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Almost Everywhere Convergence of the Spherical Partial Sums for Radial Functions

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Abstract. Let $f \in L^p(\mathbb{R}^n)$, $n \geqslant 2$, be a radial function and let $S_R f$ be the spherical partial sums operator. We prove that if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$ then $S_R f(x) \to f(x)$ a.e. as $R \to \infty$. The result is false for $p < \frac{2n}{n+1}$ and $p > \frac{2n}{n-1}$.

We define the operator S by setting

$$(Sf)\hat{}(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \|\xi\| \leqslant 1 \\ 0, & \text{if } \|\xi\| > 1 \end{cases}$$

whenever $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, $n \geqslant 2$.

On $L^p(\mathbb{R}^n)$ radial functions Herz [1] proved that S is bounded if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$; Kenig and Tomas [2] that S is not weakly bounded for $p = \frac{2n}{n+1}$; Chanillo [3] that S is restricted weak-type at the index $\frac{2n}{n+1}$.

We are interested in the a.e. convergence as $R \to \infty$ of the partial sums $S_R f(x)$ (for Fourier integrals) where $(S_R f)\hat{}(\xi) = (Sf)\hat{}\left(\frac{\xi}{R}\right)$, assuming f radial. (For a general f this is an open problem for $p = 2$, otherwise it is well known [4] that convergence a.e. fails). We define

$$Tf(x) = \operatorname{Sup}_{R>0} | S_R f(x) | .$$

[1] Partially supported by M.P.I.

We will establish the following result

Theorem. *If* $\frac{2n}{n+1} < p < \frac{2n}{n-1}$ *then T is bounded on* $L^p(\mathbb{R}^n)$ *radial functions. If* $p \leqslant \frac{2n}{n+1}$ *or* $p \geqslant \frac{2n}{n-1}$ *then T is not bounded on* $L^p(\mathbb{R}^n)$ *radial functions.*

Corollary. *For every* $f \in L^p(\mathbb{R}^n)$ *radial, if* $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, *then* $S_R f(x) \to f(x)$ *a.e. as* $R \to \infty$.

Our estimates are based on the well known weighted norm inequalities for the maximal function M, the Hilbert integral H, the maximal Hilbert transform $\tilde{H}$ and for the maximal Carleson operator, [5],

$$\tilde{C}f(x) = \mathrm{Sup}_{n>0}\,\mathrm{Sup}_{\varepsilon>0}\left|\int_{\varepsilon<|x-t|<\pi} \frac{e^{-int}f(t)}{x-t}\,dt\right|.$$

Namely we are going to make use of the following

Lemma 1. *If* $G = M,\ H,\ \tilde{H},\ \tilde{C}$ *and* $w \in A_p, p > 1$, *then* $\int |Gf(x)|^p w(x)\,dx \leqslant c_p \int |f(x)|^p w(x)\,dx.$

Moreover we shall need the following inequalities for the Bessel functions $J_k(t)$.

Lemma 2. *Let* $k \geqslant 0$ *be half of an integer. Then*

(P_1) $|J_k(t)| \leqslant c_k, \quad t \geqslant 0,$

(P_2) $J_k(t) = \sqrt{\frac{2}{\pi t}}\cos\left(t - \frac{\pi}{2}k - \frac{\pi}{4}\right) + E_k(t),$

where $|E_k(t)| \leqslant \frac{c'_k}{t^{3/2}}$, *for* $t > A(k) > 0$.

Proof. To prove (P_1) it is enough to integrate by parts ($[k+1]$) times, starting from the definition of J_k. (P_2) is a refinement of the computations in [6], p. 159 that can be easily obtained.

By c we will denote a constant not necessarily the same in all instances and possibly depending upon n.

Proof of the Theorem. T cannot be bounded for $p \leqslant \frac{2n}{n+1}$ or $p \geqslant \frac{2n}{n-1}$ due to the mentioned result of HERZ. Let us turn to the positive part of the theorem. If f is a radial function then $\hat{f}$ is also radial and

$$\hat{f}(R) = 2\pi \int_0^\infty \frac{J_{(n-2)/2}(2\pi r R)}{|rR|^{(n-2)/2}} f(r)\, r^{n-1}\, dr\,.$$

Therefore

$$S_N f(s) = \frac{c}{s^{(n-2)/2}} \int_0^\infty r^{\frac{n}{2}} f(r) \int_0^N J_{(n-2)/2}(2\pi R s)\, J_{(n-2)/2}(2\pi R r)\, R\, dR\, dr\,.$$

Denote the inner integral by $K_N(s, r)$. From [2]

$$(x^k J_k(x))' = x^k J_{k-1}(x), \quad (x^{-k} J_k(x))' = -x^{-k} J_{k+1}(x)\,.$$

This and an integration by parts give

$$K_N(s,r) = c\frac{N}{r} J_{\frac{n}{2}-2}(2\pi N s)\, J_{\frac{n}{2}}(2\pi N r) + c\frac{s}{r}\int_0^N J_{\frac{n}{2}}(2\pi R r)\, J_{\frac{n}{2}}(2\pi R s)\, R\, dR.$$

Noting the symmetry of K_n in r and s, we compute $K_N(s, r)\left(\frac{r}{s} - \frac{s}{r}\right)$ and we obtain

$$S_N f(s) = c\frac{N}{s^{(n-2)/2}} \int_0^\infty \frac{s}{s^2 - r^2} J_{(n-2)/2}(2\pi N r)\, J_{n/2}(2\pi N s)\, f(r)\, r^{n/2}\, dr -$$

$$- c\frac{N}{s^{(n-2)/2}} \int_0^\infty \frac{r}{s^2 - r^2} J_{(n-2)/2}(2\pi N s)\, J_{n/2}(2\pi N r)\, f(r)\, r^{n/2}\, dr =$$

$$= c\, T_N^0 f(s) + c\, T_N^6 f(s)\,.$$

Our estimates of T_N^0 and T_N^6 will be independent of N and will involve the operators mentioned in Lemma 1. We start with T_N^0 and break the domain of integration in three parts $\left(0, \frac{s}{2}\right)$; $\left(\frac{s}{2}, \frac{3}{2}s\right)$; $\left(\frac{3}{2}s, \infty\right)$. In correspondence we have $T_N^0 = T_N^1 + T_N^2 + T_N^3$. If $s < \frac{2A(n)}{N}$ by Lemma 2 we have

$$| T_N^1 f(s) | \leqslant c \frac{N s}{s^{(n-2)/2}} \int_0^{s/2} s^{-2} | f(r) | r^{(n-1)/2} \left(\frac{s}{2}\right)^{1/2} dr \leqslant$$

$$\leqslant c \frac{A(n)}{s^{(n-1)/2}} \frac{1}{s} \int_0^{2s} | f(r) | r^{(n-1)/2} dr = \frac{c}{s^{(n-1)/2}} M(f(r) r^{(n-1)/2})(s).$$

If instead $s > \dfrac{2A(n)}{N}$ then we further break the domain of integration in $\left(0, \dfrac{A(n)}{N}\right)$ and $\left(\dfrac{A(n)}{N}, \dfrac{s}{2}\right)$ and obtain $T_N^1 = T_N^{11} + T_N^{12}$. Again by Lemma 2 we have

$$| T_N^{11} f(s) | \leqslant \frac{N s}{s^{(n-2)/2}} \frac{c}{(N s)^{1/2}} \int_0^{A(n)/N} s^{-2} | f(r) | r^{(n-1)/2} (A(n)/N)^{1/2} dr \leqslant$$

$$\leqslant c \frac{A(n)^{1/2}}{s^{(n-1)/2}} \frac{1}{s} \int_0^{2s} | f(r) | r^{(n-1)/2} dr \leqslant \frac{c}{s^{(n-1)/2}} M(f(r) r^{(n-1)/2})(s),$$

and

$$| T_N^{12} f(s) | \leqslant \frac{N s}{s^{(n-2)/2}} \frac{c}{(N s)^{1/2}} \int_{A(n)/N}^{s/2} s^{-2} (N r)^{-1/2} | f(r) | r^{n/2} dr \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} \frac{1}{s} \int_0^{2s} | f(r) | r^{(n-1)/2} dr = \frac{c}{s^{(n-1)/2}} M(f(r) r^{(n-1)/2})(s).$$

Now let us consider T_N^3. If $2s > \dfrac{A(n)}{N}$ then

$$| T_N^3 f(s) | \leqslant \frac{N s}{s^{(n-2)/2}} \frac{c}{(N s)^{1/2}} \int_{\frac{3s}{2}}^{\infty} \frac{1}{s(s+r)} | f(r) | \frac{r^{n/2}}{(N r)^{1/2}} dr \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} \int_0^{\infty} \frac{1}{s+r} | f(r) | r^{(n-1)/2} dr = \frac{c}{s^{(n-1)/2}} H(| f(r) | r^{(n-1)/2})(s).$$

If instead $2s < \dfrac{A(n)}{N}$ then we break the domain of integration in $\left(\dfrac{3s}{2}, \dfrac{A(n)}{N}\right)$ and $\left(\dfrac{A(n)}{N}, \infty\right)$ and obtain $T_N^3 = T_N^{31} + T_N^{32}$. We have that

$$| T_N^{31} f(s) | \leqslant c \frac{N s}{s^{(n-2)/2}} \int_{\frac{3s}{2}}^{\frac{A(n)}{N}} \frac{1}{r s} | f(r) | r^{n/2} dr \leqslant$$

$$\leqslant c\frac{N}{s^{(n-2)/2}}\int_{\frac{3s}{2}}^{\frac{A(n)}{N}}|f(r)|\,r^{-1/2}r^{(n-1)/2}\,dr \leqslant$$

$$\leqslant c\frac{N}{s^{(n-1)/2}}\int_{s-\frac{A(n)}{N}}^{s+\frac{A(n)}{N}}|f(r)|\,s^{-1/2}r^{(n-1)/2}\,dr \leqslant \frac{c}{s^{(n-1)/2}}M(|f(r)|\,r^{(n-1)/2})(s)\ ,$$

and

$$|T_N^{32}f(s)| \leqslant c\frac{Ns}{s^{(n-2)/2}}\int_{\frac{A(n)}{N}}^{\infty}\frac{N}{A(n)}\frac{1}{s+r}\frac{1}{(Nr)^{1/2}}|f(r)|\,r^{n/2}\,dr \leqslant$$

$$\leqslant \frac{N^{1/2}}{s^{(n-2)/2}}\int_{\frac{A(n)}{N}}^{\infty}\frac{1}{s+r}|f(r)|\,r^{(n-1)/2}\,dr \leqslant$$

$$\leqslant c\left(\frac{N}{A(n)}\right)^{1/2}\frac{1}{s^{(n-2)/2}}H(|f(r)|\,r^{(n-1)/2})(s) \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}}H(|f(r)|\,r^{(n-1)/2})(s)\ .$$

Now we turn to the study of T_N^2. We set $s-r=t$ and we obtain

$$T_N^2f(s) =$$
$$=\frac{cN}{s^{(n-2)/2}}J_{n/2}(2\pi Ns)\int_{-s/2}^{s/2}\left(\frac{1}{t}+\frac{1}{2s-t}\right)J_{(n-2)/2}(2\pi N(s-t))f(s-t)(s-t)^{n/2}dt=$$
$$= T_N^4f(s) + T_N^5f(s)\ .$$

If $s<\dfrac{A(n)}{N}$ then for a suitable $\bar{s}$, $\dfrac{s}{10}<\bar{s}<10s$, we have

$$|T_N^4f(s)| \leqslant \frac{cN}{s^{(n-2)/2}}\left|\int_{-s/2}^{s/2}\frac{1}{t}(J_{(n-2)/2}(2\pi N(s-t)) - J_{(n-2)/2}(2\pi Ns) +\right.$$
$$\left.+ J_{(n-2)/2}(2\pi Ns))f(s-t)(s-t)^{n/2}\,dt\right| \leqslant$$
$$\leqslant \frac{cN}{s^{(n-2)/2}}\Big[\Big|\int_{-s/2}^{s/2}NJ'_{(n-2)/2}(2\pi N\bar{s})f(s-t)(s-t)^{n/2}\,dt +$$
$$+\Big|\int_{-s/2}^{s/2}\frac{1}{t}J_{(n-2)/2}(2\pi Ns)f(s-t)(s-t)^{n/2}\,dt\Big|\Big] \leqslant$$

$$\leqslant \frac{cN^2}{s^{(n-2)/2}} \int_{-s/2}^{s/2} |J'_{(n-2)/2}(2\pi N\bar{s})| \, |f(s-t)| \, |s-t|^{n/2} dt +$$

$$+ \frac{cN}{s^{(n-2)/2}} \left| \int_{-s/2}^{s/2} \frac{1}{t} f(s-t)(s-t)^{n/2} dt \right| = T_N^{41} f(s) + T_N^{42} f(s) .$$

Since $J'_k(x) = -J_{k+1}(x) + \frac{k}{x} J_k(x)$ it follows that $|J'_{\frac{n-2}{2}}(2\pi N\bar{s})| \leqslant$ $\leqslant \frac{c}{Ns}$ and therefore

$$|T_N^{41} f(s)| \leqslant \frac{cN}{s^{(n-2)/2}} \frac{1}{s} \int_{-s/2}^{s/2} |f(s-t)| \, |s-t|^{(n-1)/2} s^{1/2} dt \leqslant$$

$$\leqslant \frac{c\,A(n)}{s^{(n-1)/2}} M(f(r)\, r^{(n-1)/2})(s) .$$

Now we turn to

$$|T_N^{42} f(s)| \leqslant \frac{cN}{s^{(n-2)/2}} \left| \int_{-s/2}^{s/2} \frac{1}{t} f(s-t)(s-t)^{(n-1)/2} [(s-t)^{1/2} - s^{1/2} + s^{1/2}] \, dt \right| \leqslant$$

$$\leqslant \frac{cN}{s^{\frac{n-2}{2}}} [s^{1/2} \tilde{H}(f(r) r^{\frac{n-1}{2}})(s) + s^{-1/2} \int_{-s/2}^{s/2} |f(s-t)| \, |s-t|^{(n-1)/2} dt] \leqslant$$

$$\leqslant \frac{c\,A(n)}{s^{(n-1)/2}} (\tilde{H} + M)(f(r)\, r^{(n-1)/2})(s) .$$

Now suppose $s > \frac{A(n)}{N}$. To estimate $T_N^4 f(s)$ we write

$$J_{(n-2)/2}(r) =$$

$$= \left[J_{(n-2)/2}(r) - \frac{i \exp\left(i\left(r - \frac{\pi}{4} n - \frac{\pi}{4}\right)\right)}{\sqrt{2\pi r}} + \frac{i \exp\left(-i\left(r - \frac{\pi}{4} n - \frac{\pi}{4}\right)\right)}{\sqrt{2\pi r}} \right] +$$

$$+ \frac{i \exp\left(i\left(r - \frac{\pi}{4} n - \frac{\pi}{4}\right)\right)}{\sqrt{2\pi r}} - \frac{i \exp\left(-i\left(r - \frac{\pi}{4} n - \frac{\pi}{4}\right)\right)}{\sqrt{2\pi r}} =$$

$$= K_1(r) + K_2(r) + K_3(r)$$

and in correspondence we have $T_N^4 f(s) = T_N^{43} f(s) + T_N^{44} f(s) +$ $+ T_N^{45} f(s)$. We start by

$$| T_N^{44} f(s) | \leqslant$$

$$\leqslant \frac{cN}{s^{(n-2)/2}} | J_{n/2}(2\pi N s) \int_{-s/2}^{s/2} \frac{1}{t} K_2(2\pi N(s-t)) f(s-t)(s-t)^{n/2}\, dt | \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} | \int_{-s/2}^{s/2} \frac{\exp(-2\pi i N t)}{t} f(s-t)(s-t)^{n-1/2}\, dt | \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} \tilde{C}(f(r)\, r^{(n-1)/2})(s).$$

In a similar way $T_N^{45} f(s)$ can be estimated. Now we consider

$$T_N^{43} f(s) = \frac{cN}{s^{(n-2)/2}} J_{n/2}(2\pi N s) [\int_{-s/2}^{s/2} \frac{1}{t}(K_1(2\pi N(s-1)) -$$

$$- K_1(2\pi N s)) f(s-t)(s-t)^{n/2} dt] + \int_{-s/2}^{s/2} \frac{1}{t} K_1(2\pi N s) f(s-t)(s-t)^{n/2} dt] .$$

For a suitable $\bar{s}, s/10 < \bar{s} < 10\,s$, and using (P_2) we have

$$| T_N^{43} f(s) | \leqslant \frac{cN^2}{s^{(n-2)/2}} \frac{1}{(Ns)^{1/2}} | \int_{-s/2}^{s/2} K_1'(2\pi N \bar{s}) f(s-t)(s-t)^{n/2}\, dt | +$$

$$+ \frac{cN}{s^{(n-2)/2}} \frac{1}{Ns} | \int_{-s/2}^{s/2} \frac{1}{t} f(s-t)(s-t)^{(n-1)/2} [(s-t)^{1/2} - s^{1/2} + s^{1/2}]\, dt | \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} (M + \tilde{H})(f(r)\, r^{(n-1)/2})(s) .$$

Now we study $T_N^5 f(s)$. First suppose $s > \dfrac{A(n)}{N}$ then

$$| T_N^5 f(s) | \leqslant \frac{cNs}{s^{(n-2)/2}} \frac{1}{(Ns)^{1/2}} \int_{-s/2}^{s/2} \frac{1}{s^2} \frac{1}{(Ns)^{1/2}} | f(s-t) | \, | s-t |^{n/2}\, dt \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} M(f(r)\, r^{(n-1)/2})(s) .$$

Secondly suppose $s < \dfrac{A(n)}{N}$. Then

$$| T_N^5 f(s) | \leqslant \frac{cNs}{s^{(n-2)/2}} \int_{-s/2}^{s/2} \frac{1}{s^2} | f(s-t) | \, | s-t |^{(n-1)/2} s^{1/2}\, dt \leqslant$$

$$\leqslant c\, A(n)\, M(f(r)\, r^{(n-1)/2})(s) .$$

So far we proved that

$$|T_N^0 f(s)| \leqslant \frac{c_n}{s^{(n-1)/2}}[M + H + \tilde{H} + \tilde{C}](f(r)\, r^{(n-1)/2})(s)\,.$$

We are going to prove that the same estimate holds for

$$T_N^6 f(s) = \frac{c\,N}{s^{(n-2)/2}} J_{(n-2)/2}(2\pi N s) \int_0^\infty \frac{r}{s^2 - r^2} J_{n/2}(2\pi N r) f(r)\, r^{n/2}\, dr\,.$$

We break up T_N^6 in the same way as T_N^0 and we denote by U_N^i the operator corresponding to T_N^i. In the case of U_N^1 and U_N^5 it is enough to observe that $r < s$ and to use the estimates for T_N^1 and T_N^5. Now we consider $U_N^3 f(s)$ in the assumption $2s > \frac{A(n)}{N}$. We have

$$|U_N^3 f(s)| \leqslant \frac{c}{s^{(n-1)/2}} \int_{\frac{3s}{2}}^\infty \frac{1}{s + r} f(r)\, r^{(n-1)/2}\, dr \leqslant \frac{c}{s^{(n-1)/2}} H(f(r)\, r^{(n-1)/2})(s)\,.$$

If instead $2s < \frac{A(n)}{N}$ then we break the domain of integration in $\left(\frac{3s}{2}, \frac{A(n)}{N}\right)$ and $\left(\frac{A(n)}{N}, \infty\right)$ and we obtain

$$|U_N^3 f(s)| \leqslant \frac{c}{s^{(n-1)/2}}\left[\frac{N}{A(n)} s^{1/2} \int_{s-\frac{A(n)}{N}}^{s+\frac{A(n)}{N}} \frac{1}{r} |f(r)|\, r^{(n-1)/2} r^{1/2}\, dr + \right.$$

$$\left. + (Ns)^{1/2} \left| \int_{A(n)/N}^\infty \frac{1}{s + r} f(r)\, r^{(n-1)/2}\, dr \right| \right] \leqslant \frac{c}{s^{(n-1)/2}} (M + H)(f(r)\, r^{(n-1)/2})(s)\,.$$

Next we study $U_N^2 f(s) = U_N^4 f(s) + U_N^5 f(s)$. In case $s < A(n)/N$ proceeding as we did for T_N^4 we obtain

$$|U_N^4 f(s)| \leqslant \frac{c\,N}{s^{(n-2)/2}}\left[\left| \int_{-s/2}^{s/2} \frac{s - t}{s} J'_{n/2}(2\pi N \bar{s})\, N f(s - t)(s - t)^{n/2}\, dt \right| + \right.$$

$$\left. + \left| \int_{\frac{s}{2}}^{\frac{s}{2}} \frac{s - t}{s\,t} J_{n/2}(2\pi N s) f(s - t)(s - t)^{n/2}\, dt \right| \right] \leqslant$$

$$\leqslant \frac{c\,N}{s^{(n-2)/2}} \int_{-s/2}^{s/2} |f(s - t)|\, |s - t|^{n/2}\, |J'_{n/2}(2\pi N s)|\, dt +$$

$$+ \frac{c\,N}{s^{(n-1)/2}} \frac{1}{s^{1/2}} \left| \int_{-s/2}^{s/2} \frac{1}{t} f(s - t)(s - t)^{(n-1)/2} (s - t)^{3/2}\, dt \right| \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} (\tilde{H} + M)(f(r)\, r^{(n-1)/2})(s) .$$

Now suppose $s < A(n)/N$ and write $U_N^4 f(s) = U_N^{43} f(s) + + U_N^{44} f(S) + U_N^{45} f(s)$. We have

$$| U_N^{44} f(s) | \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} \left| \int_{-s/2}^{s/2} \frac{\exp(i 2\pi N t)}{t} (1 - t/s) f(s-t)(s-t)^{(n-1)/2} dt \right| \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} (\tilde{C} + M)(f(r)\, r^{(n-1)/2})(s) .$$

Similarly for $U_N^{45} f(s)$. Only $U_N^{43} f(s)$ is left. We have

$$| U_N^{43} f(s) | \leqslant \frac{cN}{s^{(n-2)/2}} \frac{1}{(Ns)^{1/2}} \left[\left| \int_{-s/2}^{s/2} \frac{1}{t} \frac{Nt}{(Ns)^{3/2}} \frac{s-t}{s} f(s-t)^{n/2} dt \right| + \right.$$

$$\left. + \frac{1}{(Ns)^{3/2}} \left| \int_{-s/2}^{s/2} \frac{1}{t} \frac{s-t}{s} f(s-t)(s-t)^{n/2} dt \right| \right] \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} \left[\frac{1}{s} \int_{-s/2}^{s/2} | f(s-t) |\, | s-t |^{(n-1)/2} dt + \right.$$

$$\left. + \frac{1}{s^{3/2}} \frac{1}{A(n)} \left| \int_{-s/2}^{s/2} \frac{1}{t} f(s-t)(s-t)^{(n-1)/2} (s-t)^{3/2} dt \right| \right] \leqslant$$

$$\leqslant \frac{c}{s^{(n-1)/2}} (M + \tilde{H})(f(r)\, r^{(n-1)/2}(s) .$$

So the following estimate has been proved

$$| T_N f(s) | \leqslant \frac{c_n}{s^{(n-1)/2}} (M + H + \tilde{H} + \tilde{C})(f(r)\, r^{(n-1)/2})(s) .$$

Therefore since for $-1 < \alpha < p - 1$, $s^\alpha \in A_p(\mathbb{R})$ then by Lemma 1 we have

$$\| \operatorname{Sup}_N | T_N f(x) | \|_{L^p(\mathbb{R}^n)}^p \leqslant$$

$$\leqslant c_n^p \int | (M + H + \tilde{H} + \tilde{C})(f(r)\, r^{(n-1)/2})(s) |^p s^{-\left(\frac{n-1}{2}\right)p} s^{n-1} ds \leqslant$$

$$\leqslant c_n^p \int | f(s)\, s^{(n-1)/2} |^p s^{n-1-\frac{n-1}{2}p} ds \leqslant c_n^p \| f \|_{L^p(\mathbb{R}^n)}^p$$

whenever $\dfrac{2n}{n+1} < p < \dfrac{2n}{n-1}$. This proves the theorem.

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