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On the Mathematical Structure of the B. C. S.-Model. II

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Abstract. It is shown for the degenerate B.C.S.-model how in the limit of an infinite system the exact thermal Greens-functions approach a gauge invariant average of the one's calculated with the Bogoliubov-Haag method.

§ 1. Introduction

In a previous paper [1] it was studied in which sense the B.C.S.-model is solved by the Bogoliubov-Haag [2] method in the infinite volume limit. We investigated how the B.C.S.-Hamiltonian $H_{\text{B.C.S.}}$ converges towards the Bogoliubov Hamiltonian H_{B} in the infinite tensor product representation of the field operators. It turned out that $H_{\text{B.C.S.}}$ converges only in the rather small subspace in which the gap equation holds. Only in this subspace H_{B} describes the time dependence correctly. In fact outside this subspace the time dependence is not described by a Hamiltonian at all for infinite volume since the corresponding unitary transformation is not weakly continuous. It should be stressed that this is not a mathematical pathology but corresponds to a physically completely sound situation. It is analogous to the Lamor-precession of infinitely many spins.

In this note we shall supplement these somewhat negative statements by a more useful result. We shall prove that the thermal Greens functions are correctly described by H_{B} or

$$\lim_{\Omega\to\infty} \operatorname{Tr} e^{-H_{\text{B.C.S.}}/T}\, e^{it_1 H_{\text{B.C.S.}}} A(x_1)\, e^{-it_1 H_{\text{B.C.S.}}} \ldots$$

$$\ldots e^{it_n H_{\text{B.C.S.}}} A(x_n)\, e^{-it_n H_{\text{B.C.S.}}} / \operatorname{Tr} e^{-H_{\text{B.C.S.}}/T} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \,. \qquad (1)$$

$$\operatorname{Tr} e^{-H_{\text{B}}/T}\, e^{it_1 H_{\text{B}}} A(x_1)\, e^{-it_1 H_{\text{B}}} \ldots e^{it_n H_{\text{B}}} A(x_n)\, e^{-it_n H_{\text{B}}} / \operatorname{Tr} e^{-H_{\text{B}}/T}$$

where Ω stands for the volume and the A's are field operators. ϕ is a phase angle over which we have to average to make the procedure invariant. In other words the representation furnished by thermal expectation values is one of the good ones where H_{B} gives the correct time dependence.

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For simplicity we shall use the quasi-spin formalism and consider the degenerate (strong coupling) case only. Our results strengthen previous findings [3] where it was shown that in a suitable perturbation expansion the difference of the two sides of (1) goes with $1/\Omega$ in each order. To make this argument rigorous one would have to establish the uniformity of the convergence of the perturbation expansion for $\Omega \to \infty$. We shall not have this problem since we will calculate both sides of (1) exactly.

§ 2. The Formalism

With the quasi-spin formalism one can write the B.C.S.-Hamiltonian in the form:

$$H_{\text{B.C.S.}} = -\sum_{p=1}^{\Omega} \varepsilon\sigma_p^{(z)} - \frac{2T_c}{\Omega} \sum_{p=1}^{\Omega} \sigma_p^+ \sum_{p'=1}^{\Omega} \sigma_{p'}^- . \tag{2}$$

Here the $\boldsymbol{\sigma}_p$ are a set of Ω independent spin matrices[1] and $\sigma^{\pm}$ the usual combinations $\frac{1}{2}(\sigma^{(x)} \pm i\sigma^{(y)})$. In the degenerate model ε is independent of p. We are interested in a representation of the algebra of the σ's which is furnished via the G-N-S-construction by the positive linear functional $\langle A\rangle_\Omega$ given by the thermal expectation value

$$\langle A\rangle_\Omega = \operatorname{Tr} e^{-\frac{1}{T} H_{\text{B.C.S.}}} A / \operatorname{Tr} e^{-\frac{1}{T} H_{\text{B.C.S.}}} \tag{3}$$

Since $H_{\text{B.C.S.}}$ acts in a 2^Ω dimensional space there is no problem in defining Tr. A stands for any polynomial in the σ's. The latter can be generated by

$$e^{i\sum_p \alpha_p \sigma_p^{(z)}} e^{i\sum_p \beta_p \sigma_p^{(y)}} e^{i\sum_p \gamma_p \sigma_p^{(z)}} = A . \tag{4}$$

However since $H_{\text{B.C.S.}}$ is invariant under any permutation of the σ_p it is clear that all information is already contained in[2]

$$A_\Omega(a, b, c) = e^{i\frac{a}{\Omega}\sum_{p=1}^{\Omega}\sigma_p^{(z)}} e^{i\frac{b}{\Omega}\sum_{p=1}^{\Omega}\sigma_p^{(y)}} e^{i\frac{c}{\Omega}\sum_{p=1}^{\Omega}\sigma_p^{(z)}} . \tag{5}$$

For instance, $\langle\sigma_p^{(z)}\rangle$ is independent of p and therefore

$$\langle\sigma_p^{(z)}\rangle_\Omega = \frac{\partial}{\partial ia} \langle A_\Omega\rangle_\Omega |_{a=b=c=0} . \tag{6}$$

Using $(\sigma_p^{(i)})^2 = 1$ it is easy to show that the expectation value of any polynomial can be generated by derivatives of A.

[1] We shall henceforth simply call them spins although in this model they a different physical significance.

[2] For $\Omega = \infty$ there is a difficulty in generating $\sigma^{(x)}$ this way. In this case a less familiar parametrisation than the Euler angles has to be used (F. JELINEK, to be published).

In the Bogoliubov-Haag procedure the Hamiltonian is split into

$$H_{\text{B.C.S.}} = H_B + H'$$

$$H_B = -\sum_{p=1}^{\Omega} \varepsilon \sigma_p^{(z)} - 2T_c \sum_{p=1}^{\Omega} (\sigma_p^+ \langle\sigma^-\rangle_B + \sigma_p^- \langle\sigma^+\rangle_B) \tag{7}$$

$$H' = -\frac{2T_c}{\Omega} \sum_{p=1}^{\Omega} (\sigma_p^+ - \langle\sigma^+\rangle_B) \sum_{p'=1}^{\Omega} (\sigma_{p'}^- - \langle\sigma^-\rangle_B) - 2T_c \Omega \langle\sigma^+\rangle_B \langle\sigma^-\rangle_B$$

$\langle\sigma\rangle_B$ is the expectation value of σ_p with H_B which is again independent of p. Now H' is dropped since its operator part is in some sense small and a c-number is irrelevant for expectation values. H_B can be written as

$$H_B = -T\omega \sum_p \boldsymbol{\sigma}_p \mathbf{n} \tag{8}$$

where the unit vector $\mathbf{n}$ and the constant ω is determined by calculating the expectation value of $\boldsymbol{\sigma}$.

$$\langle\boldsymbol{\sigma}\rangle_{\text{B}} = \operatorname{Tr} e^{-H_B/T} \boldsymbol{\sigma} / \operatorname{Tr} e^{-H_B/T} = \mathbf{n} \operatorname{Th} \omega\,. \tag{9}$$

Comparing (7), (8) and (9) we find that ω and the angle θ between $\mathbf{n}$ and the z-axis are determined by

$$\omega = \frac{T_c}{T} \operatorname{Th} \omega \quad \cos\theta = \frac{\varepsilon}{T\omega}\,. \tag{10}$$

The azimuthal angle ϕ of $\mathbf{n}$ remains arbitrary. This was to be anticipated since $H_{\text{B.C.S.}}$ is invariant under rotations around the z-axis. The latter corresponds to gauge transformations of the electron operators in the usual formalism. H_B is again invariant under permutations of the σ_p so that $\langle A(a, b, c)\rangle_B$ suffices to characterize the representation of the σ's. However it is immediately clear that $\langle A\rangle_\Omega \neq \langle A\rangle_B$ since H_B and therefore $\langle\ \rangle_B$ is not gauge invariant. For instance, $\langle\sigma^{(x)}\rangle_\Omega = 0$ but $\langle\sigma^{(x)}\rangle_B = n^{(x)} \operatorname{Th}\omega \neq 0$ for $\phi \neq \pi/2$. To make $\langle\ \rangle_B$ gauge invariant we have to average over ϕ and thus the best we can hope for is

$$\lim_{\Omega\to\infty} \langle A_\Omega\rangle_\Omega = \lim_{\Omega\to\infty} \frac{1}{2\pi} \int_0^{2\pi} d\phi\, \langle A_\Omega\rangle_B \tag{11}$$

where $\langle\ \rangle_B$ is taken with a H_B where $\mathbf{n}$ has the azimuthal angle ϕ. Since the spins are independent in H_B it is clear that $\langle\ \rangle_B$ becomes independent of Ω. The latter must be large enough that all σ's in A are contained in the first Ω ones. Furthermore the limit $\Omega \to \infty$ should be attained such that all derivatives at $a = b = c = 0$ are equal. We shall see that this is actually the case.

§ 3. The Right Hand Side of (11)

The evaluation of $\langle A\rangle_B$ is quite simple like the expectation value of spins in an external magnetic field in direction $\mathbf{n}$. By an elementary calculation we find for one spin

$$\frac{1}{2}\operatorname{Sp} e^{i\alpha\sigma^{(x)}} e^{i\beta\sigma^{(y)}} e^{i\gamma\sigma^{(z)}} e^{\omega\mathbf{n}\boldsymbol{\sigma}} = \operatorname{Ch}\omega\cos\beta\cos(\alpha+\gamma) + \\ + i\operatorname{Sh}\omega\big(\cos\theta\cos\beta\sin(\alpha+\gamma) + \\ + \sin\theta\sin\beta(\cos\phi\sin(\alpha-\gamma) + \\ + \sin\phi\cos(\alpha-\gamma))\big). \tag{12}$$

For Ω spins we work in the tensor product and therefore we simply multiply the expressions (12) for the individual spins together. Thus we have

$$\langle A_\Omega(a,b,c)\rangle_B = \left\{\cos\frac{b}{\Omega}\cos\frac{a+c}{\Omega} + i\operatorname{Th}\omega\left(\cos\theta\cos\frac{b}{\Omega}\sin\frac{a+c}{\Omega} + \\ + \sin\theta\sin\frac{b}{\Omega}\sin\left(\phi+\frac{a-c}{\Omega}\right)\right)\right\}^{\Omega}. \tag{13}$$

In the limit $\Omega\to\infty$ this approaches

$$\langle A_\Omega(a,b,c)\rangle_B \to e^{i\operatorname{Th}\omega((a+c)\cos\theta + b\sin\theta\sin\phi)} \tag{14}$$

uniformly for finite values of the argument. Furthermore the limits of the derivatives are the derivatives of the limit. The gauge-variant nature of this expectation value is exhibited by its ϕ-dependence which gives, f.i. $\langle\sigma^{(y)}\rangle_B = \operatorname{Th}\omega\sin\theta\sin\phi$. This vanishes on integrating over ϕ:

$$\langle A_\infty(a,b,c)\rangle_{\overline{B}} = \frac{1}{2\pi}\int\limits_0^{2\pi} d\phi\,\langle A_\infty(a,b,c)\rangle_B = J_0(b\sin\theta\operatorname{Th}\omega)\times \\ \times e^{i(a+c)\operatorname{Th}\omega\cos\theta}. \tag{15}$$

It should be noted that on averaging over ϕ correlations between the spins are introduced. They are not present in (14) since H_B is the sum of Hamiltonians for the individual spins. For instance we have

$$\langle\sigma_p^{(y)}\rangle_{\overline{B}} = 0 \tag{16}$$

$$\langle\sigma_p^{(y)}\sigma_{p'}^{(y)}\rangle_{\overline{B}} = \frac{\partial^2}{\partial(ib)^2}\langle A(a,b,c)\rangle_{\overline{B}} \neq 0 = \langle\sigma_p^{(y)}\rangle_{\overline{B}}\langle\sigma_{p'}^{(y)}\rangle_{\overline{B}}.$$

It turns out that these are exactly the correlations created by $H_{\text{B.C.S.}}$ where the spins are coupled.

§ 4. The Left Hand Side of (11)

The diagonalization of $H_{\text{B.C.S.}}$ simply amounts to diagonalizing $\mathbf{S}^2$ and S_z of the "total spin".

$$\mathbf{S} = \frac{1}{2}\sum_{p=1}^{\Omega}\boldsymbol{\sigma}_p. \tag{17}$$

Designating the eigenvalues by $S(S+1)$ and S_z resp. we have[3] $-S \leqq S_z \leqq S$, $0 \leqq S \leqq \Omega/2$. The multiplicity of the levels with (S, S_z) is found[(4)] to be $\frac{\Omega!(2S+1)}{(\Omega/2-S)!\,(\Omega/2+S+1)!}$. Thus we obtain

$$\operatorname{Tr} e^{-\frac{1}{T}H_{\text{B.C.S.}}} A_\Omega = \sum_{S=0}^{\Omega/2} \sum_{S_z=-S}^{S} \frac{\Omega!(2S+1)}{(\Omega/2-S)!\,(\Omega/2+S+1)!} \cdot \\ \cdot e^{\frac{1}{T}\left(2\varepsilon S_z + \frac{2T_c}{\Omega}(S(S+1)-S_z(S_z+1))\right)} \left(S, S_z \left| e^{\frac{2ia}{\Omega}S_z} e^{\frac{2ib}{\Omega}S_y} e^{\frac{2ic}{\Omega}S_z} \right| S, S_z\right). \tag{18}$$

The matrix element of A_Ω occuring in (18) is well-known from the representations of the rotation group and expressible in terms of a hypergeometric function [5]:

$$G_\Omega\left(\frac{2S}{\Omega}, \frac{2S_z}{\Omega}; a, b, c\right) = \left(S, S_z \left| e^{\frac{2ia}{\Omega}S_z} e^{\frac{2ib}{\Omega}S_y} e^{\frac{2ic}{\Omega}S_z} \right| S, S_z\right) \\ = \sum_\chi (-)^\chi \frac{(S+S_z)!\,(S-S_z)!\, e^{2iS_z\frac{a+c}{\Omega}}}{(S+S_z-\chi)!\,(S-S_z-\chi)!\,(\chi!)^2} \cos\frac{2S_b}{\Omega} \operatorname{tg}\frac{2\chi_b}{\Omega}. \tag{19}$$

Dividing (18) by $\operatorname{Tr} e^{-\frac{1}{T}H_{\text{B.C.S.}}}$ we see that $\langle A_\Omega\rangle$ is the average of G taken with a certain probability measure. In statistical mechanics one usually replaces such a sum by its leading term. Since we want to establish our result with certainty we justify this procedure in the following way: To approach the limit $\Omega\to\infty$ we switch over to the intensive quantities

$$\eta = \frac{2S}{\Omega}, \quad n = \frac{2S_z}{\Omega}, \quad 0 \leqq \eta \leqq 1, \quad |n| \leqq \eta. \tag{20}$$

Giving unit measure to the unit area in the η-n-plane the probability measure is

$$P_\Omega(\eta, n) = \frac{\frac{\Omega!\,(2S+1)}{(\Omega/2-S)!\,(\Omega/2+S+1)!} e^{\frac{1}{T}\left(2\varepsilon S_z + \frac{2T_c}{\Omega}(S(S+1)-S_z(S_z+1))\right)}}{\left(\frac{2}{\Omega}\right)^2 \sum_{S'=0}^{\Omega/2} \sum_{S_z'=-S'}^{S'} \frac{\Omega!\,(2S'+1)}{(\Omega/2-S')!\,(\Omega/2+S'+1)!} \times e^{\frac{1}{T}\left(2\varepsilon S_z' + \frac{2T_c}{\Omega}(S'(S'+1)-S_z'(S_z'+1))\right)}}$$

$$= \frac{e^{-\Omega\left(f(\eta_0)-f(\eta')+\frac{T_c}{2T}(n'-n_0)^2\right)} \phi_\Omega(\eta)}{\left(\frac{2}{\Omega}\right)^2 \sum_{n',n} e^{-\Omega\left(f(\eta_0)-f(\eta')+\frac{T_c}{2T}(n'-n_0)\right)} \phi_\Omega(\eta')} \tag{21}$$

[3] we shall take Ω to be even.

with

$$f(\eta)=\frac{2T_c}{T}\eta^2-\frac{1-\eta}{2}\ln(1-\eta)-\frac{1+\eta}{2}\ln(1+\eta)\,,$$

$$f'(\eta_0)=0\,,\quad \eta_0=\mathrm{Th}\frac{T_c}{T}\eta_0\,,\quad n_0=\frac{\varepsilon}{T_c}-\frac{1}{\Omega}$$

$$\phi_\Omega^2(\eta)=\frac{2(\eta+1/\Omega)^2\, e^{\eta\frac{T_c}{T}-\int\limits_0^\infty \frac{dt\,2}{e^{2\pi t}-1}\left(\operatorname{arctg}\frac{t}{\Omega/2(1+\eta)+2}+\operatorname{arctg}\frac{t}{\Omega/2(1-\eta)+1}\right)}}{\left(1-\eta+\frac{2}{\Omega}\right)\left(1+\eta+\frac{4}{\Omega}\right)^3\times\left(1+\frac{2}{\Omega(1-\eta)}\right)^{\Omega(1-\eta)}\left(1+\frac{4}{\Omega(1+\eta)}\right)^{\Omega(1+\eta)}}\,.$$

To obtain these expressions we have used Binets second formula [6] for $\Gamma(z)$. The function ϕ converges for $\Omega\to\infty$ to the harmless expression

$$\phi_\infty^2(\eta)=\frac{\eta^2 e^{2\eta\frac{T_c}{T}}}{(1-\eta^2)(1+\eta)^2 e^6} \tag{22}$$

so that the essential Ω-dependence of (21) is in the exponent. Since f has for $0\leqq\eta\leqq 1$, $|n|\leqq\eta$ one absolute maximum at (η_0, n_0) if $T<T_c$, $|n|_0\leqq\eta_0$ we expect that P goes to a δ-function: at the maximum it will behave like

$$e^{-\Omega\left((\eta-\eta_0)^2\frac{f''(\eta_0)}{2}+(n-n_0)^2\frac{T_c}{2T}\right)}$$

and thus become sharper and sharper for $\Omega\to\infty$. This intuitive argument is made rigorous by proving that the measure of any set not containing (η_0, n_0) becomes zero for $\Omega\to\infty$. For this goal we shall use the inequalities

$$(\eta-\eta_0)^2\,|f''(\eta_0)\,|\geqq|f(\eta_0)-f(\eta)|\geqq\frac{(\eta-\eta_0)^2}{4}\,|f''(\eta_0)| \tag{23}$$

valid in a neighbourhood of η_0, $|\eta-\eta_0|<\delta$, for which

$$2\inf_{|\eta-\eta_0|<\delta} f''(\eta)\geqq|f''(\eta_0)|\geqq\frac{1}{2}\sup_{|\eta-\eta_0|<\delta} f''(\eta)\,. \tag{24}$$

Summing only over the region where the exponent is >-1 we get (always assuming $T<T_c$, $|n_0|<\eta_0$)

$$\left(\frac{2}{\Omega}\right)^2\sum_{\eta,n} e^{-\Omega\left(f(\eta_0)-f(\eta)+\frac{T_c}{2T}(n-n_0)^2\right)}\phi(\eta)\geqq\frac{2^{5/2}}{e^2\Omega}\,\frac{\inf\limits_{|\eta_0-\eta'|<\delta_\Omega}\phi_\Omega(\eta')}{\sqrt{f''(\eta_0)\,T_c/T}} \tag{25}$$

if $\delta_\Omega=\frac{1}{\sqrt{f''\Omega}}<\delta$. Thus we have

$$P(\eta,n)\leqq e^{-\Omega\left(f(\eta_0)-f(\eta)+\frac{T_c}{2T}(n-n_0)^2\right)}\,\frac{\phi_\Omega(\eta)\,e^2\,\Omega\sqrt{f''(\eta_0)\,T_c/T}}{\inf\limits_{|\eta_0-\eta'|<\delta_\Omega}\phi_\Omega(\eta')\,2^{5/2}} \tag{26}$$

which goes to zero for all $(\eta, n)\neq(\eta_0, n_0)$.

Hence the average of G taken with P should just give G at η_0, n_0. There is still the slight complication that G is Ω-dependent. In fact, for $\Omega\to\infty$, the hypergeometric function converges uniformly to a Bessel function:

$$G_\infty(\eta, n; a, b, c) = \sum_{\chi=0}^{\infty} (-)^\chi \frac{(\eta^2 - n^2)^\chi}{(\chi!)^2} \left(\frac{b}{2}\right)^{2\chi} e^{\mathrm{i}n(a+c)} = J_0(b\sqrt{\eta^2 - n^2})\, e^{\mathrm{i}n(a+c)}. \tag{27}$$

Thus we anticipate the equation

$$\lim_{\Omega\to\infty} \langle A_\Omega \rangle_\Omega = G_\infty(\eta_0, n_0; a, b, c). \tag{28}$$

To demonstrate this result one has to apply the usual tricks in ε-tik.

$$\begin{aligned} &|\int d\eta\, dn\, P_\Omega(\eta, n)\, G_\Omega(\eta, n; a, b, c) - G_\infty(\eta_0, n_0; a, b, c)| \\ &\quad = |\int d\eta\, dn (P_\Omega(\eta, n)\, G_\Omega(\eta, n; a, b, c) - P_\infty(\eta, n)\, G_\infty(\eta, n; a, b, c))| \leqq \\ &\quad \leqq |\int (G_\Omega - G_\infty)\, P_\Omega\, d\eta\, dn| + |\int G_\infty (P_\Omega - P_\infty)\, d\eta\, dn|. \end{aligned} \tag{29}$$

Here both terms on the right hand side can be made arbitrarily small; the first because $G_\Omega \to G_\infty$ uniformly and the second because G_∞ is continuous and $P_\Omega \to P_\infty$ on all continuity sets. Again one sees in the same manner that all derivatives with respect to a, b, c approach the corresponding derivatives of G_∞ in a neighbourhood of the origin.

There remains just some elementary algebra to establish the identity of (15) and (28). In fact

$$\begin{aligned} \cos\theta\, \mathrm{Th}\,\omega &= \frac{\varepsilon}{T_c} = n_0 \\ \sin\theta\, \mathrm{Th}\,\omega &= \sqrt{\left(\frac{T\omega}{T_c}\right)^2 - \frac{\varepsilon^2}{T_c^2}} = \sqrt{\eta_0^2 - n^2} \end{aligned} \tag{30}$$

and thus

$$\lim_{\Omega\to\infty} \langle A_\Omega \rangle_\Omega = J_0(b \sin\theta\, \mathrm{Th}\,\omega)\, e^{i(a+c)\cos\theta\, \mathrm{Th}\,\omega} = \lim_{\Omega\to\infty} \langle A_\Omega \rangle_{\overline{B}}. \tag{31}$$

§ 5. The Time-Dependence

Our result (31) shows that the thermal expectation values of polynomials of the σ's taken with $H_{\mathrm{B.C.S.}}$ for $\Omega \to \infty$ or with H_B and averaged over ϕ agree. Speaking mathematically this means they define the same positive linear functional over the C^*-algebra. We shall now turn to (1) or the question whether they give the same time dependence. This warrants separate study in particular since for $\Omega \to \infty$ the time development leads out of the C^*-algebra. Indeed, calculating $i\dot{\sigma} = [\sigma, H]$ with $H_{\mathrm{B.C.S.}}$ we find

$$\begin{aligned} -i\dot{\sigma}^+ &= 2T_c\, \sigma^z S_\Omega^+ - 2\varepsilon\sigma^+ \\ i\dot{\sigma}^z &= 4T_c(\sigma^- S_\Omega^+ - S_\Omega^- \sigma^+) \end{aligned} \tag{32}$$

where

$$\mathbf{S}_\Omega = \frac{1}{2\Omega} \sum_{p=1}^{\Omega} \boldsymbol{\sigma}_p, \quad S_\Omega^\pm = \frac{1}{2\Omega} \sum_{p=1}^{\Omega} (\sigma^x \pm i\sigma^y). \tag{33}$$

Now the operators $\mathbf{S}_\Omega$ do not converge uniformly for $\Omega \to \infty$.

$$\left(\text{f.i. } \|\mathbf{S}_\Omega - \mathbf{S}_{2\Omega}\| = \left\| \frac{1}{4\Omega} \sum_{p=1}^{\Omega} \boldsymbol{\sigma}_p - \frac{1}{4\Omega} \sum_{p=\Omega+1}^{2\Omega} \boldsymbol{\sigma}_p \right\| = \frac{1}{2} \text{ for all } \Omega\right).$$

They converge strongly in some infinite tensor product representations or in the representations given by the thermal functionals ("thermal representation"). Thus for $\Omega \to \infty$ $\dot{\sigma}$ does not belong to the C^*-algebra. However for our purpose the existence of weak limits of S_Ω is sufficient to establish the analogue of (1) in the quasi-spin formalism. For this end consider the expectation value of S_Ω and some polynomials of the σ's.

$$\begin{aligned}
&\lim_{\Omega\to\infty} \langle \sigma_{p_1} \ldots \sigma_{p_k} 2 S^\alpha_\Omega \sigma_{p_{k+1}} \ldots \sigma_{p_m}\rangle_\Omega = \lim_{\Omega\to\infty} \left(1 - \frac{m}{\Omega}\right) \times \\
&\times \langle \sigma_{p_1} \ldots \sigma_{p_k} \sigma^\alpha_p \sigma_{p_{k+1}} \ldots \sigma_{p_m}\rangle_\Omega \frac{1}{\Omega} \sum_{j=1}^{m} \langle \sigma_{p_1} \ldots \sigma_{p_k} \sigma^\alpha_{p_j} \sigma_{p_{k+1}} \ldots \sigma_{p_m}\rangle_\Omega \to \\
&\to \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \langle \sigma_{p_1} \ldots \sigma_{p_k} \sigma^\alpha_p \sigma_{p_{k+1}} \ldots \sigma_{p_m}\rangle_B \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\phi \left\langle \sigma_{p_1} \ldots \sigma_{p_k} \frac{n^\alpha}{2} \operatorname{Th}\omega \, \sigma_{p_{k+1}} \ldots \sigma_{p_m}\right\rangle_B .
\end{aligned} \tag{34}$$

Here p is different from the $p_1 \ldots p_m$ and we have used our previous results. Thus in the limit S can be replaced by $\frac{\mathbf{n}}{2} \operatorname{Th}\omega$. In the thermal representation (which is reducible) the limit of S is not a c-number since $\mathbf{n}$ is integrated over. (e.g. $\langle S^x \rangle = 0$, $\langle (S^x)^2 \rangle \neq 0$). In the same fashion one finds that also in the expectation value of any (finite) polynomials in the σ's and S's the latter can be replaced by $\frac{\mathbf{n}}{2} \operatorname{Th}\omega$. This result suggests that H_B will give the same time dependence since calculating $i\dot{\sigma} = [\sigma, H]$ with H_B one has

$$\begin{aligned}
-i\dot{\sigma}^+ &= 2T\omega(\sigma^z n^+ - n^z \sigma^-) \\
i\dot{\sigma}^z &= 4T\omega(\sigma^- n^- - \sigma^+ n^-) .
\end{aligned} \tag{35}$$

This is identical with (32) if $\mathrm{S} \to \frac{\mathbf{n}}{2} \operatorname{Th}\omega$ since $n^z = \frac{\varepsilon}{\omega T}$, $n^\pm = \frac{1}{2} \times$ $\times (n^x \pm i n^y) \to \frac{T_c}{T\omega} S^\pm$. On iterating (32) and (35) one can generate the complete time dependence of the σ's but one has to note that S is time-dependent whereas $\mathbf{n}$ is, of course, not! In fact, from (32) follows

$$\dot{S}^z_\Omega = 0 , \quad i\dot{S}^\pm_\Omega = (2\varepsilon - 4T_c S^z_\Omega) S^\pm_\Omega$$

or (36)

$$S^z_\Omega = \text{const}, \quad S^\pm_\Omega(t) = S^\pm_\Omega(0)\, e^{-it(2\varepsilon - 4T_c S^z_\Omega)} .$$

Thus on calculating the time dependence with $H_{\text{B.C.S.}}$ we obtain the one with H_B where $\frac{\mathbf{n}}{2} \operatorname{Th}\omega$ is replaced by S plus terms containing the time derivatives of S:

$$\begin{aligned}
e^{itH_B} \boldsymbol{\sigma}_p e^{-itH_B} &= \sum_{n=0}^{\infty} t^n P_n\left(\boldsymbol{\sigma}_p, \frac{\mathbf{n}}{2} \operatorname{Th}\omega\right) \\
e^{itH_{\text{B.C.S.}}} \boldsymbol{\sigma}_p e^{-itH_{\text{B.C.S.}}} &= \sum_{n=0}^{\infty} t^n (P_n(\boldsymbol{\sigma}_p, \mathrm{S}_\Omega) + G_n(\dot{\mathrm{S}})) .
\end{aligned} \tag{37}$$

Here P_n is a polynomial of n'th order and G_n stands for the terms with the time derivatives of S. From the above discussion it follows that $\lim_{\Omega\to\infty} \langle G_n\rangle_\Omega = 0$ since in replacing S^z in $\dot{S}^+$ by $\frac{n^2}{2}\,\mathrm{Th}\,\omega$ we get $\dot{S}^+ = 0$ and also all higher derivatives. Furthermore because of (34) the two kinds of expectation values of all P_n agree. Finally $\|P_n + G_n\| \leqq \leqq \frac{(\text{const})^n}{n!}$ so that $\sum_{n=0}^{\infty}$ in (37) converges uniformly for all t in the operator norm. Hence we can safely conclude

$$\lim_{\Omega\to\infty} \langle e^{it_1 H_{\text{B.C.S.}}}\, \sigma_{p_1}\, e^{-it_1 H_{\text{B.C.S.}}} \ldots e^{it_n H_{\text{B.C.S.}}}\, \sigma_{p_n}\, e^{-it_n H_{\text{B.C.S.}}}\rangle_\Omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \langle e^{it_1 H_B}\, \sigma_{p_1}\, e^{-it_1 H_B} \ldots e^{it_n H_B}\, \sigma_{p_n}\, e^{-it_n H_B}\rangle_B\,.$$

Thus in particular for Greens-functions of gauge invariant expressions where no averaging over ϕ is necessary $H_{\text{B.C.S.}}$ is equivalent to any H_B.

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