat{h}(k_1 + \cdots + k_p) \Pi \mu(k_i)^{-1/2} dk_i \end{aligned} \tag{2.5}$$

where $\hat{h}$ is the Fourier transform of h. We assume h is in L_2, and so $\hat{h}$ is in L_2 also. Since $\mu(k) \sim |k|$ for large k, one can show that

$$\hat{h}(k_1 + \cdots + k_p) \prod_i \mu(k_i)^{-1/2} \in L_2 . \tag{2.6}$$

It is well known that (2.6) implies that each integral on the right side of (2.5) is an operator defined on the domain $\mathscr{D}(N^{p/2})$ of $N^{p/2}$. This domain is the set of $\phi = \phi_0, \phi_1, \ldots, \phi_j \in \mathscr{F}_j$ with

$$\sum \|n^{p/2} \phi_n\|^2 < \infty .$$

Thus (2.5) is an operator defined on $\mathscr{D}(N^{p/2})$. Similarly

$$H_0 + \int :P(\Phi(x)): h(x) dx$$

is an operator defined on the dense domain, $\mathscr{D}(H_0) \cap \mathscr{D}(N^{d/2})$, where d is the degree of the polynomial P.

§ 3. $\varepsilon N + \int :\Phi^2(x): h^2(x) dx$ is positive

In this section we suppose $h^2 \in L_2$ so that the operator (1.3) is defined on $\mathscr{D}(N)$. We set

$$\Phi(k) = a(k) + a^*(-k) .$$

Then

$$\int :\Phi^2(x):h^2(x)\,dx$$
$$= \int \hat{h}^2(k+l)\,\Phi(k)\,\Phi(l)\,\mu(k)^{-1/2}\mu(l)^{-1/2}\,dk\,dl \tag{3.1}$$
$$= \int \hat{h}(\eta+k)\,\hat{h}(-\eta+l)\,\Phi(k)\,\Phi^*(-l)\,\mu(k)^{-1/2}\mu(l)^{-1/2}\,dk\,dl\,d\eta\,.$$

We assume h is real, so that $\hat{h}^-(k) = \hat{h}(-k)$. Then (3.1) is equal to

$$\int :A(\eta)\,A^*(\eta):d\eta$$

where

$$A(\eta) = \int \hat{h}(\eta+k)\,\Phi(k)\,\mu(k)^{-1/2}\,dk\,.$$

Let

$$B_\varrho = \int_{|\eta|\leqq\varrho} :A(\eta)\,A^*(\eta):d\eta \tag{3.2}$$

and set (3.1) equal to

$$B_\varrho + B'_\varrho\,.$$

We assert that B_ϱ is semibounded and that B'_ϱ is small relative to N, for large ϱ. The second statement means that

$$B'_\varrho \leqq \varepsilon(N+I) \tag{3.3}$$

if $\varrho \geqq \varrho_0 = \varrho_0(\varepsilon)$. To prove (3.3) we note that B'_ϱ can be written as a sum of four integrals of the form

$$B = \int v(k_1,k_2)\,a^{\#}(k_1)\,a^{\#}(k_2)\,dk_1\,dk_2\,,$$

with $a^{\#} = a^*$ or $a^{\#} = a$ and with $v \in L_2$. Moreover v depends on ϱ and $\|v\|_2 \to 0$ as $\varrho \to \infty$. It is known that

$$(N+I)^{-1/2}\,B\,(N+I)^{-1/2} \leqq \text{const.}\,\|v\|_2 I\,,$$

and so (3.3) follows.

To prove that B_ϱ is semibounded we use the commutation relations (2.3) to remove the Wick ordering from (3.2). We find

$$:A(\eta)\,A^*(\eta): = A(\eta)\,A^*(\eta) - \int |\hat{h}(\eta+k)|^2\,\mu(k)^{-1}\,dk\,I$$

and so

$$B_\varrho = \int_{|\eta|\leqq\varrho} A(\eta)\,A^*(\eta)\,d\eta + \iint_{|\eta|\leqq\varrho} |\hat{h}(\eta+k)|^2\,\mu(k)^{-1}\,dk\,d\eta\,I\,.$$

The first term on the right is obviously positive and the second is a finite multiple of the identity. Thus B_ϱ and also

$$\varepsilon N + \int :\Phi^2(x)\,h^2(x)\,dx$$

are semibounded under the assumption that h^2 is nonnegative and in L_2.

§ 4. Reduction to a Problem with Discrete Momentum

We follow a procedure of Nelson [9] for approximating (2.5) by a finite sum.

Choose numbers γ and $\varkappa$. (Later we let $\gamma \to 0, \varkappa \to \infty$). We define

$$\Gamma = \{n\gamma : n = 0, \pm 1, \ldots\}$$

$$\Gamma_\varkappa = \{k : k \in \Gamma, |k| \leqq \varkappa\}$$

and

$$a_\gamma^\#(k) = \gamma^{-1/2} \int_0^\gamma a^\#(k+l)\, dl$$

where $a^\#$ equals a or a^* and $k \in \Gamma$. Then

$$[a_\gamma(k), a_\gamma^*(l)] = \delta_{kl} = \begin{cases} 1 & \text{if} \quad k = l \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Let

$$H_{0\gamma\varkappa} = \sum_{k \in \Gamma_\varkappa} \mu(k)\, a_\gamma^*(k)\, a_\gamma(k)\,. \tag{4.2}$$

One can check that each ϕ in $\mathscr{D}(H_0)$ is in $\mathscr{D}(H_{0\gamma\varkappa})$ also and that

$$\lim_{\substack{\gamma \to 0 \\ \varkappa \to \infty}} H_{0\gamma\varkappa}\, \phi = H_0 \phi\,. \tag{4.3}$$

Next we approximate (2.5) by

$$:\Phi_{\gamma\varkappa}^p(h): = \gamma^{p/2} \sum_{j=0}^{p} \binom{p}{j} \sum_{k_i \in \Gamma_\varkappa} a_\gamma^*(-k_1) \ldots a_\gamma^*(-k_j) \times a_\gamma(k_{j+1}) \ldots a_\gamma(k_p)\, \hat{h}_\gamma\left(\sum_i [k_i]\right) \prod_i \mu([k_i])^{-1/2} \tag{4.4}$$

where

$$\hat{h}_\gamma(k) = \int_{-\pi/\gamma}^{\pi/\gamma} e^{ikx}\, h(x)\, dx$$

and

$$[k] = \sup\{l : l \in \Gamma, l \leqq k\}$$

is the integral part of k relative to the lattice Γ. Since $h \in L_1$, $\hat{h}$ is continuous and

$$\hat{h}_\gamma(\sum [k_i]) \prod_i \mu([k_i])^{-1/2} \to \hat{h}(\sum k_i) \prod_i \mu(k_i)^{-1/2}$$

uniformly. Let $\mathscr{D}$ be the set of states $\phi = \{\phi_0, \phi_1, \ldots\}$ with $\phi_n(k_1, \ldots, k_n) = 0$ for n or $\sum |k_i|$ large. If ϕ and ψ are in $\mathscr{D}$ then

$$\lim_{\substack{\gamma \to 0 \\ \varkappa \to \infty}} (\psi, :\Phi_{\gamma\varkappa}^p(h): \phi) = (\psi, \int :\Phi^p(x): h(x)\, dx\, \phi) \tag{4.5}$$

Thus the bilinear form of

$$H_{\gamma\varkappa} = H_{0\gamma\varkappa} + \sum b_p :\Phi_{\gamma\varkappa}^p(h): \tag{4.6}$$

converges to H on $\mathscr{D} \times \mathscr{D}$ where $b_0, b_1, \ldots$ are the coefficients of $x^0, x^1, \ldots$ in the polynomial $P(x)$. Hence if the $H_{\gamma\varkappa}$ are semibounded with a lower bound independent of γ and $\varkappa$ then H is semibounded also.

Let $\mathscr{F}_\gamma$ be the subspace of $\mathscr{F}$ consisting of functions which are piecewise constant between lattice points. In other words,

$$\phi = \phi_0, \phi_1, \ldots \in \mathscr{F}_\gamma \text{ if}$$
$$\phi_n(k_1, \ldots, k_n) = \phi([k_1], \ldots, [k_n]) .$$

Let $\mathscr{F}_{\gamma\varkappa}$ be the subspace of $\mathscr{F}_\gamma$ defined by the restriction

$$\phi_n(k_1, \ldots, k_n) = 0 \quad \text{if} \quad [k_i] \notin \Gamma_\varkappa$$

for some i, $1 \leqq i \leqq n$.

The operators $a_\gamma^*(k)$ and $a_\gamma(k)$, $k \in \Gamma_\varkappa$, leave $\mathscr{F}_{\gamma\varkappa}$ invariant and act irreducibly on $\mathscr{F}_{\gamma\varkappa}$. We set $\gamma = 2^{-\nu}$, $\varkappa = 2^\nu$, and observe that $\mathscr{F}_{2^{-\nu}, 2^\nu}$ increases monotonically with ν and that

$$\mathscr{D}' = \mathscr{D} \cap \bigcup_\nu \mathscr{F}_{2^{-\nu}, 2^\nu}$$

is dense in $\mathscr{F}$ and

$$H \subset (H|\mathscr{D}')^- .$$

Here $H \mid \mathscr{D}'$ denotes the restriction of H to $\mathscr{D}'$. Thus it is sufficient to prove the semiboundedness of

$$H_{\gamma\varkappa} \mid (\mathscr{D}(H_0) \cap \mathscr{D}(N^{d/2}) \cap \mathscr{F}_{\gamma\varkappa})$$

with a lower bound independent of γ and $\varkappa$.

§ 5. Diagonalizing the Potential

So far we have used a representation of the Hilbert space in which H_0 is diagonalized, or is represented as a multiplication operator. In this section we give a new representation of $\mathscr{F}_{\gamma\varkappa}$ in which the interaction term $:\Phi^p(h):$ is a multiplication operator while the free Hamiltonian becomes more complicated. Let

$$\begin{aligned}
q_{|k|} &= (1/4\,\mu(k))^{1/2}[a_\gamma(k) + a_\gamma^*(k) + a_\gamma(-k) + a_\gamma^*(-k)] \\
q_{-|k|} &= i(1/4\,\mu(k))^{1/2}[-a_\gamma(|k|) + a_\gamma^*(|k|) + a_\gamma(-|k|) - a_\gamma^*(-|k|)] \\
p_{|k|} &= i(\mu(k)/4)^{1/2}[a_\gamma(k) - a_\gamma^*(k) + a_\gamma(-k) - a_\gamma^*(-k)] \\
p_{-|k|} &= (\mu(k)/4)^{1/2}[a_\gamma(|k|) + a_\gamma^*(|k|) - a_\gamma(-|k|) - a_\gamma^*(-|k|)]
\end{aligned}$$

for $0 \neq k \in \Gamma$ and let

$$\begin{aligned}
q_0 &= (1/2\,\mu_0)^{1/2}[a_\gamma(0) + a_\gamma^*(0)] \\
p_0 &= i(\mu_0/2)^{1/2}[a_\gamma(0) - a_\gamma^*(0)] .
\end{aligned}$$

One can compute that

$$H_{0\gamma\varkappa} = \sum_{k \in \Gamma_\varkappa} 2^{-1}[p(k)^2 + \mu(k)^2 q(k)^2 - \mu(k)] . \tag{5.1}$$

As in [9, § 3–4] we replace p_k and q_k by unitarily equivalent operators. Let

$$\mathscr{H}_{\gamma\varkappa} = \bigotimes_{k \in \Gamma_\varkappa} \mathscr{H}_k .$$

where $\mathscr{H}_k$ is L_2 of the real line with respect to the Gaussian measure

$$\phi_k^2(q)\,dq = (\mu(k)/\pi)^{1/2}\exp(-\mu(k)q^2)\,dq\,. \tag{5.2}$$

There is a unitary equivalence between $\mathscr{H}_{\gamma\varkappa}$ and $\mathscr{F}_{\gamma\varkappa}$ which sends q_k into multiplication by q in the factor $\mathscr{H}_k$ and p_k into the operator

$$\phi_k^{-1}\, i\,(d/dq)\,\phi_k$$

again acting in the factor $\mathscr{H}_k$. The proof of this statement is essentially von Neumann's uniqueness theorem for irreducible representations of the commutation relations. We identify $\mathscr{H}_{\gamma\varkappa}$ and $\mathscr{F}_{\gamma\varkappa}$ and we identify q_k, etc. with its image, multiplication by q, etc. Let

$$\begin{aligned} H_{\mu(k)} &= 2^{-1}\phi_k^{-1}\,[-(d/dq)^2 + \mu(k)q^2 - \mu(k)]\,\phi_k \\ &= -\,2^{-1}(d/dq)^2 + \mu(k)\,q\,(d/dq)\,, \end{aligned} \tag{5.3}$$

acting on $\mathscr{H}_k$. Now $-\mathscr{H}_{\mu(k)}$ is the infinitesimal generator of a known Markoff process and furthermore the operator $e^{-tH_{\mu(k)}}$ is an integral operator and the kernel can be computed explicitly, [10]. In particular

$$(e^{-tH_{\mu(k)}}\psi)\,(q) = \int p^t(q,q')\ \psi(q')\,\phi_k^2(q')\,dq' \tag{5.4}$$

for $\psi\in\mathscr{H}_k$, where

$$p^t(q,q') = (1-e^{-2\mu t})^{-1/2}\exp\left[-\frac{\mu(q'-e^{-\mu t}q)^2}{1-e^{-2\mu t}} + \mu\,q'^2\right] \tag{5.5}$$

Let q now denote a variable in a Euclidean space $R_\varkappa$ and let q have coordinates q_k, $k\in\Gamma_\varkappa$. Then

$$\phi_\varkappa^2(q)\,dq = \prod_{k\in\Gamma_\varkappa}\phi_\varkappa^2(q_k)\,dq_k \tag{5.6}$$

is the product of the measures (5.2) and

$$H_{\gamma\varkappa} = L_2(\phi_\varkappa^2(q)\,dq)\,.$$

In addition to the function space L_2, we will have to consider

$$L_r = L_r(\phi_\varkappa^2(q)\,dq)\,.$$

Since $\int\phi_\varkappa^2(q)\,dq = 1$, we have $L_{r_2}\subset L_{r_1}$ if $r_1\leqq r_2$.

Lemma 5.1. *$\exp(-tH_{0\gamma\varkappa})$ is a contraction operator on L_r, $1\leqq r\leqq\infty$. When $T\leqq t$, $1<p$ and $r<\infty$ it is a contraction from L_p to L_r, for some T not depending on γ or $\varkappa$. If p is bounded away from one and r is bounded then T does not depend on p or r.*

Proof. If $\psi\in L_\infty$, then by a change of variables,

$$\begin{aligned} &|(e^{-tH_\mu}\,\psi)\,(q)| \\ &\quad\leqq (\mu/\pi)^{1/2}(1-e^{-2\mu t})^{-1/2}\int\exp\left[\frac{-\mu(q_k'-e^{-2\mu t}\,q_k)^2}{1-e^{-2\mu t}}\right]\ \psi(q')\,dq_k' \\ &\quad\leqq \pi^{-1/2}\,\|\psi\|_\infty\int\exp(-q_k'^2)\,dq_k' = \|\psi\|_\infty\,, \end{aligned}$$

and so $\exp(-tH_{0\gamma\varkappa})$ is a contraction on L_∞. Since $H_{0\gamma\varkappa}\geqq 0$, $\exp(-tH_{0\gamma\varkappa})$

is a contraction on L_2 and thus on L_r, $2 \leqq r \leqq \infty$, by the Riesz Thorin convexity theorem [8]. Let $\|\exp(-tH_{0\gamma\varkappa})\|_{pr} = C_{pr}$ be the norm of $\exp(-tH_{0\gamma\varkappa})$ as an operator from L_p to L_r. The cited theorem states that $\log C_{pr}$ is a convex function of p^{-1} and r^{-1} in the square

$$0 \leqq p^{-1}, r^{-1} \leqq 1 .$$

Now $\exp(-tH_{0\gamma\varkappa})$ is symmetric in the L_2 inner product, since $p^t(q_k, q_k') = \exp[-b(e^{-\mu t}q_k'^2 - 2q_k q_k' + e^{-\mu t}q_k^2)]$ with

$$b = \mu e^{-\mu t}(1 - e^{-2\mu t})^{-1} .$$

Thus by duality, $\exp(-tH_{0\gamma\varkappa})$ is a contraction from L_r to L_r, $1 < r \leqq 2$. It is a contraction on L_1 by taking limits as $r \to 1$.

Let

$$a^2(q_k) = \int p^t(q_k, q_k')^2 \phi_k^2(q_k)\, dq_k .$$

Then

$$|(e^{-tH_\mu}\psi)(q_k)| \leqq a(q_k)\, \|\psi\|_2$$

by the Schwartz inequality, so $\|a(q_k)\|_r$ bounds the norm of $\exp(-tH_\mu)$ as an operator from L_2 to L_r. For $r - 1 < e^{2\mu t}$, $\|a(q_k)\|^r$ is finite and has a bound independent of $\mu \geqq \mu_0$ and $t \geqq T$, for some T, as computed in [10]. Let $\mathscr{H}_k'$ be the orthogonal compliment of 1 in $\mathscr{H}_k$. Since the norm of $\exp(-tH_\mu)$, as an operator from $\mathscr{H}_k'$ to $\mathscr{H}_k'$, tends to zero as $t \to \infty$, the norm of

$$\exp(-tH_\mu) : \mathscr{H}_k' \to L_4$$

also tends to zero as $t \to \infty$, uniformly in k. Let $\psi' \in \mathscr{H}_k'$ and let $\psi = 1 + \psi'$. Then

$$\begin{aligned}(\exp(-tH_\mu)\,\psi)^2 &= 1 + 2\exp(-tH_\mu)\,\psi' + (\exp(-tH_\mu)\,\psi')^2\\ &= [1 + c] + [2\exp(-tH_\mu)\,\psi' + (\exp(-tH_\mu)\,\psi')^2 - c] .\end{aligned}$$

We choose

$$c = \big(1, (\exp(-tH_\mu)\,\psi')^2\big) = \|\exp(-tH_\mu)\,\psi'\|_2^2$$

and then the terms in the brackets above are orthogonal. Thus

$$\begin{aligned}\|\exp(-tH_\mu)\,\psi\|_4^4 &= \|(\exp(-tH_\mu)\,\psi)^2\|_2^2\\ &\leqq (1 + c)^2 + \big(2c^{1/2} + \|(\exp(-tH_\mu)\,\psi')^2\|_2\big)^2\\ &\leqq (1 + c)^2 + 8c + 2\,\|(\exp(-tH_\mu)\,\psi')^2\|_2^2\\ &= (1 + c)^2 + 8c + 2\,\|\exp(-tH_\mu)\,\psi'\|_4^4 .\end{aligned}$$

For large t,

$$\|\exp(-tH_\mu)\;\psi\|_4^4 \leqq 1 + 2\,\|\psi'\|_2^2 + \|\psi'\|_2^4 = \|\psi\|_2^4 .$$

and so $\exp(-tH_\mu)$ and $\exp(-tH_{0\gamma\varkappa})$ are contractions from L_2 to L_4. The lower bound T on t does not depend on $\varkappa$ or γ.

We apply the Riesz Thorin convexity theorem to the maps

$$\exp(-tH_{0\gamma\varkappa}) : L_2 \to L_4$$
$$\exp(-tH_{0\gamma\varkappa}) : L_\infty \to L_\infty$$

and conclude that for large t

$$\exp(-tH_{0\gamma\varkappa}) : L_r \to L_{2r}$$

is a contraction, $r \geqq 2$, We take products of several of these maps and conclude that $\exp(-tH_{0\gamma\varkappa})$ is a contraction from L_2 to L_r for any $r < \infty$, if t is large. By duality it is a contraction from L_p to L_2 and hence from L_p to L_r for any $p, r \in (1, \infty)$, and the lower bound on t is independent of $\gamma, \varkappa, p$ and r if p is bounded away from 1 and r is bounded.

Now we show that the interaction term $:\phi^p_{\gamma\varkappa}(h):$ is a polynomial in the $q's$.

Let

$$\phi_{\gamma\varkappa}(x) = \gamma^{1/2} \sum_{k \in \Gamma_\varkappa} (a_\gamma(k) + a^*_\gamma(-k))\, \mu(k)^{-1/2} e^{ikx} . \tag{5.7}$$

Since

$$(a_\gamma(k) + a^*_\gamma(-k))\, \mu(k)^{-1/2} = \begin{cases} q_{|k|} + i q_{-|k|} & \text{if} \quad k > 0 \\ 2^{1/2}\, q_0 & \text{if} \quad k = 0 \\ q_{|k|} - i q_{-|k|} & \text{if} \quad k < 0 , \end{cases}$$

$\phi_{\gamma\varkappa}(x)$ and also $\phi^p_{\varkappa\gamma}(x)$ are polynomials in the $q's$. We use the formula

$$\phi^p_{\gamma\varkappa}(x) = \sum_{j=0}^{[p/2]} \frac{p!}{(p-2j)!\,j!}\, 2^{-j} c^j_\varkappa : \phi^{p-2j}_{\gamma\varkappa}(x) : \tag{5.8}$$

to conclude by induction on p that $:\phi^p_{\gamma\varkappa}(x):$ is also a polynomial in the $q's$. In (5.8) the coefficient

$$\frac{p!}{(p-2j)!\,j!}\, 2^{-j}$$

is just the number of ways of selecting j unordered pairs from p objects and $c_\varkappa$ is defined by the formula

$$c_\varkappa = \gamma \sum_{k \in \Gamma_\varkappa} \mu(k)^{-1} ;$$

we have the bound

$$c_\varkappa \leqq K_1 \ln \varkappa \tag{5.9}$$

where K_1 is independent of γ and $\varkappa$. Thus

$$:\phi^p_{\gamma\varkappa}(h): = \int_{-\pi/\gamma}^{\pi/\gamma} :\phi^p_{\gamma\varkappa}(x): h(x)\, dx$$

is a polynomial in the $q's$, as desired.

Let

$$P(x) = b_0 + b_1 x + \cdots + b_d x^d$$

be the polynomial in (1.2) and let

$$V_{\gamma\varkappa} = \sum b_p : \phi^p_{\gamma\varkappa}(h) : \tag{5.10}$$

denote our approximate interaction term, as in (4.6).

Lemma 5.2. *For some constant K_2, independent of γ and $\varkappa$, we have*

$$-(\ln\varkappa)^{d/2}K_2 \leqq V_{\gamma\varkappa}\,. \tag{5.11}$$

Proof. We use (5.8) to remove the Wick ordering in (5.10) and obtain

$$V_{\gamma\varkappa} = \sum a_p(c_\varkappa) \int_{\pi/\gamma}^{\pi/\gamma} \phi_{\gamma\varkappa}^{\,p}(x)\, h(x)\, dx$$

where a_p is a polynomial in $c_\varkappa$ of degree at most $[(d-p)/2]$. The coefficients of a_p depend only on the coefficients of P, and so we have an estimate

$$|a_p(c_\varkappa)| \leqq K' c_\varkappa^{[(d-p)/2]}\,.$$

Since $a_d = b_d > 0$ and since d is even by hypothesis, it follows that $0 < \sum a_p(c_\varkappa)x^p$ for $K''(c_\varkappa + 1) < |x|^2$ and

$$-c_\varkappa^{\,d/2}\,K''' \leqq \sum a_p(c_\varkappa)x^p$$

for all x. We bound $c_\varkappa$ by (5.9) and the proof is complete.

Lemma 5.3. *$V_{\gamma\varkappa} \in L_r$ for all $r < \infty$ and if $\lambda \leqq \varkappa$*

$$\|V_{\gamma\varkappa} - V_{\gamma\lambda}\|_{2j}^{2j} \leqq (dj)\,!K_3^j\lambda^{-j}\,, \tag{5.12}$$

where K_3 is a constant which is independent of γ, λ and $\varkappa$.

Proof. We use the particle representation, $\mathscr{F}_{\gamma\varkappa}$, in place of the representation $\mathscr{H}_{\gamma\varkappa} = L_2(\phi_k^2 dq)$. Now $1 \in \mathscr{H}_{\gamma\varkappa}$ corresponds to the vacuum state

$$\Omega = 1, 0, \ldots \in \mathscr{F}_{\gamma\varkappa}\,,$$

so

$$\begin{aligned}\|V_{\gamma\varkappa} - V_{\gamma\lambda}\|_{2j}^{2j} &= \int (V_{\gamma\varkappa} - V_{\gamma\lambda})^{2j}\phi_k^2 dq\\ &= ((V_{\gamma\varkappa} - V_{\gamma\lambda})^j\Omega, (V_{\gamma\varkappa} - V_{\gamma\lambda})^j\Omega)\\ &= \|V_{\gamma\varkappa} - V_{\gamma\lambda})^j\Omega\|^2\,.\end{aligned} \tag{5.13}$$

We set $\lambda = 0$ above and get

$$\|V_{\gamma\varkappa}\|_{2j}^{2j} = \|V_{\gamma\varkappa}^{2j}\Omega\|^2 < \infty$$

and so $V_{\gamma\varkappa} \in L_r$ for all $r < \infty$. We return to (5.13) and note that $V_{\gamma\varkappa} - V_{\gamma\lambda}$ is a sum of $d2^d$ terms of the form

$$A = b_p\,\gamma^{p/2} \sum_{k_i} \left[\hat{h}_\gamma\left(\sum_{i=1}^{p} k_i\right) : \prod_{i=1}^{p} (a_\gamma^{\#}(\pm k_i)\,\mu(k_i)^{-1/2}) :\right] \tag{5.14}$$

where in the summation over k_i we have

$$k_i \in \Gamma_\varkappa \quad \text{for} \quad 1 \leqq i \leqq p, p \leqq d, \quad \text{and} \quad k_i \notin \Gamma_\lambda$$

for at least one i. Summing again over the same range of k_i, we have

$$\gamma^p \sum_{k_i} \left[\left|\hat{h}_\lambda\left(\sum_{i=1}^{p} k_i\right)\right|^2 \prod_{i=1}^{p} \mu(k_i)^{-1}\right] \leqq K_4\lambda^{-1} \tag{5.15}$$

and K_4 is independent of λ, $\varkappa$ and γ.

Let ψ be a state with at most l particles. It follows from (5.15) and the form of A that

$$\|A\,\psi\|^2 \leqq (l+p)!/l!)\,K_4\lambda^{-1}\,\|\,\psi\|^2$$

and furthermore $A\,\psi$ is a state with at most $l+p$ particles. Thus if we have operators $A_1, \ldots, A_j$ of the form (5.14),

$$\|A_1 \ldots A_j \Omega\|^2 \leqq (dj)!\,K_4^j \lambda^{-1}\,.$$

Hence

$$\|(V_{\gamma\varkappa} - V_{\gamma\lambda})^j\,\Omega\|^2 \leqq (dj)!\,(d\,2^d K_4)^j\,\lambda^{-j} = (dj)!\,K_3^j\,\lambda^{-j}\,,$$

and the proof is complete.

§ 6. Path space

Let C be the space of continuous paths $q = q(s)$,

$$q(s) \in R_\varkappa,\ 0 \leqq s < \infty\,.$$

There is a measure on C intrinsically associated with the semigroup $\exp(-tH_{0\gamma\varkappa})$. To define this measure we set

$$p_k^t(q_k, q_k')\,\phi_k^2(q_k')\,dq_k' = \Pr\{q_k(t) = q_k' \mid q_k(0) = q_k\}\,, \qquad (6.1)$$

the probability that $q_k(t) = q_k'$ if it is known that $q_k(0) = q_k \cdot p_k^t$ is defined by (5.5); we have added a subscript k to indicate the dependence on $\mu = \mu(k)$. Let

$$p_\varkappa^t(q, q') = \prod_{k \in \Gamma_\varkappa} p_k^t(q_k, q_k')\,. \qquad (6.2)$$

The σ-field of measurable subsets of C is generated by the sets

$$q(s_i) \in B_i,\ 1 \leqq i \leqq j\,, \qquad (6.3)$$

where B_i is a Borel subset of $R_\varkappa$. The measure of (6.3) is

$$\int_{B_j \times \cdots \times B_j} \prod_{i=2}^{j} p_\varkappa^{s_i - s_{i-1}}(q(s_{i-1}), q(s_i))\,\phi_\varkappa^2\,(q(s_i))\,dq(s_i)\,\phi_\varkappa^2(q(0))\,dq(0) \quad (6.4)$$

if $s_1 = 0 < s_2 < \ldots < s_j$. The definition (6.4) is forced by the definition (6.1) together with the Markov character of the process, the stipulation that each coordinate q_k of q defines an independent process and the specification of $\phi_k^2(q)dq$ as the probability distribution of the initial point $q(0)$ of the path q.

If $V_1, \ldots, V_j \in L_j(R_\varkappa, \phi_\varkappa^2 dQ)$ then we compute

$$\begin{aligned}\int \prod_i V_i(q(s_i))\,dQ = \int V_1(q(0))\,\phi_\varkappa^2(q(0))\,dq(0) \\ \cdot\,[\exp((s_0 - s_1)\,H_{0\gamma\varkappa})\,V_2\,\exp((s_1 - s_2)\,H_{0\gamma\varkappa}) \\ \cdot\,(\ldots(V_{j-1}\exp((s_{j-1} - s_j)\,H_{0\gamma\varkappa})\,V_j)\ldots)]\,(q(0))\end{aligned} \qquad (6.5)$$

and

$$\left|\int \prod_i V_i(q(s_i))\,dQ\right| \leqq \prod_i \|V_i\|_j \qquad (6.6)$$

using (6.4) and the fact that $\exp(-tH_{0\gamma\varkappa})$ is a contraction on L_r. Furthermore (6.5) and (6.6) remain valid when some of the times s_i coincide.

Lemma 6.1. *Let V be a polynomial function on $R_\varkappa$. Then $\int_0^t V(q(s))\,ds \in L_p(C, dQ)$ for all $p < \infty$ and*

$$\left\| \int_0^t V(q(s))\,ds \right\|_j \leqq t\, \|V\|_j$$

for j an even positive integer.

Proof. For a given path q,

$$I = \int_0^t V(q(s))\,ds = \lim_n \sum_l (t/n)\, V(q(lt/n))$$

and since each Riemann approximating sum to I is a measurable function of q, $I = I(q)$ is measurable also. Let $I_n(q)$ be the n^{th} Riemann approximating sum to I. Then

$$\left|\int I_n(q)^j\, dQ\right| \leqq t^j\, \|V\|_j^j$$

by (6.6) and

$$|I| \leqq \text{const.}\; t(M_t(q))^d$$

where d is the degree of V and

$$M_t(q) = \max_{0 \leqq s \leqq t} |q(s)|\,.$$

Thus the lemma will follow from the Lebesgue bounded convergence theorem once we show that $M \in L_p(C, dQ)$ for all $p < \infty$.

However the dQ probability of the set

$$\{|q_k(t)| \leqq a \mid |q_k(0)|\}$$

dominates the Wiener probability of the same set if $|q_k(0)| \leqq a$. It follows that the L_p norm of M_t is dominated by the Wiener L_p norm of M_t. This latter norm is finite by known properties of Wiener measure, [4, p. 25, 26].

Lemma 6.2. *Let $r \in [1, 2)$. There is a T independent of γ and $\varkappa$ such that if $t \geqq T$ and if ϕ and $\psi \in L_2(\phi_\varkappa^2 dq)$ then $\phi(q(0))\;\psi(q(t)) \in L_r(C, dQ)$* and

$$\|\phi(q(0))\;\psi(q(t))\|_r \leqq \|\phi\|_2\, \|\psi\|_2$$

The T can be chosen independently of r provided r is bounded away from 2.

Proof.

$$\begin{aligned}\|\phi(q(0))\;\psi(q(t))\|_r^r &= \int |\phi(q(0))|^r\, |\psi(q(t))|^r\, dQ \\ &= \int |\phi|^r\, [\exp(-tH_{0\gamma\varkappa})\, |\psi|^r]\, \phi_\varkappa^2\, dq \\ &\leqq \|\, |\phi|^r\|_{2/r}\, \|\exp(-tH_{0\gamma\varkappa})\, |\psi|^r\|_p\end{aligned}$$

where p is the conjugate index to $2/r$; $(r/2) + p^{-1} = 1$. However

$$\|\, |\phi|^r\|_{2/r} = \|\phi\|_2^r$$

and

$$\|\exp(-tH_{0\gamma\varkappa})\,|\psi|^r\|_p \leqq \|\,|\psi|^r\|_{2/r} = \|\psi\|_2^r$$

for large t, by Lemma 5.1.

§ 7. The Feynman Kac formula

The Feynman Kac formula states that

$$(\phi, \exp(-tH_{\gamma\varkappa})\psi) = \int \phi(q(0))^- \exp\left(-\int_0^t V_{\gamma\varkappa}(q(s))\,ds\right) \psi(q(t))dQ\,. \tag{7.1}$$

The right member of (7.1) is bounded by

$$\|\phi(q(0))^-\,\psi(q(t))\|_{p'}\left\|\exp\left(-\int_0^t V_{\gamma\varkappa}(q(s))\,ds\right)\right\|_p$$

$$\leqq \|\phi\|_2\,\|\psi\|_2\left\|\exp\left(-\int_0^t V_{\gamma\varkappa}(q(s))\,ds\right)\right\|_p$$

for $p > 2$ and for t large, by Lemma 6.2. Thus

$$\|\exp(-tH_{\gamma\varkappa})\| \leqq \left\|\exp\left(-\int_0^t V_{\gamma\varkappa}(q(s))\right)\right\|_p$$

and

$$-t^{-1}\,ln\left[\left\|\exp\left(-\int_0^t V_{\gamma\varkappa}(q(s))\right)\right\|_p\right] \leqq H_{\gamma\varkappa}\,. \tag{7.2}$$

Let

$$I_\lambda = \int_0^t V_{\gamma\lambda}(q(s))\,ds\,.$$

Then

$$-\,K_2t(ln\,\lambda)^{d/2} \leqq I_\lambda$$

by Lemma 5.2. Let $K_5, \ldots$ denote positive constants depending only on t and the polynomial P and let Pr denote the measure defined by dQ. Then

$$\Pr\{I_\varkappa \leqq -K_2t(ln\,\lambda)^{d/2} - 1\} \leqq \Pr\{|I_\varkappa - I_\lambda| \geqq 1\} \tag{7.3}$$

$$\leqq \int |I_\varkappa - I_\lambda|^{2j}\,dQ \leqq t^{2j}\,\|V_{\gamma\varkappa} - V_{\gamma\lambda}\|_{2j}^{2j}$$

(by Lemma 6.1)

$$\leqq (dj)!\,t^{2j}\,K_3^j\,\lambda^{-j}$$

(by Lemma 5.3)

$$\leqq (dj)^{dj}\,e^{-d(j+1)}\,K_5^j\,\lambda^{-j}$$

by Stirling's formula. We choose j so that

$$j \leqq d^{-1}\,K_5^{-1/d}\,\lambda^{1/d} < j + 1\,.$$

Then

$$e^{-d(j+1)} \leqq \exp(-K_5^{-1/d}\lambda^{1/d})$$

is a bound for (7.3) and so

$$\Pr\{I_\varkappa \leqq -X-1\} \leqq \exp(-K_6\, e^{K_7 X^{2/d}})\,.$$

Thus

$$\int |e^{-I_\varkappa}|^p\, dQ = \int e^{-pI_\varkappa}\, dQ \leqq e^{-2p} + \sum_{n \geqq 1} \exp(p(n+2)) \exp(-K_6\, e^{K_7 n^{2/d}})$$

is bounded independently of γ and $\varkappa$ and combining this with (7.2) we have $H_{\gamma\varkappa}$ bounded below by a constant which is independent of $\varkappa$ and γ; according to § 4, this proves Theorem A.

The formula (7.1) can be proved by standard methods. See for example [6, p. 168–171], where a similar formula is derived.

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