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Plane-wave, coordinate-space, and moment techniques in the operator-product expansion: equivalence, improved methods, and the heavy quark expansion

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Abstract. We compare plane-wave, coordinate-space and moment methods for evaluating operator-product expansion (OPE) coefficients of the light-quark and gluon condensates. Equivalence of these methods for quark condensate contributions is proven to all orders in the quark mass parameter m. The three methods are also shown to yield equivalent gluon condensate contributions to two-current correlation functions, regardless of the gauge chosen for external gluon fields in the coordinate space approach. An improved method for evaluating quark-condensate OPE coefficients is presented for several (two-current) correlation functions. Gauge-dependent Green functions are also discussed. It is shown that contradictory expressions for the gluon-condensate contribution to the quark propagator occurring from the plane-wave and coordinate-space approaches yield identical relations between the heavy-quark and gluon condensates, as anticipated from the gauge invariance of the heavy-quark expansion.

1 Introduction

QCD condensates, which characterize the non-perturbative content of the QCD vacuum, are an essential element in determining hadronic properties through sum-rule methods [1–3]. In this approach, each condensate makes a distinct contribution to the operator-product expansion (OPE) of a two-current correlation function.

There are several different techniques that have been developed to evaluate the contribution of a particular condensate in any process [1–6]. The plane-wave method begins with Wilson's operator identity [7] where the coefficients are perturbatively calculable at sufficiently large momentum. After forming a vacuum expectation value (vev) of the operator relation, the resulting condensates then describe the nonperturbative content of the vacuum. In the plane-wave method [1, 2] the operator identity is "sandwiched" with appropriately chosen states to single out the contribution of a given operator (condensate).

In contrast, the coordinate-space method does not begin with the operator identity [2, 4, 5]. Instead, the Wick expansion is applied to the Green function of interest leading to two contributions: purely perturbative propagators, and residual normal-ordered terms which are non-trivial because of the (nonperturbative) properties of the vacuum. The residual normal ordered terms are then expanded in a coordinate space series containing local vevs of composite operators, namely the condensates. Another way of viewing this process is through background (or external) fields. Although the end result of this process is that condensates are contributing to a given correlation function, there is no direct connection with the operator identity in the plane-wave method.

Yet another method favoured by Lavelle and co-workers is the moment technique [6]. In this approach a Green function is written in terms of two-point functions (propagators) which consist of a sum of perturbative and nonperturbative contributions. The Green function is then arranged as moments of the non-perturbative portion of the propagator, and these moments are each identified with appropriate condensates via gap equations.

It is clear that these three techniques have profoundly different conceptual formulations of how nonperturbative effects arise, and it is certainly not evident that the three approaches should lead to identical results. Since each method has been extensively applied, it is important to establish that the three approaches are fundamentally identical, consolidating the progress that has been obtained through sum-rule applications.

The gluon condensate in particular illustrates the technical issues that can arise when comparing the above methods. In coordinate-space techniques, the fixed-point gauge [8] is used to evaluate the two-gluon non-local vev. However, this gauge violates translation invariance, and could in principle conflict with the covariant gauge used to formulate QCD. This is a serious issue which can only be resolved by demonstrating that the coordinate space

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(fixed-point gauge) method is identical to the other techniques, and thus no adverse effect of the gauge choice would remain. It is also important to demonstrate that other gauge choices for the non-local gluon vev [20] do not affect the result for the gluon condensate OPE coefficient. Indeed there is an example in the literature where (fixed-point) coordinate-space and plane wave results for the gluon condensate contributions to the quark propagator are in disagreement [5, 11, 14]. The question as to whether this is a reflection of gauge dependence or a fundamental difference between the techniques further motivates our investigation.

Gluon condensate calculations also illustrate that the (quark mass) chiral limit of the OPE coefficients is non-trivial. With coordinate space methods, if the gluon condensate contribution is calculated directly for $m=0$, then the result disagrees with the $m \to 0$ limit [13]. This problem can only be resolved by computing the quark condensate coefficient with arbitrary quark mass dependence and then taking the $m \to 0$ limit weighted by the $\alpha_s/(12\pi m)$ factor from the heavy quark expansion. Thus there is a "misplaced" gluon condensate contribution that cannot be found in an expansion about the chiral limit, but must be obtained from the mass dependence of the quark condensate contribution. The dependence of the OPE coefficients on the light quark mass is also of importance for their infrared behaviour because of potentially large logarithmic contributions for light quarks.

However, complications arise in all methods when the mass dependence of the light quark condensate $\langle \bar{q}q \rangle$ effects are considered. In coordinate space techniques, the $\langle \bar{q}q \rangle$ OPE coefficient is determined as a power series in m^2/p^2 (p is the external momentum), and the resulting series must be summed and analytically continued [9, 10, 13]. A similar difficulty arises in plane-wave methods where it becomes necessary to average over momentum directions by expanding propagator factors in a series and then re-summing to obtain a final result [11, 13]. Moment techniques are typically applied to lowest order in the quark mass [6], and so they must be extended to higher orders. The behaviour of the $\langle \bar{q}q \rangle$ contributions is not only important for the infrared and chiral limiting properties as discussed above, but also for the formulation of Laplace sum-rules as contour integrals in the momentum plane [12]. These technical problems associated with existing coordinate-space, plane-wave and moment methods suggest the desirability of an all-orders analytic approach towards the evaluation of quark condensate contributions to OPE coefficients.

To summarize, the purpose of the present paper is to study the relation between plane-wave, coordinate space, and moment methods for evaluating the quark and gluon condensate OPE coefficients. We also develop an improved analytic approach, as opposed to the order-by-order series techniques discussed above, for the evaluation of the OPE coefficients of the quark condensate.

In Sect. 2, we demonstrate the equivalence of existing methods for evaluating the quark condensate OPE coefficient of gauge invariant correlation functions. This analysis leads naturally to an efficient analytic approach for evaluating quark condensate effects to all orders in the quark mass. This new approach, which can be phrased in the language of a new Feynman rule, is applied in Sect. 3 to correlation functions of scalar, vector and axial vector currents, illustrating the method and confirming the results of previous calculations. In Appendix A we illustrate how this method is applied to one-loop divergent integrals as would appear in a typical baryonic correlation function. Appendix B extends the calculations of Sect. 3 to the case of off-diagonal currents.

In Sect. 4, we demonstrate the equivalence of all techniques for the gluon condensate OPE coefficient for gauge invariant correlation functions. For coordinate-space techniques, we show that the value of the OPE coefficients is insensitive to the gauge chosen for the external gluon fields, an important property since in principle the gauge for the non-local vev could conflict with the covariant gauge used to formulate QCD. We also consider a different approach to the non-local two-gluon vev that maintains translation invariance.

Finally, Sect. 5 will consider some applications to the quark propagator – a gauge dependent Green function – in order to address some controversies about coordinate-space and plane-wave approaches. For the quark condensate contribution, plane-wave calculations find a (physical momentum) "freeze-out" [11] that does not occur in the coordinate space approach [14]. We demonstrate that the new technique developed in Sect. 3 leads to the same freeze-out effect. Furthermore, contradictory fixed-point and covariant gauge expressions for the gluon condensate contribution to the quark propagator (respectively applicable to standard coefficient-space and plane-wave approaches) are nevertheless seen to yield the same gauge invariant relationship between the heavy-quark and gluon condensate. To our knowledge, this is the first instance that the heavy quark expansion has been derived outside of the (plane-wave) operator mixing approach [1, 2]. This calculation demonstrates how gauge dependence at an intermediate stage of a calculation does not affect a genuinely gauge invariant quantity, and illustrates that the discrepancy is a reflection of the propagator's gauge dependent nature rather than a failure of one of the methods.

2 Equivalence of OPE techniques for the quark condensate

To examine the relation between plane-wave, coordinate-space and moment approaches, it is first necessary to review each method.

Coordinate space methods [2, 4, 5] begin with the Wick expansion of a time-ordered product of currents. The time-ordered product is expanded in a perturbation series, and then the non-perturbative vacuum $|\Omega\rangle$ is used to obtain the correlation function. Since the vacuum is non-perturbative, residual normal-ordered contributions to the Wick expansion of a time-ordered product of currents can have a non-zero vev. Consequently, correlation functions are expressed in terms of non-local vevs of fundamental quark and gluon fields, which are then expanded in a coordinate-space series whose coefficients are the local composite operators known as QCD condensates.

A given non-local vev can only generate certain condensates in its coordinate space expansion. If only quark condensate effects are of interest, then the only non-local vev that needs to be considered [14] is $\langle\Omega|:\bar{\psi}(z)\psi(y):|\Omega\rangle$, where the normal ordering refers to the perturbative vacuum.

It has been shown elsewhere [9, 13, 14] that the quark condensate projection of the two-quark non-local vev is

$$\begin{aligned}&\langle\Omega|:\bar{\psi}^{\xi}(z)\psi^{\eta}(y):|\Omega\rangle\\&=\frac{\delta^{\eta\xi}}{3}\langle\bar{q}q\rangle\sum_{n=0}^{\infty}\frac{(-im)^{2n}(y-z)^{2n}}{n!(n+1)!4^{n+1}}\\&+\frac{\delta^{\eta\xi}}{3}\langle\bar{q}q\rangle\sum_{n=0}^{\infty}\frac{(-im)^{2n+1}\gamma\cdot(y-z)(y-z)^{2n}}{2(n+2)!n!4^{n+1}},\end{aligned}\tag{1}$$

where ξ,η represent colour indices and m is the quark mass parameter. Denoting $x=(y-z)$, the above expression is merely a series representation of Bessel functions:

$$\begin{aligned}&\langle\Omega|:\bar{\psi}^{\xi}(z)\psi^{\eta}(y):|\Omega\rangle\\&=\frac{\delta^{\xi\eta}}{6m}\langle\bar{q}q\rangle\left[\frac{1}{\sqrt{x^2}}J_1(m\sqrt{x^2})-\frac{i\gamma\cdot x}{x^2}J_2(m\sqrt{x^2})\right]\\&=\frac{\delta^{\xi\eta}}{6m^2}\langle\bar{q}q\rangle(i\partial\!\!\!/+m)\left[\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}\right].\end{aligned}\tag{2}$$

An important property of the quark condensate projection of the two-quark vev is that it satisfies the free equation of motion

$$(i\partial\!\!\!/-m)\langle\Omega|:\bar{\psi}^{\xi}(0)\psi(x):|\Omega\rangle=0\,.\tag{3}$$

This property allows a simple method for obtaining a D-dimensional version of (2) simply by seeking the solution of the equation of motion which reduces to $\langle\bar{q}q\rangle/12$ at $x=0$. The result is

$$\begin{aligned}\langle\Omega|:\bar{\psi}^{\xi}(z)\psi^{\eta}(y):|\Omega\rangle&=\delta^{\xi\eta}\frac{\Gamma(D/2)2^{D/2-1}}{12m}\langle\bar{q}q\rangle(i\partial\!\!\!/+m)\\&\times\left[\frac{J_{D/2-1}(m\sqrt{x^2})}{(m^2x^2)^{D/4-1/2}}\right].\end{aligned}\tag{4}$$

The functional form (2) has not previously been used in coordinate space methods, but the series (1) allows an evaluation of quark condensate effects as a series in m^2/p^2.

The apparent free-field nature of the non-local vev is initially perplexing. A priori it appears unlikely that a coordinate-space approach based upon (2) would lead to the same OPE coefficient as in the plane-wave method delineated below. On further reflection however, it is important not to ascribe too much physical meaning to (4), since it is simply used to extract OPE coefficients of the quark condensate. Since the OPE coefficients are perturbatively calculable, any amplitude used for this purpose will be a perturbative expansion containing free-field propagators and by analogy, it appears reasonable that the free-field expression should occur for the non-local vev. All the infrared nature of the OPE will then be contained in the condensates themselves, although this factorization of low-energy effects is difficult to establish [15, 16]. The equivalence of the various OPE methods, as will be shown below, supports this interpretation of the non-local vev.

Thus the coordinate-space approach for obtaining quark condensate effects consists of retaining residual normal-ordered terms $\langle\Omega|:\bar{\psi}(z)\psi(y):|\Omega\rangle$ from the Wick expansion of a given amplitude and then utilizing (1) for the $\langle\bar{q}q\rangle$ projection of this non-local vev.

The plane-wave method will now be reviewed for the particular example of a scalar current correlation function. This allows the relation between plane-wave and coordinate-space methods to be explored without the complication of non-trivial Lorentz and Dirac structures.

Consider the OPE for the correlation function of two scalar currents $j(x)$ (e.g. $j(x)=\bar{\psi}(x)\psi(x)$),

$$\begin{aligned}&i\int d^4x\,e^{ip\cdot x}T(j(x)j(0))\\&=\mathscr{I}(p)+\mathscr{D}^0(p):\bar{\psi}\psi:+\cdots+\mathscr{D}^l(p):\bar{\psi}D^{2l}\psi:+\cdots\\&\quad+\mathscr{E}_n^l(p)p^{\alpha_1}\cdots p^{\alpha_{2n}}:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n}}D^{2l}\psi:+\cdots\\&\quad+\mathscr{O}_n^l(p)p^{\alpha_1}\cdots p^{\alpha_{2n+1}}:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n+1}}D^{2l}\psi:+\cdots\\&\quad+\textit{equation of motion operators}+\text{BRS }\textit{variations}\\&\quad+\textit{operators not leading to }\langle\bar{q}q\rangle,\end{aligned}\tag{5}$$

where D_α is the gauge covariant derivative. The OPE must have this form, since for a product of BRS invariant currents, only gauge invariant operators appear apart from the trivial contributions from BRS variations and equations of motion [17]. Perturbative contributions to (5) are contained in $\mathscr{I}(p)$, and operators which contain anything but two quark fields have been neglected since they do not ultimately contribute to the $\langle\bar{q}q\rangle$ effects in the correlation function. Non-trivial Dirac structures such as $:\bar{\psi}\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}\psi:$ have also been neglected since they are not independent objects after a vev is formed, thereby contributing in a similar fashion as the operators already considered.

A correlation function is now formed form (5) by evaluating an expectation value in the non-perturbative vacuum:

$$\begin{aligned}\Pi(p^2)&=i\int d^4x\,e^{ip\cdot x}\langle\Omega|T(j(x)j(0))|\Omega\rangle\\&=\mathscr{I}(p)+\cdots+\mathscr{D}^l(p)\langle\Omega|:\bar{\psi}D^{2l}\psi:|\Omega\rangle+\cdots\\&\quad+\mathscr{E}_n^l(p)p^{\alpha_1}\cdots p^{\alpha_{2n}}\langle\Omega|:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n}}D^{2l}\psi:|\Omega\rangle\\&\quad+\cdots+\mathscr{O}_n^l(p)p^{\alpha_1}\cdots p^{\alpha_{2n+1}}\\&\quad\times\langle\Omega|:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n+1}}D^{2l}\psi:|\Omega\rangle+\cdots\\&\quad+0\ (\textit{equation of motion},\ \text{BRS }\textit{variation})\\&\quad+\textit{operators not leading to }\langle\bar{q}q\rangle.\end{aligned}\tag{6}$$

Since we are dealing with a scalar current the vevs in the above expression can be simplified (the symmetrization of indices imposed by contraction into products with momenta is important):

$$\langle\Omega|:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n+1}}D^{2l}\psi:|\Omega\rangle=0\tag{7a}$$

$$\begin{aligned}&\langle\Omega|:\bar{\psi}D_{\alpha_1}\ldots D_{\alpha_{2n}}D^{2l}:|\Omega\rangle\\&=S_{\alpha_1\cdots\alpha_{2n}}\langle\Omega|:\bar{\psi}D^{2(l+n)}\psi:|\Omega\rangle\\&\quad+\textit{operators not leading to }\langle\bar{q}q\rangle,\end{aligned}\tag{7b}$$

where $S_{\alpha_1\cdots\alpha_{2n}}$ is a completely symmetric tensor normalized so that $S_{\alpha_1}^{\alpha_1}\cdots{}_{\alpha_n}^{\alpha_n}=1$.

If attention is restricted to the $\langle\bar{q}q\rangle$ contribution to $\Pi(p^2)$ then the equation of motion

$$D^2\psi=(\gamma\cdot D\gamma\cdot D-(i/2)\sigma_{\mu\nu}G_{\mu\nu})\psi$$
$$=-m^2\psi+\textit{irrelevant operators}=-m^2\psi \quad (8)$$

can be used, since the neglected gauge-field contributions from the complete equation of motion do not generate the quark condensate [14]. After vacuum averaging, the $\langle\bar{q}q\rangle$ component of $\Pi(p^2)$ is thus given by

$$\Pi(p^2)=i\int d^4x e^{ip\cdot x}\langle\Omega|T(j(x)j(0))|\Omega\rangle$$
$$=\langle\bar{q}q\rangle[\cdots+\mathscr{D}_n(p)(-m^2)^n+\cdots$$
$$+\mathscr{E}_n^l(p)(-m^2)^{n+l}p^{\alpha_1}\cdots p^{\alpha_{2n}}S_{\alpha_1\cdots\alpha_{2n}}]$$
$$\equiv\mathscr{C}(p^2)\langle\bar{q}q\rangle. \quad (9)$$

The plane-wave method is designed to single out the quark condensate effects and to simulate the effect of vacuum averaging. If a matrix element of (5) is formed between on-shell one-quark states $|k,s\rangle$ where k is the quark momentum and s represents the spin state, then only operators with two quark fields will be non-zero (in particular D_α can be replaced with ∂_α), and since the states are on-shell the equation of motion operators will not contribute. Consequently, we find that

$$i\int d^4x e^{ip\cdot x}\langle k,s|T(j(x)j(0))|k,s\rangle$$
$$=\mathscr{D}^0(p)\langle k,s|:\bar{\psi}\psi:|k,s\rangle+\cdots$$
$$+\mathscr{D}^l(p)\langle k,s|:\bar{\psi}\partial^{2l}\psi:|k,s\rangle+\cdots$$
$$+\mathscr{E}_n^l p^{\alpha_1}\cdots p^{\alpha_{2n}}\langle k,s|:\bar{\psi}\partial_{\alpha_1}\ldots\partial_{\alpha_{2n}}\partial^{2l}:|k,s\rangle+\cdots$$
$$+\mathscr{O}_n^l(p)p^{\alpha_1}\cdots p^{\alpha_{2n+1}}\langle k,s|:\bar{\psi}\partial_{\alpha_1}\ldots\partial_{\alpha_{2n+1}}\partial^{2l}\psi:|k,s\rangle+\cdots. \quad (10a)$$

$$i\int d^4x e^{ip\cdot x}\langle k,s|T(j(x)j(0))|k,s\rangle$$
$$=\mathscr{D}^0(p)\langle k,s|:\bar{\psi}\psi:|k,s\rangle+\cdots$$
$$+\mathscr{D}^l(p)(-k^2)^l\langle k,s|:\bar{\psi}\psi:|k,s\rangle+\cdots$$
$$+\mathscr{E}_n^l(p)(-ip\cdot k)^{2n}(-k^2)^l\langle k,s|:\bar{\psi}\psi:|k,s\rangle+\cdots$$
$$+\mathscr{O}_n^l(p)(-ip\cdot k)^{2n+1}(-k^2)^l\langle k,s|:\bar{\psi}\psi:|k,s\rangle+\cdots. \quad (10b)$$

Averaging over the momentum directions $\hat{k}$ simulates the effect of the vacuum as illustrated by the following identities:

$$\int d\hat{k}k_{\alpha_1}\ldots k_{\alpha_{2n+1}}=0$$
$$\int d\hat{k}k_{\alpha_1}\ldots k_{\alpha_{2n}}=(k^2)^n S_{\alpha_1\cdots\alpha_{2n}}. \quad (11)$$

An identical plane-wave approach is to average over the external momentum p, an option which is more convenient for the gluon condensate [13].

Since the quark states are on-shell, k^2 is replaced with m^2 after averaging, and a sum over spin states (and implicitly a colour trace) is performed, thereby leading to the plane-wave determination of the $\langle\bar{q}q\rangle$ OPE coefficient $\mathscr{C}(p^2)$:

$$\mathscr{C}(p^2)\equiv[\mathscr{D}^0(p)\cdots+\mathscr{D}_n(p)(-m^2)^n+\cdots$$
$$+\mathscr{E}_n^l(p)(-m^2)^{n+l}p^{\alpha_1}\cdots p^{\alpha_{2n}}S_{\alpha_1\cdots\alpha_{2n}}]$$
$$=\tfrac{1}{6}\int d^4x e^{ip\cdot x}\int d\hat{k}\sum_s\langle k,s|T(j(x)j(0))|k,s\rangle. \quad (12)$$

At this point it should be remarked that no distinction between bare and renormalized fields has been made, implicitly assuming that we are working to leading order. The only modification that occurs in higher-loop calculations is the effects of renormalizing the composite operators that appear in the OPE. One then identifies the next-to-leading contribution to the OPE coefficient of the *renormalized* composite operators as a combination of a next-to-leading amplitude and the leading amplitude with a renormalization effect (see [16] for details of this procedure for a particular calculation). This would occur in an identical fashion both for plane-wave and coordinate-space methods. The only complication that can occur is the failure in infrared singularities to cancel from the OPE coefficients [15]; a property which, if desired, can be explicitly verified at the relevant order [16]. Thus our analysis applies to higher loop effects provided that the appropriate renormalization factors are taken into account.

Before reviewing moment techniques, we will now demonstrate the equivalence of plane-wave and coordinate-space techniques. To obtain the connection between these methods, consider the possible contributions from the Wick expansion of the final line of (12). Clearly only terms containing two (normal-ordered) quark fields: $\bar{\psi}(z)\psi(y)$: can contribute to the matrix element with $\langle k,s|$. All other terms will either be disconnected processes or will have a zero matrix element because of their field content. Focussing upon a single term of this type occurring in the final line of (12) leads us to consider the following contribution:

$$\tfrac{1}{6}\int d\hat{k}\sum_s\langle k,s|:\bar{\psi}^\eta(z)\psi^\xi(y):|k,s\rangle, \quad (13)$$

where η and ξ are colour indices. Since the Wick expansion is a perturbative expansion of free fields, this matrix element can be easily evaluated by expanding the quark fields in terms of their free-particle solutions.

$$\tfrac{1}{6}\int d\hat{k}\sum_s\langle k,s|:\bar{\psi}^\eta(z)\psi^\xi(y):|k,s\rangle$$
$$=\frac{\delta^{\eta\xi}}{12m}\int d\hat{k}(\not{k}+m)e^{-ik\cdot(y-z)}$$
$$=\frac{\delta^{\eta\xi}}{12m}(i\not{\partial}+m)\int d\hat{k}e^{-ik\cdot x},\quad x\equiv y-z. \quad (14)$$

The only remaining complication is the averaging process over $\hat{k}$, which is facilitated through utilization of the D

dimensional partial wave expansion [4]

$$e^{-ik\cdot x}=\Gamma\left(\frac{D}{2}-1\right)\sum_{n=0}^{\infty}(-i)^n\left(n+\frac{D}{2}-1\right)\times C_n^{(D/2-1)}(\hat{x}\cdot\hat{k})\left[\frac{x^2k^2}{4}\right]^{n/2}\times\left(\frac{2}{\sqrt{k^2x^2}}\right)^{n+D/2-1}J_{n+D/2-1}(\sqrt{k^2x^2}). \quad (15)$$

The RHS of (14) is then evaluated utilizing the orthogonality of the Gegenbauer polynomials and the $k^2=m^2$ mass-shell conditions:

$$\frac{1}{6}\int d\hat{k}\sum_s\langle k,s|:\bar{\psi}^\eta(z)\psi^\xi(y):|k,s\rangle$$

$$=\Gamma(D/2)2^{D/2-1}\frac{\delta^{\xi\eta}}{12m}(i\partial\!\!\!/+m)\left[\frac{J_{D/2-1}(m\sqrt{x^2})}{(m^2x^2)^{D/4-1/2}}\right] \quad (16a)$$

$$=\frac{\delta^{\xi\eta}}{6m^2}(i\partial\!\!\!/+m)\left[\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}\right],\quad D=4. \quad (16b)$$

Equation (16b) is identical to the coordinate space representation for the two-quark non-local vev (4), apart from the $\langle\bar{q}q\rangle$ factor absorbed into the definition (9) of $\mathscr{C}(p^2)$, thereby demonstrating the equivalence of plane-wave and coordinate-space methods for evaluating the $\langle\bar{q}q\rangle$ OPE coefficient.

We now consider the connection between coordinate-space and moment methods. Moment techniques [6] begin with a separation of the full quark propagator $S(k)$ into a perturbative and non-perturbative portion.

$$S^{NP}(k)=S(k)-S^P(k). \quad (17)$$

To interpret this relation, consider the definition of the quantities on the RHS of the previous equation

$$iS^{NP}(k)=\int d^Dx\,e^{ik\cdot x}\langle\Omega|T(\psi(x)\bar{\psi}(0))|\Omega\rangle-\langle O|T(\psi(x)\bar{\psi}(0))|O\rangle. \quad (18)$$

But the Wick expansion

$$T(\psi(x)\bar{\psi}(0))=\langle O|T(\psi(x)\bar{\psi}(0))|O\rangle+:\psi(x)\bar{\psi}(0):, \quad (19)$$

applied to the RHS of (18) for a normalized vacuum $|\Omega\rangle$ yields

$$iS^{NP}(k)=\int d^Dx\,e^{ik\cdot x}\langle\Omega|:\psi(x)\bar{\psi}(0):|\Omega\rangle$$

$$\langle\Omega|:\psi(x)\bar{\psi}(0):|\Omega\rangle=\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot x}iS^{NP}(k). \quad (20)$$

Moment methods [6] then relate the local condensate $\langle\bar{q}q\rangle$ to integrals of the non-perturbative propagator $S^{NP}(k)$ in a fashion similar to a gap equation:

$$\int\frac{d^Dk}{(2\pi)^D}i\mathrm{Tr}\left[S^{NP}(k)\right]=-\frac{\langle\bar{q}q\rangle}{12}. \quad (21)$$

The Dirac structures of $S^{NP}(k)$ have moments which are related to the quark condensate. In particular, with the help of the $k\!\!\!/=m$ equation of motion we see that

$$S^{NP}(k)=k\!\!\!/\tilde{A}(k^2)+\tilde{B}(k^2)$$

$$\int\frac{d^Dk}{(2\pi)^D}k^2\tilde{A}(k^2)=\int\frac{d^Dk}{(2\pi)^D}m\tilde{B}(k^2)=-\frac{m\langle\bar{q}q\rangle}{12}. \quad (22a)$$

Moreover, the $\langle\bar{q}q\rangle$ component of higher order moments is found by having each additional integrand factor of k^2 correspond to multiplying the RHS of (22a) by an extra factor of m^2:

$$\int\frac{d^Dk}{(2\pi)^D}(k^2)^{s+1}\tilde{A}(k^2)=\int\frac{d^DK}{(2\pi)^D}m(k^2)^s\tilde{B}(k^2)=-(m^2)^s\frac{m\langle\bar{q}q\rangle}{12}. \quad (22b)$$

It is now straightforward to extract the non-local version of the two-quark vev by returning to (20):

$$\langle\Omega|:\psi(x)\bar{\psi}(0):|\Omega\rangle=i\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot x}k\!\!\!/\tilde{A}(k^2)+\tilde{B}(k^2)=i\partial\!\!\!/\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot x}\tilde{A}(k^2)+i\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot x}\tilde{B}(k^2). \quad (23)$$

The partial wave expansion (15) is then used to evaluate the angular integrals in (23), and we find that

$$\langle\Omega|:\psi(x)\bar{\psi}(0):|\Omega\rangle=i\partial\!\!\!/\int\frac{d^Dk}{(2\pi)^D}\tilde{A}(k^2)\Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\sqrt{k^2x^2}}\right)^{D/2-1}J_{D/2-1}(\sqrt{k^2x^2})+i\int\frac{d^Dk}{(2\pi)^D}\tilde{B}(k^2)\Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\sqrt{k^2x^2}}\right)^{D/2-1}J_{D/2-1}(\sqrt{k^2x^2}). \quad (24)$$

One can use the series representation of the Bessel functions in (24) in conjunction with (22b) in order to find the result

$$\langle\Omega|:\psi(x)\bar{\psi}(0):|\Omega\rangle=-\frac{\Gamma(D/2)2^{D/2-1}}{12m}\langle\bar{q}q\rangle(i\partial\!\!\!/+m)\left[\frac{J_{D/2-1}(m\sqrt{x^2})}{(m^2x^2)^{D/4-1/2}}\right], \quad (25)$$

identical to the coordinate-space expression (4).

Consequently we have demonstrated the equivalence of plane-wave, coordinate-space and moment methods for evaluating the $\langle\bar{q}q\rangle$ OPE coefficients. In the next section, a new analytic method for evaluating OPE coefficients to all orders in m will be applied to several examples.

3 Momentum space approach to coordinate space methods

Scalar current correlation function

Coordinate space methods generally involve calculating configuration space integrals over each term in the series (1). In this section, we utilize the full functional form of the two-quark non-local vev (2) to obtain a momentum space integral for the lowest order $\langle \bar{q}q \rangle$ contribution to the scalar current correlation function:

$$\Pi(p^2) \equiv i \int d^4 x\, e^{ip\cdot x} \langle \Omega | T(j(x) j(0)) | \Omega \rangle,$$

$$j(x) \equiv \bar{\psi}(x)\psi(x). \tag{26}$$

The quark condensate component of $\Pi(p^2)$ is obtained by retaining a two-quark vev from the Wick expansion of the correlation function

$$\Pi(p^2) = -2i \int d^4 x\, e^{ip\cdot x} \mathrm{Tr}[\langle O | T(\psi(0)\bar{\psi}(x)) | O \rangle \times \langle \Omega | : \psi(x)\bar{\psi}(0) : | \Omega \rangle]. \tag{27}$$

Transformation to momentum space is now made using the following free-field expressions.

$$\langle O | T(\psi(0)\bar{\psi}(x)) | O \rangle = i \int \frac{d^4 q}{(2\pi)^4} e^{iq\cdot x} \frac{\not{q} + m}{q^2 - m^2 + i\varepsilon} \tag{28a}$$

$$\langle \Omega | : \psi(x)\bar{\psi}(0) : | \Omega \rangle = -\frac{\langle \bar{q}q \rangle}{6m^2} (i\not{\partial} + m) \left[\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \right] \equiv \int d^4 k\, e^{-ik\cdot x} (\not{k} + m) \mathscr{F}(k) \tag{28b}$$

$$\int d^4 k\, e^{-ik\cdot x} \mathscr{F}(k) \equiv -\frac{\langle \bar{q}q \rangle}{6m^2} \frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \tag{28c}$$

The Feynman rule for the normal ordered term in (27) is given by (28b), and is illustrated in Fig. 1. Working directly from (27) and (28), or applying the Feynman rules for the diagram in Fig. 2, the scalar current correlation function can now be expressed as the momentum space integral after evaluating traces.

$$\Pi(p^2) = 24 \int d^4 k\, [m^2 - p\cdot k + k^2] \frac{\mathscr{F}(k)}{(p-k)^2 - m^2 + i\varepsilon}. \tag{29a}$$

An important property of the Fourier transform $\mathscr{F}(k)$ is its on-shell behaviour

$$(k^2 - m^2)\mathscr{F}(k) = 0, \tag{29b}$$

which follows from substituting both sides of (28b) into (3). Making use of this property in the integrand of (29) we find that

$$\Pi(p^2) = 12 \int d^4 k [4m^2 - p^2] \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} + 12 \int d^4 k\, \mathscr{F}(k). \tag{30}$$

The second integral in (30) is determined by the $x \to 0$ limit of (28c):

$$\lim_{x\to 0} \int d^4 k\, e^{-ik\cdot x} \mathscr{F}(k) = -\frac{\langle \bar{q}q \rangle}{6m^2} \lim_{x\to 0} \frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}$$

$$\int d^4 k\, \mathscr{F}(k) = -\frac{\langle \bar{q}q \rangle}{12m}. \tag{31}$$

Now consider exponentiating the propagator factor occurring in the first integral of (30):

$$\int d^4 k \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} = -i \int d^4 k\, \mathscr{F}(k) \int_0^\infty d\eta\, e^{i\eta(p^2 - 2p\cdot k + i\varepsilon)}. \tag{32}$$

The k integral can now be done using the definition (28c) of $\mathscr{F}(k)$:

$$\int d^4 k \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} = \frac{i\langle \bar{q}q \rangle}{12m^2\sqrt{p^2}} \int_0^\infty \frac{d\eta}{\eta} e^{i\eta p^2 - \varepsilon\eta} J_1(2\eta m\sqrt{p^2}). \tag{33a}$$

This final integral is tabulated [18],

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) \frac{dx}{x} = \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^\nu}{\nu\beta^\nu}$$

$$\mathrm{Re}\,\nu > 0;\ \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|, \tag{33b}$$

and when the $\varepsilon \to 0$ limit is taken, the following result is obtained:

$$\int d^4 k \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} = -\frac{\langle \bar{q}q \rangle}{6mp^2} [1 + \sqrt{1 - 4m^2/p^2}]^{-1}. \tag{34}$$

The results of (31) and (34) are now substituted into (30). Thus the lowest-order quark condensate contribution to the scalar current correlation function for Euclidean momenta Q^2 is then found to be

$$\Pi(Q^2) = -\frac{\langle \bar{q}q \rangle}{m} \frac{(1-v)(1+2v)}{1+v}, \quad v \equiv \sqrt{1 + 4m^2/Q^2}, \tag{35}$$

in agreement with previous work [13].

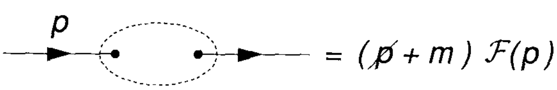

Fig. 1. The momentum-space Feynman rule for the vacuum expectation value of normal-ordered quark fields, as in the integrand of (28b)

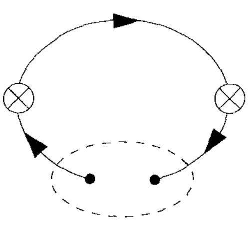

Fig. 2. The Feynman diagram for (lowest-order) quark condensate contributions to two-current correlation functions

Vector current correlation function

The vector current correlation function of light quarks

$$\Pi_{\mu\nu}(p) = i \int d^4 x\, e^{ip\cdot x} \langle \Omega | T(j_\mu(x) j_\nu(0)) | \Omega \rangle,$$

$$j_\mu(x) \equiv \bar{\psi}(x)\gamma_\mu\psi(x) \tag{36}$$

is extremely important in applications because of the relation between its imaginary part and the ubiquitous quantity $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \text{muons})$.

The leading quark condensate contribution to $\Pi_{\mu\nu}(p)$ devolves from a two-quark vev in the Wick expansion.

$$\Pi_{\mu\nu}(p) = -2i \int d^4x\, e^{ip\cdot x} \mathrm{Tr}\left[\langle O|T(\psi(0)\bar\psi(x))|O\rangle\gamma_\mu \times \langle\Omega|{:}\psi(x)\bar\psi((0){:}|\Omega\rangle\gamma_\nu\right] \quad (37)$$

A momentum-space expression for $\Pi_{\mu\nu}$ can be obtained using either (28) or by applying the Feynman rules to Fig. 2.

$$\Pi_{\mu\nu}(p) = 2\int d^4k\, \mathrm{Tr}\left[(\not k - \not p + m)\gamma_\mu(\not k + m)\gamma_\nu\right] \times \frac{\mathscr{F}(k)}{(k-p)^2 - m^2 + i\varepsilon}. \quad (38)$$

The correlation function $\Pi_{\mu\nu}$ must be transverse, as required by charge conservation. To see this explicitly, we contract $p^\mu p^\nu$ into the correlation function, and make liberal use of (29b) in order to find that

$$\begin{aligned} p^\mu p^\nu \Pi_{\mu\nu}(p) &= 24\int d^4k\,\left[(m^2-k^2)p^2 + 2(p\cdot k)^2 - p^2 p\cdot k\right] \times \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + k^2 - m^2 + i\varepsilon} \\ &= -24\int d^4k\, p\cdot k\,\mathscr{F}(k). \end{aligned} \quad (39)$$

We note from (28c) that

$$\begin{aligned} \int d^4k\, p\cdot k\,\mathscr{F}(k) &= i\lim_{\xi\to 0}\frac{d}{d\xi}\int d^4k\, e^{-i\xi k\cdot p}\mathscr{F}(k) \\ &= -i\frac{\langle\bar q q\rangle}{6m}\lim_{\xi\to 0}\frac{d}{d\xi}\frac{J_1(m\xi\sqrt{p^2})}{m\xi\sqrt{p^2}} = 0, \end{aligned} \quad (40)$$

thereby verifying the transversality of $\Pi_{\mu\nu}$. If we contract $g^{\mu\nu}$ into both sides of (38) we find, upon evaluating traces and imposing the on-shell constraint (29a), that

$$\begin{aligned} \Pi^\mu_\mu(p^2) &= 6\int d^4k\,\left[16m^2 - 8(k^2 - p\cdot k)\right]\frac{\mathscr{F}(k)}{(p-k)^2 - m^2 + i\varepsilon} \\ &= 24(2m^2 + p^2)\int d^4k\,\frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} \\ &\quad - 24\int d^4k\,\mathscr{F}(k). \end{aligned} \quad (41)$$

The integrals appearing in (41) were evaluated in (34) and (31). Using the transversality of $\Pi_{\mu\nu}(p)$, the result for the vector current correlation function

$$\Pi_{\mu\nu}(p) = \frac{\langle\bar q q\rangle}{3m^3}\left[1 + \left(1 + \frac{2m^2}{p^2}\right)\sqrt{1 - \frac{4m^2}{p^2}}\right](p_\mu p_\nu - p^2 g_{\mu\nu}), \quad (42)$$

is easily obtained. This result agrees with previous work using the coordinate-space series methods [9], and with plane-wave calculations [13].

Axial-vector current correlation function

The axial-vector current correlation function is of relevance to properties of the pion and their PCAC relations. The correlation function for axial currents is

$$\Pi^A_{\mu\nu}(p) = i\int d^4x\, e^{ip\cdot x}\langle\Omega|T(j^5_\mu(x)J^5_\nu(0))|\Omega\rangle$$

$$j^5_\mu \equiv \bar\psi(x)\gamma_\mu\gamma_5\psi(x), \quad \gamma_5^2 = 1 \quad (43)$$

Following familiar procedures, a momentum-space expression for $\Pi^A_{\mu\nu}$ is obtained. Since the axial current is not conserved, the correlation function contains a longitudinal component

$$\begin{aligned} \Pi^A_{\mu\nu}(p) &\equiv p_\mu p_\nu \Pi^A_L(p^2) + (p_\mu p_\nu - p^2 g_{\mu\nu})\Pi^A_T(p^2) \\ &= 2\int d^4k\,\mathrm{Tr}\left[(\not k - \not p + m)\gamma_\nu(\not k - m)\gamma_\mu\right] \times \frac{\mathscr{F}(k)}{(p-k)^2 - m^2 + i\varepsilon}. \end{aligned} \quad (44)$$

The longitudinal part Π^A_L of the correlation function is obtained by contracting (44) with $p^\mu p^\nu$, evaluating traces, and imposing the on-shell condition as before:

$$p^\mu p^\nu \Pi^A_{\mu\nu} = -48m^2p^2\int d^4k\,\frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon} - 24\int d^4k\, p\cdot k\,\mathscr{F}(k). \quad (45)$$

Recalling the results of the integrals in (31) and (40) then finds the longitudinal part of $\Pi^A_{\mu\nu}$ to be

$$\Pi^A_L(p^2) = \frac{2m\langle\bar q q\rangle}{m^2p^2}\left[1 - \sqrt{1 - \frac{4m^2}{p^2}}\right]. \quad (46)$$

The transverse component is found by contracting (44) with $g_{\mu\nu}$:

$$\Pi^{A\mu}_\mu(p) = p^2\Pi_L(p^2) - 3p^2\Pi_T(p^2). \quad (47a)$$

$$= 48\int d^4k\,\frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + i\varepsilon}\left(-3m^2 + \tfrac{1}{2}p^2\right) - 24\int d^4k\,\mathscr{F}(k). \quad (47b)$$

Evaluating the integrals and solving for Π_T leads to the final result for $\Pi^A_{\mu\nu}$,

$$\begin{aligned} \Pi^A_{\mu\nu}(p) &= \frac{2m\langle\bar q q\rangle}{m^2p^2}\left[1 - \sqrt{1 - \frac{4m^2}{p^2}}\right]p_\mu p_\nu \\ &\quad + \frac{m\langle\bar q q\rangle}{3m^4}\left[1 - \frac{6m^2}{p^2} - \left(1 - \frac{4m^2}{p^2}\right)^{3/2}\right] \\ &\quad \times (p_\mu p_\nu - p^2 g_{\mu\nu}). \end{aligned} \quad (48)$$

confirming earlier results [10, 13] obtained through use of conventional coordinate-space and plane-wave techniques. (The relevant equation in [13] has an easily-identified typographical error.)

4 Equivalence of OPE techniques for the gluon condensate

The equivalence of plane-wave, coordinate-space, and moment methods for the OPE coefficient of the gluon condensate ($\langle\alpha_s G^2\rangle$) appears a priori unlikely because of the fixed-point gauge employed for the non-local two-gluon vev in coordinate-space methods. The issue is further complicated by the possibility of operator mixing in the OPE, as illustrated by the ST identities for the gluon propagator [19]. Nevertheless, for the evaluation of the product of gauge invariant currents as in QCD sum-rule applications, then the restricted class of (gauge invariant) operators appearing in the OPE allows a demonstration of the equivalence between the three methods.

To demonstrate this equivalence, we first review plane-wave, coordinate space, and moment methods as applied to the gluon condensate. Consider the OPE of the product of two (gauge invariant) scalar currents:

$$\begin{aligned} i\int d^4x\, e^{ip\cdot x}\, T(j(x)j(0)) &= \mathscr{I}(p^2)+\mathscr{C}(p^2):G^a_{\mu\nu}G^a_{\mu\nu}: \\ &\quad +\mathscr{D}(p^2)p^\lambda p^\rho\left[:G^a_{\mu\lambda}G^a_{\mu\rho}: -\tfrac{1}{4}g^{\lambda\rho}:G^a_{\mu\nu}G^a_{\mu\nu}:\right] \\ &\quad +\textit{operators not leading to } \langle\alpha_s G^2\rangle . \end{aligned} \tag{49}$$

A non-trivial complication in (49) occurs for coordinate-space methods when the (light) quark mass is taken to be non-zero in order to deal with infrared problems. In this case, the $m=0$ limit does not agree with the direct calculation with massless quarks because of an operator mixing between the quark and gluon condensates (at lowest order) when $m\neq 0$ [13]. The resolution of this problem is closely related to the heavy quark expansion (the heavy quark expansion will be discussed in Sect. 5). For now, we will assume that all calculations of the gluon condensate effects involve massless quarks. This problem does not occur in plane-wave methods because the OPE can be interpreted in terms of non-normal-ordered operators [22].

The correlation function $\Pi(p^2)$ is formed by taking a vev using the non-perturbative vacuum $|\Omega\rangle$. Consequently, the gluonic condensate contributions to $\Pi(p^2)$ are given by

$$\begin{aligned} \Pi(p^2) &= i\int d^4x\, e^{ip\cdot x}\langle\Omega|T(j(x)j(0))|\Omega\rangle \\ &= \mathscr{C}(p^2)\langle G^2\rangle+\mathscr{D}(p^2)p^\lambda p^\rho \\ &\quad \times\left[\langle\Omega|:G^a_{\mu\lambda}G^a_{\mu\rho}:|\Omega\rangle-\tfrac{1}{4}g^{\lambda\rho}\langle G^2\rangle\right], \end{aligned} \tag{50}$$

where $\langle G^2\rangle\equiv\langle\Omega|:G^a_{\mu\nu}G^a_{\mu\nu}:|\Omega\rangle$. The process of vacuum averaging annihilates the term proportional to $\mathscr{D}(p^2)$

$$\langle\Omega|:G^a_{\mu\lambda}G^a_{\mu\rho}:|\Omega\rangle=\tfrac{1}{4}g_{\lambda\rho}\langle G^2\rangle, \tag{51}$$

so that the gluon condensate contribution to $\Pi(p^2)$ is given entirely by $\mathscr{C}(p^2)$:

$$\Pi(p^2)=\mathscr{C}(p^2)\langle G^2\rangle+\textit{terms not leading to } \langle G^2\rangle . \tag{52}$$

The plane-wave method is designed to extract the gluon condensate by forming a (connected) matrix element of the OPE with one-gluon states $|\varepsilon, k\rangle$:

$$\begin{aligned} &\int d^4x\, e^{ip\cdot x}\langle\varepsilon,k|T(j(x)j(0))|\varepsilon,k\rangle \\ &=4\mathscr{C}(p^2)[\varepsilon^2k^2-(\varepsilon\cdot k)^2] \\ &\quad +4\mathscr{D}(p^2)[k^2(p\cdot\varepsilon)^2-p\cdot kp\cdot\varepsilon k\cdot\varepsilon-\tfrac{1}{4}p^2(\varepsilon^2k^2-(\varepsilon\cdot k)^2)]. \end{aligned} \tag{53}$$

At this point, the origin of the operator mixing is clearly evident in the plane-wave approach, since when $m\neq 0$ the matrix element $\langle\varepsilon,k|\bar{\psi}\psi|\varepsilon,k\rangle$ is non-trivial, leading to the misidentification of gluon condensate effects as discussed in [13].

As with the quark condensate, the behaviour of the vacuum is simulated by averaging over the directions of the external momentum p

$$\int d\hat{p}\, p_\alpha p_\beta=\tfrac{1}{4}p^2 g_{\alpha\beta}, \tag{54}$$

in which case the term proportional to $\mathscr{D}(p^2)$ is once again annihilated:

$$\begin{aligned} &\int d\hat{p}\int d^4x\, e^{ip\cdot x}\langle\varepsilon,k|T(j(x)j(0))|\varepsilon,k\rangle \\ &\quad =4\mathscr{C}(p^2)[\varepsilon^2k^2-(\varepsilon\cdot k)^2]. \end{aligned} \tag{55}$$

The invariant amplitude for the LHS of (55) is constrained by gauge invariance ($\varepsilon_\alpha\to\varepsilon_\alpha+k_\alpha$) to be transverse to the momentum k, resulting in two possible terms:

$$\begin{aligned} T_{\alpha\beta}(p,k) &= \int d^4x\, e^{ip\cdot x}\langle\alpha,k|T(j(x)j(0))|\beta,k\rangle \\ &= T^{(1)}(p,k)\,[k^2g_{\alpha\beta}-k_\alpha k_\beta] \\ &\quad +T^{(2)}(p,k)[k^2p_\alpha p_\beta-k\cdot pk_\alpha p_\beta-k\cdot pk_\beta p_\alpha \\ &\quad +(k\cdot p)^2g_{\alpha\beta}-\tfrac{1}{2}p^2(k^2g_{\alpha\beta}-k_\alpha k_\beta)]. \end{aligned} \tag{56}$$

The term proportional of $T^{(2)}$ is zero after averaging over p, leaving dependence only on $T^{(1)}$:

$$\begin{aligned} 4\mathscr{C}(p^2)[\varepsilon^2k^2-(\varepsilon\cdot k)^2] &= \varepsilon^\alpha\varepsilon^\beta\int d\hat{p}\, T_{\alpha\beta}(p,k) \\ &= T^{(1)}(p,0)(k^2g_{\alpha\beta}-k_\alpha k_\beta)\varepsilon^\alpha\varepsilon^\beta. \end{aligned} \tag{57}$$

Thus in the plane-wave method, the OPE coefficient of the gluon condensate $\langle G^2\rangle$ is given by one of the components of the invariant amplitude evaluated at zero external gluon momenta,

$$\mathscr{C}(p^2)=\tfrac{1}{4}T^{(1)}(p,0). \tag{58}$$

As with the quark condensate, coordinate-space techniques for the gluon condensate involve a non-local vev originating in the Wick expansion of the time-ordered product of currents. The only non-local vev that can contribute to the gluon condensate contains two gluon fields $\langle\Omega|:A^a_\mu(x)A^b_\nu(y):|\Omega\rangle$ [14]. Since this quantity is gauge dependent, it must be demonstrated that such gauge dependence does not affect the gluon condensate contributions to gauge invariant correlation functions. Furthermore, the equivalence between coordinate-space and plane-wave methods must be established.

First consider a covariant gauge representation of the two-gluon non-local vev [20]

$$\begin{aligned} &\langle\Omega|:A^a_\mu(x)A^b_\nu(y):|\Omega\rangle \\ &=\frac{\delta^{ab}}{N_c^2-1}[C(x-y)_\mu(x-y)_\nu+E(x-y)^2g_{\mu\nu}] \\ &\quad +\textit{terms leading to higher dimension condensates}. \end{aligned} \tag{59}$$

The coefficients C and E are related to the dimension-four gluonic condensates.

$$C-2E=\tfrac{1}{24}\langle G^2\rangle \tag{60a}$$

$$5C+2E=-\tfrac{1}{4}\langle\Omega|:(\partial\cdot A)^2:|\Omega\rangle, \tag{60b}$$

The momentum space representation of (59) is

$$\begin{aligned} &\langle\Omega|:A^a_\mu(x)A^a_\nu(y):|\Omega\rangle \\ &=-\int d^4k\, e^{-ik\cdot(x-y)}\left[C\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu}+Eg_{\mu\nu}\frac{\partial}{\partial k_\lambda}\frac{\partial}{\partial k^\lambda}\right]\delta^4(k). \end{aligned} \tag{61}$$

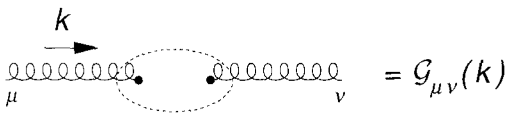

Fig. 3. The momentum-space Feynman rule for the vacuum expectation value of normal-ordered gluon fields, corresponding to the integrand of (61) in the limit $y-x \to 0$

As with the quark condensate, the normal-ordered term can be identified with a Feynman rule as illustrated in Fig. 3. Thus the contribution of this non-local vev from the Wick expansion of $\Pi(p^2)$ is related to the invariant amplitude

$$\Pi(p^2) = -\int d^4k\,\delta^4(k)\left[C\frac{\partial}{\partial k^\alpha}\frac{\partial}{\partial k^\beta} + E g_{\alpha\beta}\frac{\partial}{\partial k^\lambda}\frac{\partial}{\partial k_\lambda}\right]\frac{1}{2}T_{\alpha\beta}(p,k). \tag{62}$$

The symmetry factor of 1/2 originates in the double-counting when joining the external gluon lines in the amplitude. Using the expression (56) for $T_{\alpha\beta}(p,k)$ and performing the delta function integration leads to the following contribution to $\Pi(p^2)$:

$$\begin{aligned}\Pi(p^2) = &-\int d^4k\,\delta^{(4)}(k)\left(C\frac{\partial}{\partial k^\alpha}\frac{\partial}{\partial k^\beta} + E g_{\alpha\beta}\frac{\partial}{\partial k^\lambda}\frac{\partial}{\partial k_\lambda}\right)\\ &\times[T^{(1)}(p,k)(k^2 g_{\alpha\beta} - k_\alpha k_\beta)\\ &+T^{(2)}(p,k)(k^2 p_\alpha p_\beta - k\cdot p\, k_\alpha p_\beta - k\cdot p\, k_\beta p_\alpha\\ &+(k\cdot p)^2 g_{\alpha\beta} - \tfrac{1}{2}p^2(k^2 g_{\alpha\beta} - k_\alpha k_\beta))]\\ =&\,6(C-2E)\,T^{(1)}(p,0).\end{aligned} \tag{63}$$

Recalling the relations (60) between C, E and the gluon condensates leads to a coordinate-space determination of the gluon condensate component of the scalar current correlation function.

$$\begin{aligned}&\Pi(p^2) = \tfrac{1}{4}\langle G^2\rangle\, T^{(1)}(p,0)\\ &\mathscr{C}(p^2) = \tfrac{1}{4}T^{(1)}(p,0).\end{aligned} \tag{64}$$

This result is identical to (58), demonstrating the equivalence of plane-wave and coordinate-space approaches for covariant gauge representations of the non-local two-gluon vev.

A similar procedure exists for fixed-point gauges. In this case the non-local vev violates translation invariance, and the connection with plane-wave methods requires distinct momenta for the external gluon lines. The requirement is similar to the modified non-zero momentum (NZI) plane-wave method [19]. The modified NZI plane-wave approach has proved to be useful in resolving questions of operator mixing and in analyzing the infrafed finiteness of OPE coefficients.

In the fixed-point gauge, the gluon condensate component of the non-local two-gluon vev is

$$\begin{aligned}&\langle\Omega|:A^a_\alpha(y)A^b_\beta(z):|\Omega\rangle\\ &=\frac{1}{48}y^\rho z^\tau\left(\frac{\delta^{ab}}{N_c^2-1}\right)[g_{\rho\tau}g_{\alpha\beta} - g_{\rho\beta}g_{\alpha\tau}]\langle G^2\rangle.\end{aligned} \tag{65}$$

Since this vev is not translation invariant, its momentum-space version depends on two momenta:

$$\begin{aligned}&\langle\Omega|:A^a_\alpha(y)A^a_\beta(z):|\Omega\rangle\\ &=-\frac{\langle G^2\rangle}{48}\int d^4k\int d^4\ell\, e^{-ik\cdot y}e^{-i\ell\cdot z}\\ &\quad\times\left(g_{\alpha\beta}\frac{\partial}{\partial k^\lambda}\frac{\partial}{\partial \ell_\lambda} - \frac{\partial}{\partial k^\beta}\frac{\partial}{\partial \ell^\alpha}\right)\delta^{(4)}(k)\delta^{(4)}(\ell).\end{aligned} \tag{66}$$

The invariant amplitude is also modified by the presence of different momenta for the external gluon lines.

$$\begin{aligned}\mathscr{T}_{\alpha\beta}(p,k,\ell) &\equiv \int d^4x\, e^{ip\cdot x}\langle O|T(j(x)j(0))|\alpha,k;\beta,\ell\rangle\\ &=\mathscr{T}^{(1)}(p,k,\ell)[-k\cdot\ell\, g_{\alpha\beta} + k_\beta\ell_\alpha]\\ &\quad+\mathscr{T}^{(2)}(p,k,\ell)[-k\cdot\ell\, p_\alpha p_\beta + p\cdot k\,\ell_\alpha p_\beta\\ &\quad+p\cdot\ell\, k_\beta p_\alpha - \tfrac{1}{2}(-k\cdot\ell\, g_{\alpha\beta} + k_\beta\ell_\alpha)].\end{aligned} \tag{67}$$

Gauge invariance is again satisfied by this amplitude since it is transverse with respect to k^α, ℓ^β. In the limit $\ell = -k$, the above expression must reduce to (56), implying that

$$\begin{aligned}&\mathscr{T}_{\alpha\beta}(p,k,-k) = T_{\alpha\beta}(p,k),\\ &\mathscr{T}^{(1)}(p,k,-k) = T^{(1)}(p,k),\\ &\mathscr{T}^{(2)}(p,k,-k) = T^{(2)}(p,k).\end{aligned} \tag{68}$$

Using the above results, the contribution of the fixed-point gauge non-local vev to the correlation function $\Pi(p^2)$ is then given by

$$\begin{aligned}\Pi(p^2) = &-\frac{\langle G^2\rangle}{48}\int d^4k\int d^4\ell\,\delta^{(4)}(k)\delta^{(4)}(\ell)\\ &\times\left[g_{\alpha\beta}\frac{\partial}{\partial k^\lambda}\frac{\partial}{\partial \ell_\lambda} - \frac{\partial}{\partial k^\beta}\frac{\partial}{\partial \ell^\alpha}\right]\frac{1}{2}\mathscr{T}_{\alpha\beta}(p,k,\ell).\end{aligned} \tag{69}$$

Using the expression (67) for the amplitude $\mathscr{T}_{\alpha\beta}$ and recalling (68) provides the final result for the gluon condensate component of $\Pi(p^2)$,

$$\Pi(p^2) = \tfrac{1}{4}\langle G^2\rangle\,\mathscr{T}^{(1)}(p,0,0) = \tfrac{1}{4}\langle G^2\rangle\, T^{(1)}(p,0). \tag{70}$$

Comparison of (58), (64) and (70) reveals that coordinate-space and plane-wave methods are equivalent for determining the gluon condensate OPE coefficient in correlation functions of gauge invariant currents. Furthermore, the coordinate-space techniques are independent of the gauge chosen for evaluating the non-local two-gluon vev. This latter point is obviously a concern when combining the non-local vev with perturbative covariant-gauge gluon propagators.

As with the quark condensate, moment techniques for the gluon condensate identify integrals of the non-perturbative gluon propagator with the gluon condensate:

$$\begin{aligned}&D^{NP}_{\mu\nu}(k) = D_{\mu\nu}(k) - D^{pert}_{\mu\nu}(k) = \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\tilde{D}(k^2)\\ &i\int\frac{d^4k}{(2\pi)^4}k^2\tilde{D}(k^2) = \frac{\langle G^2\rangle}{48}.\end{aligned} \tag{71}$$

Thus the relevant (gluon condensate) contribution to $\Pi(p^2)$ is related to the invariant amplitude

$$\Pi(p^2)=i\int\frac{d^4k}{(2\pi)^4}4D^{NP}_{\alpha\beta}(k)\,T_{\alpha\beta}(p,k), \tag{72}$$

where the factor of 4 comes from a colour trace and a symmetry factor to prevent double counting. Substituting for $D^{NP}_{\alpha\beta}(k)$ from (71), recalling that $T_{\alpha\beta}$ is transverse to the momentum k, and using the explicit form (56) for the invariant amplitude leads to the expression

$$\begin{aligned}\Pi(p^2)&=i\int\frac{d^4k}{(2\pi)^4}4\tilde{D}(k^2)\,T^{\alpha}_{\alpha}(p,k)\\&=i\int\frac{d^4k}{(2\pi)^4}4\tilde{D}(k^2)\left[3k^2T^{(1)}(p,k)\right.\\&\quad\left.+\left(-\frac{1}{2}p^2k^2+2(p\cdot k)^2\right)T^{(2)}(p,k)\right].\end{aligned} \tag{73}$$

The angular integration annihilates the term containing $T^{(2)}$, and the relation (71) between the moments and $\langle G^2\rangle$ gives a final result

$$\Pi(p^2)=\tfrac{1}{4}\langle G^2\rangle\,T^{(1)}(p,0) \tag{74}$$

identical to the previous methods. This completes our demonstration of the equivalence of the plane-wave, coordinate-space and moment methods for evaluating the OPE coefficient of the gluon condensate. As mentioned earlier, it has been assumed that massless quarks have been used in all calculations for the gluon condensate (see [13] for details on dealing with the case when $m\neq 0$).

5 Aspects of gauge dependence: the quark propagator

An important element of the analysis of Sect. 4 is the nature of the OPE. For products of gauge invariant currents, the OPE can only obtain gauge invariant, equation of motion, or BRS variation operators. To lowest order in the gluon condensate only the gauge invariant operators contribute to the correlation functions, in which case the problems associated with operator mixing and renormalization of composite operators do not occur. However, for gauge dependent correlation functions there is no restriction on the operators appearing in the OPE and hence the various approaches to evaluating the gluon condensate contribution will differ. An example of such gauge dependence is provided by the quark propagator.

An unresolved issue in the literature is the gluon condensate contributions to the quark propagator, where the results of plane-wave [11] and fixed-point coordinate space techniques [5, 11, 14] disagree. It is essential that such dependence disappear from calculations of (gauge invariant) physical quantities. We show below that when the different expressions for the quark propagator are used in the heavy quark expansion for the quark condensate, all remnants of gauge dependence cancel, and standard results [2, 21] are obtained.

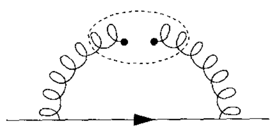

Fig. 4. The Feynman diagram for the (lowest-order) gluon condensate contributions to the quark propagator

First consider the plane-wave and fixed-point results for the $\langle\alpha_s G^2\rangle$ portion of the heavy quark self-energy $\Sigma(p)$:

$$\Sigma(p)=\frac{\pi m\not{p}(\not{p}-m)}{3(p^2-m^2)^3}\langle\alpha_s G^2\rangle\ \text{(fixed-point)}, \tag{75a}$$

$$\Sigma(p)=\frac{\pi[(p^2-3m^2)\not{p}+4m^3]}{9(p^2-m^2)^3}\langle\alpha_s G^2\rangle\ \text{(plane-wave)}. \tag{75b}$$

The parameter m is the heavy quark mass, and the self-energy is related to the quark propagator $\Delta S(p)$ in the usual fashion:

$$\begin{aligned}i\Delta S(p)&\equiv\int d^4x\,e^{ip\cdot x}\langle\Omega|T(\psi(x)\bar{\psi}(0))|\Omega\rangle\\&=(\not{p}+m)\,\Sigma(p)\,(\not{p}+m)\frac{i}{(p^2-m^2)^2}.\end{aligned} \tag{76}$$

We will illustrate the gluonic condensate contributions to the quark propagator in the covariant gauge. The term from the Wick expansion that contains the non-local gluon vev is

$$\begin{aligned}i\Delta S(p)=&-\frac{g^2}{4}\int d^4x\,e^{ip\cdot x}\int d^4y\int d^4z\,\langle O|T(\psi(x)\bar{\psi}(y))|O\rangle\\&\times\gamma^\mu\lambda^a\langle\Omega|:A^a_\mu(x)A^b_\nu(y):|\Omega\rangle\langle O|T(\psi(y)\bar{\psi}(z))|O\rangle\\&\times\gamma^\nu\lambda^b\langle O|T(\psi(z)\bar{\psi}(0))|O\rangle,\end{aligned} \tag{77}$$

as illustrated in Fig. 4 (the Feynman rule for the normal-ordered product is shown in (61) and Fig. 3). The covariant gauge expression (59) for the non-local gluon vev leads, upon converting to momentum space, to the self-energy

$$\Sigma(p)=-\frac{g^2}{6}\gamma^\mu\left[C\frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p^\nu}+Eg_{\mu\nu}\frac{\partial}{\partial p^\tau}\frac{\partial}{\partial p_\tau}\right]\frac{\not{p}+m}{p^2-m^2}\gamma^\nu. \tag{78}$$

Evaluating the derivatives, and recalling the definitions of C, E from (60) leads to the covariant gauge result for the gluonic condensate portion of the heavy-quark self-energy

$$\begin{aligned}\Sigma(p)=&\frac{\pi m^3}{3(p^2-m^2)}\alpha_s\langle\Omega|:(\partial\cdot A)^2:|\Omega\rangle\\&+\frac{\pi[(p^2-3m^2)\not{p}+3m^3]}{9(p^2-m^2)^3}\langle\alpha_s G^2\rangle.\end{aligned} \tag{79}$$

The explicit appearance of the $(\partial\cdot A)^2$ operator is a consequence of the OPE for a gauge dependent current.

Clearly the gluon condensate component of the quark self-energy is gauge dependent. However, when calculating a gauge invariant quantity such as the heavy-quark condensate, such gauge dependence must cancel. The connection with the heavy quark expansion for the quark condensate $\langle\bar{\psi}\psi\rangle$ is made through the gluonic contributions to the self-energy:

$$-\langle\bar{\psi}\psi\rangle \equiv \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\,[i\Delta S(p)]. \tag{80}$$

Substituting the different versions of the self-energy leads to plane-wave (pw), fixed-point (fp), and covariant gauge (cg) expressions for the heavy quark condensate:

$$-\langle\bar{\psi}\psi\rangle_{pw} = i\frac{12}{9}\pi\langle\alpha_s G^2\rangle \int \frac{d^4p}{(2\pi)^4} \times \frac{4m^3(p^2+m^2)+2mp^2(p^2-3m^2)}{(p^2-m^2)^5}. \tag{81a}$$

$$-\langle\bar{\psi}\psi\rangle_{fp} = i4\pi m\langle\alpha_s G^2\rangle \int \frac{d^4p}{(2\pi)^4}\frac{p^2}{(p^2-m^2)^4}. \tag{81b}$$

$$-\langle\bar{\psi}\psi\rangle_{cg} = i4\pi\alpha_s m^3\langle\Omega|:(\partial\cdot A)^2|\Omega\rangle \int \frac{d^4p}{(2\pi)^4}\frac{p^2+m^2}{(p^2-m^2)^5} + i\frac{4}{3}\pi m\langle\alpha_s G^2\rangle \int \frac{d^4p}{(2\pi)^4}\frac{2p^4-3m^2p^2+3m^4}{(p^2-m^2)^5}. \tag{81c}$$

All the momentum integrals are finite, and can be calculated using the integral [4]

$$\int \frac{d^4k}{(2\pi)^4}\frac{(k^2)^\alpha}{(k^2-m^2)^\beta} = \frac{i}{16\pi^2}(-m^2)^{\alpha-\beta+2}\frac{\Gamma(2+\alpha)\Gamma(\beta-\alpha-2)}{\Gamma(\beta)\Gamma(2)}. \tag{82}$$

The final (standard [2, 21]) result for the heavy quark expansion contains only gauge invariant operators (i.e. the coefficient of $\langle\Omega|:(\partial\cdot A)^2:|\Omega\rangle$ is zero), and as expected, is independent of the gauge used to evaluate the quark self-energy:

$$-m\langle\bar{\psi}\psi\rangle = \frac{\langle\alpha_s G^2\rangle}{12\pi}. \tag{83}$$

This is an explicit illustration of the equivalence of plane-wave and coordinate space methods when a gauge dependent correlation function is used as an intermediate stage in the calculation of a gauge-invariant quantity.

Another example of an apparent discrepancy between plane-wave and coordinate-space methods is the (light) quark condensate contribution to the quark self-energy. The difference between these methods occurs for physical momenta where the plane-wave (pw) technique observes a "freeze-out" below $p^2=m^2$ [11] that is inaccessible in the explicit m^2/p^2 coordinate-space (cs) series [14].

$$\Sigma(p)_{cs} = \frac{g^2}{9p^2}\langle\bar{q}q\rangle\left[(3+a)-a\frac{m\not{p}}{p^2}\right]. \tag{84a}$$

$$\Sigma(p)_{pw} = \frac{g^2}{9p^2}\langle\bar{q}q\rangle\left[(3+a)-a\frac{m\not{p}}{p^2}\right]\theta(p^2-m^2) + \frac{g^2}{9m^2}\langle\bar{q}q\rangle\left[(3+a)-a\frac{\not{p}}{m}\right]\theta(m^2-p^2) \tag{84b}$$

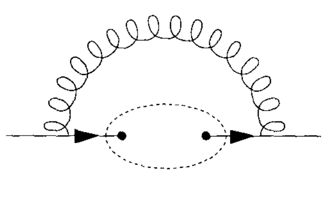

Fig. 5. The Feynman diagram for (lowest-order) quark condensate contributions to the quark propagator

The coordinate space series approach, leading to a series in powers of m^2/p^2, is insensitive to the non-analytic freeze-out. We show below, however, that the improved technique for the coordinate-space approach developed in Sect. 3 yields the same freeze-out as observed via plane-wave methods.

Before continuing, some comments on the physical interpretation of the freeze-out are required. Since the Euclidean momentum expression must be analytic, there can be only one analytic continuation of the Euclidean result to physical momentum. Phrased another way, only one of the two segments in (84b) can be analytically continued to Euclidean momentum, namely the large p^2 portion, leading to the physical OPE coefficient which is analytic for large Euclidean momentum ($m^2/p^2 \leqslant 0$). However, the freeze-out is unavoidable in the plane-wave approach [11], and thus if the OPE methods are truly equivalent then this effect must also be observed in coordinate-space techniques.

The relevant term in the Wick expansion of the quark propagator containing the two-quark vev is

$$i\Delta S(p) = \frac{g^2}{4}\int d^4x\,d^4y\,d^4z\,e^{ip\cdot x}\langle O|T(\psi(x)\bar{\psi}(y))|O\rangle\gamma^\mu\lambda^a \times\langle\Omega|:\bar{\psi}(z)\psi(y):|\Omega\rangle\gamma^\nu\lambda^b\langle O|T(A^a_\mu(y)A^b_\nu(z))|O\rangle \times\langle O|T(\psi(z)\bar{\psi}0))|O\rangle, \tag{85}$$

as illustrated in Fig. 5. Converting (85) to momentum space (via (28) or through use of the Feynman rules) yields the following quark self-energy

$$\Sigma(p) = \frac{4}{3}g^2\int d^4k\frac{\mathcal{F}(k)}{(p-k)^2}\gamma_\mu(\not{k}+m)\gamma_\nu \times\left(-g^{\mu\nu}+(1-a)\frac{(p-k)^\mu(p-k)^\nu}{(p-k)^2}\right). \tag{86}$$

The self-energy is now expressed in terms of its distinct Dirac structures

$$\Sigma(p) = A(p^2)\not{p}+B(p^2), \tag{87a}$$

$$\mathrm{Tr}(\Sigma(p)) = 4B(p^2), \tag{87b}$$

$$\mathrm{Tr}(\not{p}\Sigma(p)) = 4p^2A(p^2). \tag{87c}$$

The functions $A(p^2)$ and $B(p^2)$ are found from (86) to be

$$B(p^2) = -\frac{4}{3}(3+a)mg^2\int d^4k\frac{\mathcal{F}(k)}{(p-k)^2} \tag{88a}$$

$$A(p^2) = \frac{4g^2}{3p^2}\int d^4k\,\mathcal{F}(k)\frac{2p\cdot k}{(p-k)^2}+\frac{4g^2}{3p^2}(1-a) \times\int d^4k\,[(p^2+m^2)p\cdot k-2p^2m^2]\frac{\mathcal{F}(k)}{(p-k)^4}. \tag{88b}$$

The integral determining $B(p^2)$ is evaluated by exponentiating the denominator and utilizing (29b) in the integrand of (88a). After evaluating the k integral

$$\int d^4 k \frac{\mathscr{F}(k)}{(p-k)^2} = \int d^4 k \frac{\mathscr{F}(k)}{p^2 - 2p\cdot k + m^2} = \frac{i\langle\bar{q}q\rangle}{12m^2\sqrt{p^2}} \int_0^\infty \frac{d\eta}{\eta} e^{i\eta(p^2+m^2+i\varepsilon)} J_1(2m\eta\sqrt{p^2}). \tag{89a}$$

the remaining integral is tabulated [18] (see (33b)). Carefully taking the $\varepsilon \to 0^+$ limit in the tabulated integral leads to a result containing a freeze-out for physical momenta,

$$\int d^4 k \frac{\mathscr{F}(k)}{(p-k)^2} = -\frac{\langle\bar{q}q\rangle}{12mp^2}\theta(p^2-m^2) - \frac{\langle\bar{q}q\rangle}{12m^3}\theta(m^2-p^2). \tag{90}$$

Using (90), the result for $B(p^2)$ in the physical momentum region is

$$B(p^2) = \frac{g^2}{9}(3+a)\langle\bar{q}q\rangle\left[\frac{1}{p^2}\theta(p^2-m^2) + \frac{1}{m^2}\theta(m^2-p^2)\right]. \tag{91}$$

Now consider evaluating the integrals determining $A(p^2)$. The first integral in (88b) is simple to evaluate using (90) and (31):

$$\begin{aligned}\int d^4 k \mathscr{F}(k) \frac{2p\cdot k}{(p^4-k)^2} &= \int d^4 k \mathscr{F}(k) \frac{p^2+m^2}{p^2+m^2-2p\cdot k} - \int d^4 k \mathscr{F}(k) \\ &= \frac{\langle\bar{q}q\rangle}{12m}\left\{1 - (p^2+m^2)\left[\frac{1}{p^2}\theta(p^2-m^2) + \frac{1}{m^2}\theta(m^2-p^2)\right]\right\}.\end{aligned} \tag{92}$$

The second integral of (88b) can be expressed as a linear combination of (90) and a new integral:

$$\begin{aligned}\int d^4 k \mathscr{F}(k) \frac{(p^2+m^2)p\cdot k - 2p^2m^2}{(p^2+m^2-2p\cdot k)^2} &= -\frac{1}{2}(p^2+m^2)\int d^4 k \frac{\mathscr{F}(k)}{(p-k)^2} + \frac{1}{2}(p^2-m^2)^2 \int d^4 k \frac{\mathscr{F}(k)}{(p-k)^4} \\ &= \frac{p^2+m^2}{24m}\langle\bar{q}q\rangle\left[\frac{1}{p^2}\theta(p^2-m^2) + \frac{1}{m^2}\theta(m^2-p^2)\right] \\ &\quad + \frac{1}{2}(p^2-m^2)^2\int d^4 k \frac{\mathscr{F}(k)}{(p-k)^4}.\end{aligned} \tag{93}$$

The final integral in (93) is evaluated by exponentiating the denominator, using the on-shell constraint, and then performing the k integral:

$$\begin{aligned}\int d^4 k \frac{\mathscr{F}(k)}{(p^2+m^2-2p\cdot k)^2} &= -\int d^4 k \int_0^\infty d\eta\, \eta e^{i\eta(p^2+m^2-2p\cdot k+i\varepsilon)} \mathscr{F}(k) \\ &= \frac{\langle\bar{q}q\rangle}{12m^2\sqrt{p^2}} \int_0^\infty d\eta\, e^{i\eta(p^2+m^2+i\varepsilon)} J_1(2m\eta\sqrt{p^2}).\end{aligned} \tag{94a}$$

The remaining integral is tabulated [18],

$$\int_0^\infty dx e^{-\alpha x} J_\nu(\beta x) = \frac{(\sqrt{\alpha^2+\beta^2}-\alpha)^\nu}{\beta^\nu\sqrt{\alpha^2+\beta^2}},$$

$$\mathrm{Re}\,\nu > -1;\ \mathrm{Re}(\alpha \pm i\beta) > 0, \tag{94b}$$

and after carefully evaluating the $\varepsilon \to 0^+$ limit a freeze-out is again observed:

$$\int d^4 k \frac{\mathscr{F}(k)}{(p^2+m^2-2p\cdot k)^2} = \frac{\langle\bar{q}q\rangle}{24m^3p^2}\left[1 - \frac{p^2+m^2}{p^2-m^2}\theta(p^2-m^2) - \frac{p^2+m^2}{m^2-p^2}\theta(m^2-p^2)\right]. \tag{95}$$

Substituting (92), (93) and (95) back into (88b) leads to

$$A(p^2) = -\frac{g^2}{9} am\langle\bar{q}q\rangle\left[\frac{1}{p^4}\theta(p^2-m^2) + \frac{1}{m^4}\theta(m^2-p^2)\right], \tag{96}$$

in which case, the quark condensate component of the light-quark self-energy

$$\Sigma(p) = \frac{g^2}{9p^2}\langle\bar{q}q\rangle\left[(3+a) - a\frac{m\not{p}}{p^2}\right]\theta(p^2-m^2) + \frac{g^2}{9m^2}\langle\bar{q}q\rangle\left[(3+a) - a\frac{\not{p}}{m}\right]\theta(m^2-p^2), \tag{97}$$

is found to be in agreement with the plane-wave result [11].

Thus the improved method developed in Sect. 3 for the coordinate-space approach is identical to the plane-wave result, leading to a freeze-out in the quark self-energy at the physical momentum point $p^2 = m^2$.

Conclusions

The equivalence of plane-wave, coordinate space and moment methods has been demonstrated for the determination of the $\langle\bar{q}q\rangle$ and $\langle\alpha_s G^2\rangle$ coefficients in the OPE of gauge invariant currents. An important conclusion of this analysis is that the fixed point gauge commonly employed in coordinate space applications to the gluon condensate does not affect the result of the OPE for gauge invariant currents.

Sources of disagreement exist between the various OPE techniques only when gauge dependent quantities are considered. In Sect. 5, however, such gauge dependence is shown not to affect the calculation of a gauge invariant quantity such as the heavy quark expansion relating the heavy-quark and gluon condensates.

A new approach for evaluating quark condensate effects to all orders in the quark mass has been developed in Sect. 3 and applied to products of gauge invariant currents of physical significance. It is hoped that this new technique will be of use in future calculations.

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Appendix A: One-loop baryonic integrals

In evaluating the lowest order quark condensate contributions to Baryonic currents, the following typical one-loop integrals will occur in analogy to (38).

$$I(q^2)=\nu^{4-D}\int\frac{d^Dl}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{(k-l)^2-m^2+i0}\times\frac{1}{(q-k)^2-m^2+i0}\mathscr{F}(l). \quad (A1)$$

This is a one-loop divergent calculation, so to apply dimensional regularization we need the D-dimensional version of (28c) as obtained from (4).

$$\langle\Omega|\psi(x)\bar{\psi}(0)|\Omega\rangle=\int d^Dk\,e^{-ik\cdot x}(\not{k}+m)\,\mathscr{F}(k)$$

$$\int d^Dl\,e^{-il\cdot x}\,\mathscr{F}(l)=-\frac{\Gamma(D/2)2^{D/2-1}}{12m}\langle\bar{q}q\rangle\frac{J_{D/2-1}(m\sqrt{x^2})}{(m^2x^2)^{D/4-1/2}}. \quad (A2)$$

Following the procedures developed in Sect. 3, the propagator factors are exponentiated, and the momentum variable k is shifted to eliminate the $k\cdot l$ and $k\cdot q$ cross-terms. The result is

$$I(q^2)=-\nu^{4-D}\int_0^\infty d\eta\int_0^\infty d\xi\exp\left\{i\left[\xi\left(1-\frac{\xi}{\eta+\xi}\right)q^2+\eta\left(1-\frac{\eta}{\eta+\xi}\right)m^2-(\xi+\eta)m^2+i0\right]\right\}\times\int\frac{d^Dl}{(2\pi)^D}\exp\left\{-2i\frac{\eta\xi}{\eta+\xi}q\cdot l\right\}\mathscr{F}(l)\times\int\frac{d^Dk}{(2\pi)^D}\exp\{i(\eta+\xi)k^2\} \quad (A3)$$

where the $l^2=m^2$ constraint has been imposed as in Sect. 3. The l integral is calculated using (A2), and the k integral can be accomplished with the following result:

$$\int\frac{d^Dk}{(2\pi)^D}e^{iak^2}=i\frac{a^{-D/2}}{(4\pi)^{D/2}}e^{-i(\pi/4)D} \quad (A4)$$

After performing the momentum integrals, the ξ and η parametric integrations can be simplified by changing variables.

$$\eta=\lambda(1-x)\qquad\xi=\lambda x;\qquad\left|\frac{\partial(\eta,\xi)}{\partial(\lambda,x)}\right|=\lambda \quad (A5)$$

The resulting expression for $I(q^2)$ with $D=4-2\varepsilon$ is

$$I(q^2)=-i\frac{2^{1-\varepsilon}e^{i\pi\varepsilon/2}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(2-\varepsilon)\nu^{2\varepsilon}\langle\bar{q}q\rangle}{12m}\int_0^1 dx\,\beta^{-1+\varepsilon}\times\int_0^\infty d\lambda\,\lambda^{-2+2\varepsilon}e^{-\alpha\lambda}J_{1-\varepsilon}(\beta\lambda);$$

$$\alpha=0-i[x(1-x)(q^2+m^2)-m^2];$$

$$\beta=2m\sqrt{q^2}\,x(1-x) \quad (A6)$$

In the $\varepsilon\to0$ limit the λ integral is divergent since for small λ, the integrand behaves like $\lambda^{-1+\varepsilon}$. Isolating this divergent structure leads to two integrals.

$$\beta^{-1+\varepsilon}\int_0^\infty d\lambda\,\lambda^{-2+2\varepsilon}e^{-\alpha\lambda}J_{1-\varepsilon}(\beta\lambda)=\beta^{-1+\varepsilon}\left\{\int_0^\infty d\lambda\,\lambda^{-2+2\varepsilon}e^{-\alpha\lambda}\frac{(\beta\lambda/2)^{1-\varepsilon}}{\Gamma(2-\varepsilon)}+\int_0^\infty d\lambda\,\lambda^{-2+2\varepsilon}e^{-\alpha\lambda}\left[J_{1-\varepsilon}(\beta\lambda)-\frac{(\beta\lambda/2)^{1-\varepsilon}}{\Gamma(2-\varepsilon)}\right]\right\} \quad (A7)$$

The first integral is divergent and leads to the Gamma function, while the second integral is finite in the limit as $\varepsilon\to0$. Since we can replace $\varepsilon=0$, the second integral is

$$\mathscr{J}(\alpha/\beta);\quad\text{where }\mathscr{J}(\xi)=\int_0^\infty du\frac{e^{-\xi u}}{u^2}\left[J_1(u)-\frac{u}{2}\right] \quad (A8)$$

which can be evaluated by differentiating with respect to ξ, using (33b) and then solving the differential equation for $\mathscr{J}(\xi)$. The result of this procedure is

$$\mathscr{J}(\xi)=\frac{\xi^2}{2}\left\{1-\sqrt{1+\frac{1}{\xi^2}}\right\}-\frac{1}{2}\log\left\{1+\sqrt{1+\frac{1}{\xi^2}}\right\}+\frac{1}{4}+\frac{1}{2}\log 2 \quad (A9)$$

Substituting the results of (A7–A9) into (A6) leads to the following result for Euclidean momenta $Q^2=-q^2>0$.

$$I(Q^2)=-\frac{i\langle\bar{q}q\rangle}{(4\pi)^2\,12m}\int_0^1 dx\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\frac{1}{2}+\log 2-\log Q^2/\nu^2-\log\left[x(1-x)+\frac{m^2}{Q^2}(x^2-x+1)\right]+\frac{1}{\xi^2}[1-\sqrt{1+\xi^2}]-\log[1+\sqrt{1+\xi^2}]\right)$$

$$\xi^2=\frac{4x^2(1-x)^2(m^2/Q^2)}{[x(1-x)+(m^2/Q^2)(x^2-x+1)]^2}. \quad (A10)$$

The final x integral is convergent. This illustrates how the methods developed in Sect. 3 for evaluating the quark condensate OPE coefficient can be extended to one-loop (divergent) integrals which would typically occur in baryonic currents.

Appendix B: charged-current correlation functions

For off-diagonal vector and axial vector charged currents (such as $\bar{d}\gamma_\mu\gamma_5 u$ characterizing charged pions, or in charged current weak interactions), vector and axial vector correlation functions corresponding to Fig. 2 are given by

$$\Pi_{\mu\nu}^{V,A}(p) \equiv \begin{pmatrix} V_T(p^2) \\ A_T(p^2) \end{pmatrix} [p^2 g_{\mu\nu} - p_\mu p_\nu] + \begin{pmatrix} V_T(p^2) \\ A_L(p^2) \end{pmatrix} p_\mu p_\nu$$
$$= 2\int d^4 k \frac{\mathrm{Tr}[(\not{k} - \not{p} + m_1)\Gamma_\mu(\not{k} + m_2)\Gamma_\nu]\mathscr{F}_2(k)}{(k-p)^2 - m_1^2 + i\varepsilon} \quad \text{(B1)}$$

where $\Gamma_\mu = \gamma_\mu$ for vector (V) and $\gamma_\mu \gamma_5$ for axial-vector (A) currents. The mass of the condensing fermion in Fig. 2 is m_2 [with the corresponding mass shell constraint $k^2 \mathscr{F}_2(k) = m_2^2 \mathscr{F}_2(k)$], and differs from the mass m_1 of the unbroken fermion line. We then find the following expressions for the transverse and longitudinal components of $\Pi_{\mu\nu}^{V,A}$:

$$p^4 \begin{pmatrix} V_L(p^2) \\ A_L(p^2) \end{pmatrix} = p^\mu p^\nu \Pi_{\mu\nu}^{V,A}$$
$$= -24 \int d^4 k\, p \cdot k \mathscr{F}_2(k) - 12(m_2^2 - m_1^2) \int d^4 k \mathscr{F}_2(k)$$
$$+ 12 [(m_2^2 - m_1^2)^2 - p^2(m_1 \mp m_2)^2]$$
$$\times \int d^4 k \frac{\mathscr{F}_2(k)}{p^2 - 2p \cdot k + m_2^2 - m_1^2 + i\varepsilon}; \quad \text{(B2)}$$

$$\begin{pmatrix} V_T(p^2) \\ A_T(p^2) \end{pmatrix} = \frac{1}{3p^2} g^{\mu\nu} \Pi_{\mu\nu}^{V,A} - \frac{1}{3} \begin{pmatrix} V_L(p^2) \\ A_L(p^2) \end{pmatrix}, \quad \text{(B3)}$$

$$g^{\mu\nu} \Pi_{\mu\nu}^{V,A} = -24 \int d^4 k \mathscr{F}_2(k) + 24 [p^2 - m_1^2 - m_2^2 \pm 4 m_1 m_2]$$
$$\times \int d^4 k \frac{\mathscr{F}_2(k)}{p^2 - 2p \cdot k + m_2^2 - m_1^2 + i\varepsilon}. \quad \text{(B4)}$$

Except for the last integral appearing in (B2) and (B4), all other integrals have been calculated [see (31) and (40)]. Evaluation of this new integral proceeds as follows:

$$I(p^2) = \int d^4 k \frac{\mathscr{F}_2(k)}{p^2 - 2p \cdot k + m_2^2 - m_1^2 + i\varepsilon}$$
$$= -i \int_0^\infty d\eta \exp[i\eta(p^2 + m_2^2 - m_1^2)] \int d^4 k \mathscr{F}_2(k) e^{-i2\eta k \cdot p}$$
$$= \frac{i\langle \bar{q}_2 q_2 \rangle}{12 m_2^2 \sqrt{p^2}} \int_0^\infty \frac{d\eta}{\eta} \exp[i\eta(p^2 + m_2^2 - m_1^2)]$$
$$\times J_1[2 m_2 \eta \sqrt{p^2}], \quad \text{(B5)}$$

where (28c) has been used. The last integral is tabulated [18], leading to the following result:

$$I(p^2) = -\frac{\langle \bar{q}_2 q_2 \rangle}{24 m_2^3 p^2} [p^2 + m_2^2 - m_1^2$$
$$- \sqrt{(p^2 - m_2^2)^2 - m_1^2(2p^2 + 2m_2^2 - m_1^2)}]. \quad \text{(B6)}$$

It is simple to verify that as required, the results for these correlation functions in the $m_2 = m_1$ limit agree with those obtained in Sect. 3. It is of some interest to note that the above correlation functions develop imaginary parts in the physical momentum region for $(m_1 - m_2)^2 < p^2 < (m_1 + m_2)^2$ corresponding to below-kinematic-threshold (as opposed to unitarity-driven) branch cuts also exhibited for $0 < p^2 < 4m^2$ in (34), (42), (46) and (48).

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