Commun. math. Phys. 35, 257—264 (1974)

On the Time Evolution Automorphisms of the CCR-Algebra for Quantum Mechanics

M. Fannes* and A. Verbeure

Katholieke Universiteit Leuven, Belgium

Received September 13, 1973

Abstract. In ordinary quantum mechanics for finite systems, the time evolution induced by Hamiltonians of the form $H = \frac{P^2}{2m} + V(Q)$ is studied from the point of view of *-automorphisms of the CCR C^*-algebra $\bar{\Delta}$ (see Ref. [1, 2]). It is proved that those Hamiltonians do not induce *-automorphisms of this algebra in the cases: a) $V \in \bar{\Delta}$ and b) $V \in L^\infty(\mathbb{R}, dx) \cap L^1(\mathbb{R}, dx)$, except when the potential is trivial.

I. Introduction

Consider the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n, dx^n)$ of square integrable functions on $\mathbb{R}^n$. For notational convenience we restrict ourselves to the case $n = 1$. The general case is a trivial extension.

Define the Schrödinger position and momentum operators respectively by: for $\phi \in \mathscr{H}$, $x \in \mathbb{R}$.

$$(Q\phi)(x) = x\,\phi(x),$$

$$(P\phi)(x) = \frac{1}{i}\frac{\partial}{\partial x}\phi(x); \qquad (\hbar = 1).$$

They satisfy the commutation relations $[Q, P] \subsetneqq i$. Denote $\delta_{p,q} = \exp i(pQ + qP)$; $p, q \in \mathbb{R}$. Form the *-algebra Δ, generated by the unitary operators $\delta_{p,q}$ on $\mathscr{H}$ by taking the finite linear combinations of them, the *-operation is defined by $(\delta_{p,q})^* = \delta_{-p,-q}$ and the product rule is given by

$$\delta_{p,q}\delta_{p',q'} = \delta_{p+p',q+q'} \exp\left\{-\frac{i}{2}(pq' - qp')\right\}.$$

The operator norm closure $\bar{\Delta}$ of Δ is the CCR C^*-algebra, realized as a concrete C^*-algebra in $\mathscr{B}(\mathscr{H})$ (all bounded operators on $\mathscr{H}$). It is equivalent with the one considered in Refs. [1] and [2]. We take this algebra as the basic C^*-algebra for an algebraic formulation of quantum mechanics for finite systems.

* Aspirant N.F.W.O., Belgium.

In this work we are concerned with the time-evolution as a *-automorphism of the algebra of observables. This point of view was mostly accepted as well in the algebraic formulation of field theory [3] as in the algebraic formulation of equilibrium statistical mechanics [4].

It has been proved to hold for spin systems for a large class of potentials [5]. We study if this property holds for ordinary quantum mechanics. Of course the choice of C^*-algebra of observables is very important. We take the smallest C^*-algebra $\bar{\Delta}$ containing the Weyl operators (see [1]). This is not only mathematically interesting, but also the suitable C^*-algebra to introduce plane wave states in quantum mechanics (see Refs. [6] and [7]).

We restrict ourselves to automorphisms induced by quantum mechanical Hamiltonians of the form $H = \frac{P^2}{2m} + V(Q)$ where P and Q are the canonical variables and prove that they never induce *-automorphisms of the C^*-algebra $\bar{\Delta}$ except when the potential is trivial, see Theorems II.5 and II.6 below.

II. Hamiltonians and Time Automorphisms

The quantum mechanical Hamiltonian H_λ is supposed to be given by

$$H_\lambda = \frac{P^2}{2} + \lambda V(Q)\,; \qquad \lambda \in \mathbb{R}$$

(the mass is put equal to one). $V(Q)$ is the potential satisfying:

$$V = V^*$$

$$(V(Q)\phi)(x) = V(x)\,\phi(x)\,; \qquad \phi \in \mathscr{H}, \qquad x \in \mathbb{R}$$

$$\sup_x |V(x)| < \infty, \quad \text{hence} \quad V \in \mathscr{B}(\mathscr{H})\,.$$

As the momentum operator P is self-adjoint, also H_λ is self-adjoint and $\exp(iH_\lambda t)$, $t \in \mathbb{R}$, is a unitary operator on $\mathscr{H}$. Furthermore denote

$$\alpha_t^\lambda(A) = \exp(itH_\lambda)A \exp -(itH_\lambda),. \quad A \in \mathscr{B}(\mathscr{H})$$

$(\alpha_t^\lambda)_t$ is a one-parameter *-automorphism group of $\mathscr{B}(\mathscr{H})$. The main result of this work is the answer to the question: is $(\alpha_t^\lambda)_t$ restricted to the C^*-algebra $\bar{\Delta}$ a *-automorphism group of $\bar{\Delta}$?

First we prove a few Lemma's; remark that $(\alpha_t^0)_t$ is a *-automorphism group of $\bar{\Delta}$, because

$$\alpha_t^0\,\delta_{p,q} = \delta_{p,q+pt}\,.$$

This *-automorphism group is not strongly continuous with respect to the parameter t, as is well known, however we have the following continuity property.

Lemma II.1. *For all $A \in \mathscr{B}(\mathscr{H})$, the map $t \to \alpha_t^0(A)$ is ultrastrongly continuous.*

Proof. Let $U_t^0 = \exp \frac{it P^2}{2}$ then for $\phi \in \mathscr{H}$

$$\begin{aligned} &\|\alpha_t^0(A)\phi - \alpha_{t_0}^0(A)\phi\| \\ &\leqq \|U_t^0 A U_{-t}^0 \phi - U_t^0 A U_{-t_0}^0 \phi\| + \|U_t^0 A U_{-t_0}^0 \phi - U_{t_0}^0 A U_{-t_0}^0 \phi\| \\ &\leqq \|A\| \; \|U_{-t}^0 \phi - U_{-t_0}^0 \phi\| + \|U_t^0 \psi - U_{t_0}^0 \psi\| \end{aligned}$$

where $\psi = A U_{-t_0}^0 \phi$. By the strong continuity of $t \to U_t^0$, the strong continuity of the map $t \to \alpha_t^0(A)$ follows. Because $\|\alpha_t^0(A)\| = \|A\|$ we have also the ultrastrong continuity. Q.E.D.

Lemma II.2. *For all $A \in \mathscr{B}(\mathscr{H})$,*

$$\alpha_t^\lambda(A) = \alpha_t^0(A) + \sum_{n \geqq 1} (i\lambda)^n \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq t} \cdots \int ds_1 \ldots ds_n \; [\alpha_{s_1}^0(V), \ldots [\alpha_{s_n}^0(V), \alpha_t^0(A)] \ldots]; \qquad t \geqq 0$$

where the series and the integrals are in the ultrastrong sense. An analogous series expansion exists for $t \leqq 0$.

Proof. The existence of the integrals in the right hand side of the equality is garantueed by Lemma II.1. The rest of the proof is a matter of verification. Q.E.D.

Lemma II.3. *With the notations of above, if $(\alpha_t^\lambda)_{t \in \mathbb{R}}$ maps $\bar{\Delta}$ into itself, i.e. $\alpha_t^\lambda \bar{\Delta} \subseteq \bar{\Delta}$ for all real λ, then for all $A \in \bar{\Delta}$ and $t \in \mathbb{R}$, and all $t \in \mathbb{R}$, there exists an element $B \in \bar{\Delta}$ such that*

$$B = i \int_0^t ds [\alpha_s^0(V), \alpha_t^0(A)]$$

where again the integral is taken in the ultrastrong sense.

Proof. From Lemma II.2 for all $\phi \in \mathscr{H}$:

$$\begin{aligned} &\left\{ \frac{1}{\lambda} (\alpha_t(A) - \alpha_t^0(A)) - i \int_0^t ds [\alpha_s^0(V), \alpha_t^0(A)] \right\} \phi \\ &= i \sum_{n \geqq 2}^{\infty} (i\lambda)^{n-1} \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq t} \cdots \int ds_1 \ldots ds_n \\ &\qquad [\alpha_{s_1}^0(V), \ldots [\alpha_{s_n}^0(V), \alpha_t^0(A)] \ldots] \phi \end{aligned}$$

or

$$\left\| \frac{1}{\lambda} (\alpha_t^\lambda(A) - \alpha_t^0(A))\phi - i \int_0^t ds [\alpha_s^0(V), \alpha_t^0(A)]\,\phi \right\|$$
$$\leqq 2\|A\|\,\|V\|\,(\exp(2\|A\|\,\|V\|\,\lambda) - 1)\,\|\phi\|$$

and

$$\sup_{\phi \in \mathscr{H}} \frac{1}{\|\phi\|} \left\| \frac{1}{\lambda} (\alpha_t^\lambda(A) - \alpha_t^0(A))\phi - i \int_0^t ds [\alpha_s^0(V), \alpha_t^0(A)]\,\phi \right\|$$
$$\leqq 2\|A\|\,\|V\|\,(\exp(2\|A\|\,\|V\|\,\lambda) - 1) \qquad (*)$$

As $\alpha_t^\lambda(A) \in \bar{\Delta}$ for all $A \in \bar{\Delta}$, and $\lambda \neq 0$, $t \in \mathbb{R}$, then also $\frac{1}{\lambda}(\alpha_t^\lambda(A) - \alpha_t^0(A)) \in \bar{\Delta}$ and together with $(*)$ we get

$$\lim_{\lambda \to 0} \frac{1}{\lambda} (\alpha_t^\lambda(A) - \alpha_t^0(A)) \equiv B$$

exists as an element of $\bar{\Delta}$, moreover

$$B = i \int_0^t ds [\alpha_s^0(V), \alpha_t^0(A)]\,. \quad \text{Q.E.D.}$$

In the following Lemma a characterization of the elements of the C^*-algebra $\bar{\Delta}$ is given:

Lemma II.4. *Each element A of $\bar{\Delta}$ can be written in the form*

$$A = \sum_{p,q} \mu(p,q)\delta_{p,q}$$

where $\mu(p,q) = \omega_0(\delta_{-p,-q}A)$; ω_0 is the central state [1] *on $\bar{\Delta}$, defined by*

$$\omega_0(\delta_{pq}) = 0 \quad \text{if} \quad q^2 + p^2 \neq 0$$
$$= 1 \quad \text{if} \quad q^2 + p^2 = 0\,.$$

The convergence is in the l^2-sence.

Proof. Let π_0, $\mathscr{H}_0$, Ω_0 be respectively the cyclic representation, representation space and cyclic vector induced by the central state ω_0. Consider the map

$$\phi : A \in \bar{\Delta} \to \pi_0(A)\Omega_0 \in \mathscr{H}_0\,.$$

As the state ω_0 is faithful [1], the map ϕ is a bijection and as the set $\{\pi_0(\delta_{p,q})\Omega_0 \mid p, q \in R\}$ is an orthonormal basis of $\mathscr{H}_0$ we have

$$\pi_0(A)\Omega_0 = \sum_{p,q} (\pi_0(\delta_{p,q})\Omega_0, \pi_0(A)\Omega_0)\,\pi_0(\delta_{p,q})\Omega_0$$
$$= \sum_{p,q} \omega_0(\delta_{-p,-q}A)\,\pi_0(\delta_{p,q})\Omega_0$$

hence the Lemma follows. Q.E.D.

Denote $\mathscr{H}^2 = \mathscr{H} \otimes \mathscr{H}$. The following map π of $\bar{\Delta}$ into $\mathscr{B}(\mathscr{H}^2)$ extends to a *-representation of $\bar{\Delta}$ ([1], Proposition 3.4):

$$\pi(\delta_{p,q}) = \delta_{\frac{p}{\sqrt{2}}, \frac{q}{\sqrt{2}}} \otimes \delta_{-\frac{q}{\sqrt{2}}, \frac{p}{\sqrt{2}}} . \tag{1}$$

For any pair of elements $\psi, \phi \in \mathscr{H}$ such that $\|\psi\| = \|\phi\| = 1$, consider the vector state $\omega_{\phi,\psi}$ defined by

$$\omega_{\phi,\psi}(X) = (\phi \otimes \psi, \pi(X)\, \phi \otimes \psi), \qquad X \in \bar{\Delta} . \tag{2}$$

As the map

$$(p, q) \to \omega_{\phi,\psi}(\delta_{p,q}) = \left(\phi, \delta_{\frac{p}{\sqrt{2}}, \frac{q}{\sqrt{2}}} \phi\right) \left(\psi, \delta_{-\frac{q}{\sqrt{2}}, \frac{p}{\sqrt{2}}} \psi\right)$$

is continuous, the state $\omega_{\phi,\psi}$ is a Weyl state of the canonical commutation relations. By von Neumann's uniqueness theorem [8], the representation $\pi_{\phi,\psi} = \pi$ induced by the state $\omega_{\phi,\psi}$ is a direct sum of copies of the Schrödinger representation. Hence the map

$$X \in \bar{\Delta} \to (\pi(Y)\phi \otimes \psi, \pi(X)\phi \otimes \psi)$$

for all $Y \in \bar{\Delta}$ is ultrastrongly continuous ([9], p. 54), and π can be continuously extended to the ultrastrong closure $\mathscr{B}(\mathscr{H})$ of $\bar{\Delta}$. This extension is used in the proof of the following main Theorems.

Theorem II.5. *If the potential V belongs to the algebra $\bar{\Delta}$, then for all real $\lambda \neq 0$ and real t, the *-automorphism α_t^λ of $\mathscr{B}(\mathscr{H})$ is not a *-automorphism of the C *-subalgebra $\bar{\Delta}$, except for V a multiple of the unity operator.*

Proof. Suppose that α_t^λ is a *-automorphism of $\bar{\Delta}$ then by Lemma II.3 there exists an element B of the algebra $\bar{\Delta}$ such that

$$B = i \int_0^t ds [\alpha_s^0(V), \delta_{pq}] , \tag{3}$$

where the integral is taken in the ultrastrong sense.

The essential part of the proof consists in showing that $B = 0$ independent of the choice of t, p and q.

In that case, it follows that

$$[\alpha_t^0(V), \delta_{p,q}] = 0$$

for all, t, p, and q; this means that V commutes with all elements of $\bar{\Delta}$ and hence with $\mathscr{B}(\mathscr{H})$. It follows that V is a multiple of the unity operator, and the theorem is proved.

Now we proceed in proving that $B=0$.

Apply the representation π of $\mathscr{B}(\mathscr{H})$ constructed above to the equality (3):

$$\pi(B)=\pi\left(i\int_0^t ds[\alpha_s^0(V),\delta_{p,q}]\right).$$

Perform the substitution $\psi=\phi=\frac{1}{\sqrt{2n}}\chi_n$ in formula (2); χ_n is the characteristic function of the interval $[-n,n]$. Formula (2) becomes

$$\omega_{\chi_n,\chi_n}(\delta_{-p_0,-q_0}B)$$
$$=\frac{1}{4n^2}\left(\chi_n\otimes\chi_n,\pi(\delta_{-p_0,-q_0})\,\pi\left(i\int_0^t ds[\alpha_s^0(V),\delta_{p,q}]\right)\chi_n\otimes\chi_n\right).$$

Because of the ultra strong continuity of π and the integral

$$\omega_{\chi_n,\chi_n}(\delta_{-p_0,-q_0}B)$$
$$=\frac{1}{4n^2}\left(\pi(\delta_{p_0,q_0})\,\chi_n\otimes\chi_n,\,i\int_0^t ds[\pi(\alpha_s^0(V),\pi(\delta_{p,q}))]\,\chi_n\otimes\chi_n\right).$$

As $V\in\bar{\Delta}$, by Lemma II.4 the potential is of the form

$$V=\sum_k\mu(k)\,\delta_{k,0}$$

and by an explicit calculation we get:

$$\begin{aligned}&\omega_{\chi_n,\chi_n}(\delta_{-p_0,-q_0}B)\\&=-2\int_0^t ds\sum_k\mu(k)\sin\frac{1}{2}(ps-q)k\\&\quad\cdot\exp\frac{i}{2}[p_0(ks+q)-q_0(k+p)]\\&\quad\cdot\frac{1}{2n}\left(\chi_n,\delta_{\frac{k+p-p_0}{\sqrt{2}},\frac{ks+q-q_0}{\sqrt{2}}}\chi_n\right)\\&\quad\cdot\frac{1}{2n}\left(\chi_n,\delta_{\frac{-ks+q-q_0}{\sqrt{2}},\frac{k+p-p_0}{\sqrt{2}}}\chi_n\right).\end{aligned}$$

Using the fact that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2n}(\chi_n,\delta_{p,q}\chi_n)&=0\quad\text{for}\quad p\neq 0\\&=1\quad\text{for}\quad p=0\end{aligned}$$

we get

$$\lim_{n\to\infty} \omega_{\chi_n,\chi_n}(\delta_{-p_0,-q_0}B)=0 \quad \text{for all} \quad p_0, q_0 . \tag{4}$$

Again, as $B \in \bar{\Delta}$, by Lemma II.4, B is of the form

$$B=\sum_{p,q} \beta(p,q)\delta_{p,q}$$

and from (4) it follows that $\beta(p,q)=0$ for all $p,q \in R$, hence $B=0$. Q.E.D.

Next we prove an other theorem with an even negative result. As in Theorem II.5 if V belongs to the algebra $\bar{\Delta}$, then $V(x)$ is an almost periodic function of the position variable x. One may guess that a potential, which goes to zero at infinity fast enough, may save the situation. That this is not true is proved in the following.

Theorem II.6. *Let V be any multiplication operator on $\mathscr{H}$, such that $V \in L^1(\mathbb{R}, dx) \cap L^\infty(\mathbb{R}, dx)$, then for all real $\lambda \neq 0$ and real t, the *-automorphism α_t^λ of $\mathscr{B}(\mathscr{H})$ is not a *-automorphism of the C^*-subalgebra $\bar{\Delta}$, except for $V=0$.*

Proof. The proof of this theorem goes exactly along the same lines as that of Theorem II.5, therefore we restrict ourselves to indicating the points where the proof differs.

The potential V does not belong to the C^*-algebra $\bar{\Delta}$, but as $V \in L^1(\mathbb{R}, dx) \cap L^\infty(\mathbb{R}, dx)$ it has a Fourier transform $\tilde{v}$ such that

$$V=\int_R \tilde{v}(k)\delta_{k,0}\,dk$$

and an argument analogous as that in the proof of Lemma II.1 yields the existence of the integral in the ultrastrong sense. It follows that

$$\pi(\alpha_s^0(V))=\int_R dk\,\tilde{v}(k)\,\pi(\alpha_s^0(\delta_{k,0})) .$$

The rest of the proof is obtained by substituting $\sum_k \ldots$ into $\int_R dk \ldots$. Q.E.D.

Remark. As it was not our aim to prove Theorem II.6 with the minimal conditions on the potential, we remark that they can easily be weakened yielding the same result.

References

1. Manuceau, J., Sirugue, M., Testard, D., Verbeure, A.: Commun. math. Phys. **32**, 231 (1973)
2. Slawny, J.: Commun. math. Phys. **24**, 151 (1972)
   Manuceau, J.: Ann. Inst. Henri Poincaré **8**, 139 (1968)
3. Haag, R., Kastler, D.: J. Math. Phys. **5**, 848 (1964)

4. Haag, R., Hugenholtz, N. M., Winnink, M.: Commun. math. Phys. **5**, 215 (1967)
5. Robinson, D. W.: Commun. math. Phys. **7**, 337 (1968)
6. Sirugue, M.: Improper states of canonical commutation relations for a finite number of degrees of freedom; CNRS. Marseille, preprint 71/p. 421
7. Fannes, M., Verbeure, A., Weder, R.: On momentum states in quantum mechanics; Leuven preprint, July 1973
8. Von Neumann, J.: Math. Ann. **104**, 570 (1931)
9. Dixmier, J.: Les algèbres d'opérateurs dans l'espace hilbertien; Paris: Gauthier-Villars 1957

Communicated by W. Hunziker

M. Fannes
A. Verbeure
Institute voor Theoretische Natuurkunde
Celestijnenlaan 200 D
B-3030 Heverlee/Belgium