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# Limit Theorems for Sums of Partial Quotients of Continued Fractions

By

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*In memory of Wilfried Nöbauer*

**Abstract.** We prove that the partial quotients $a_j$ of the regular continued fraction expansion cannot satisfy a strong law of large numbers for any reasonably growing norming sequence, and that the $a_j$ belong to the domain of normal attraction to a stable law with characteristic exponent 1. We also show that the $a_j$ satisfy a central limit theorem if a few of the largest ones are trimmed.

**1. Introduction.** Let $\omega \in (0, 1)$ be irrational and let $\omega = [a_1(\omega), a_2(\omega), \ldots]$ be its regular continued fraction expansion. The $a_k(\omega) \in \mathbb{N}$ and are called the partial quotients. Denote by

$$S_N(\omega) = \sum_{k \leqslant N} a_k(\omega) \tag{1.1}$$

their $N$-th partial sum. Since $\int_0^1 a_1(\omega)\, d\omega = \sum_{j=1}^{\infty} j \frac{1}{j(j+1)} = \infty$ none of the classical limit theorems in probability, such as the weak and the strong law of large numbers and the central limit theorem hold for these sums. However, with different norming constants the weak law of large numbers holds, according to an old theorem of Khinchin [8].

**Theorem A.** [8] *Let $\lambda$ denote Lebesgue measure. Then for any $\varepsilon > 0$*

$$\lambda\left\{\omega \in (0, 1): \left|\frac{S_N}{N \log N} - \frac{1}{\log 2}\right| \geqslant \varepsilon\right\} \to 0$$

*i.e. $S_N/N \log N$ converges in measure to* $1/\log 2$.

For the strong law of large numbers the situation is entirely different. No matter, how a sequence $\{\sigma(n), n \geqslant 1\}$ of norming constants is chosen (as long as it is reasonably regular) we cannot have almost everywhere convergence to a finite nonzero limit.

**Theorem 1.** *Let* $\{\sigma(n), n \geqslant 1\}$ *be a sequence of positive numbers such that*

$$\sigma(n)/n \text{ is non-decreasing.} \tag{1.2}$$

*Then*

$$\lim_{N\to\infty} \frac{S_N}{\sigma(N)} = 0 \quad or \quad \limsup \frac{S_N}{\sigma(N)} = \infty \text{ a.e.}$$

*according as*

$$\sum \frac{1}{\sigma(N)} < \infty \quad or \ = \infty \ .$$

*Remark.* For the divergence part no regularity condition on $\sigma$ is needed.

However, in the absence of (1.2) the limes superior can be non-zero and finite a.e., as has been shown by DIAMOND and VAALER [3, Corollary 3]. Here is a quick proof. By Theorem A $S_N/(N \log N)$ converges a.e. to $1/\log 2$ along some subsequence $N_k$. Then with $\sigma(N) = N_k \log N_k$ for $N_{k-1} < N \leqslant N_k$ we have $S_N/\sigma(N) \leqslant S_{N_k}/(N_k \log N_k)$ and so the limes superior equals $1/\log 2$ a.e.

Since

$$\lambda\{\omega \in (0,1) : a_n(\omega) \geqslant x\} = \frac{1}{x}(1 + o(1)) \quad x \to \infty$$

the classical theory for independent random variables suggests that $\{a_n, n \geqslant 1\}$ belongs to the domain of attraction to a stable law with characteristic exponent 1.

This is indeed the case and a result to this effect has been stated by P. LÉVY [10]. Recent results of SAMUR [13] apply directly to the case of continued fractions provided one can show that $\varphi(1) < 1$. (For this notation see (2.1)* below.) In his lecture SAMUR [14] has announced a new proof of Lévy's theorem along these lines. It is easy to generalize these results to measures $\mu$ absolutely continuous with respect to Lebesgue measure and this is done in the following theorem.

**Theorem 2.** *Let* $\mu$ *be any probability measure absolutely continuous with respect to* $\lambda$. *Then the* $\mu$*-measure of the set of all* $\omega \in (0,1)$ *with*

$$\frac{S_N}{N/\log 2} - \log N \leqslant \alpha$$

*converges to a distribution function $F(\alpha)$ with characteristic function*

$$f(t) = \exp\left(-\frac{\pi}{2}|t| - it\log|t| + i\gamma t\right).$$

*Here, $\gamma$ is given by*

$$\gamma = \lim_{N\to\infty}\left(\sum_{k\leqslant N/\log 2} k\log\left(1+\frac{1}{k(k+2)}\right) - \log N\right).$$

On the other hand, as DIAMOND and VAALER [3] recently showed, if the largest $a_k$ is trimmed or omitted from $S_N$ then the strong law of large numbers with norming constants $N\log N$ holds for $S_N$.

**Theorem B.** [3] *For almost all $\omega$ there is a number $N_0 = N_0(\omega)$ such that*

$$S_N = \frac{1+o(1)}{\log 2} N\log N + \vartheta_+ \max_{1\leqslant k\leqslant N} a_k$$

*holds for all $N \geqslant N_0$.*

Discarding or trimming the outliers of a sample is common practice in statistics. In a recent paper HAHN and KUELBS [5] showed that after trimming a few of the largest members of a sample of independent identically distributed random variables the properly normalized partial sums converge to a Gaussian distribution. In Section 4 of this note we prove such a result for sums of the form (1.1).

To state this theorem we recall some notation from [5]. For $N \geqslant 1$, $1 \leqslant j \leqslant N$ let

$$F_N(j) = \operatorname{card}\{i\colon a_i > a_j \text{ for } 1\leqslant i\leqslant N \text{ or } a_i = a_j \text{ for } 1\leqslant i\leqslant j\}.$$

If $F_N(j) = k$, set $a_N^{(k)} = a_j$, i.e. $a_j$ is the $(k+1)$-th largest element of the sample $\{a_1,\ldots,a_N\}$ when $F_N(j) = k$. Let $\{r_n, n\geqslant 1\}$ and $\{\xi_n, n\geqslant 1\}$ be two sequences of integers with

$$r_n \to \infty,\ \xi_n \geqslant 7\ . \tag{1.3}$$

We set

$$S_N^* = S_N - \sum_{j<r_N\xi_N} a_N^{(j)} 1\{a_N^{(j)} > N/r_N\}, \tag{1.4}$$

i.e. $S_N^*$ denotes the $N$-th partial sum with the $r_N\xi_N$ largest terms trimmed provided that they exceed $N/r_N$.

**Theorem 3.** *Let $\mu$ be any probability measure on* $(0, 1)$ *absolutely continuous with respect to Lebesgue measure. Then the $\mu$-measure of the set of all* $\omega \in (0, 1)$ *with*

$$S_N^* - \frac{1}{\log 2} N \log(N/r_N) \leqslant \alpha N (r_N \log 2)^{-1/2}$$

*converges to*

$$G(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-\tfrac{1}{2} t^2)\, dt\,.$$

If the random variables $a_j$ were independent this result would follow immediately from [5, Theorem 1] except that there $\xi_N$ needs to go to infinity. Here we only require $\xi_N \geqslant 7$. However, according to a theorem of CHATTERJI [2] if the random variables $a_j$ are independent with respect to some probability measure on $(0, 1)$ then this measure is necessarily singular with respect to Lebesgue measure.

With respect to the Gauss measure $\{a_j, j \geqslant 1\}$ is a stationary $\psi$-mixing sequence of random variables (see Section 2 below.) In a second paper HAHN and KUELBS [6] prove a similar general result for such sequences. However, their Theorem 1 does not apply to the present situation since in view of Theorem A above their condition (1.6) is not satisfied. What is needed here is an extension of their Theorem 1 to triangular arrays with stationary rows of mixing random variables. But all things considered it is much quicker to prove Theorem 3 directly using the idea of their proof and a theorem of SAMUR [13].

**2. Preliminaries.** Let $T: (0, 1) \to (0, 1)$ be defined by $T\omega = \{1/\omega\}$ where $\{\cdot\}$ denotes the fractional part. We restrict ourselves to irrational $\omega$ from now on. This is no loss of generality since the rational numbers have Lebesgue measure 0. Then $a_{n+1}(\omega) = a_1(T^n \omega)$, $n \geqslant 1$. The intervals $\{\omega \in (0, 1) : a_1(\omega) = r_1, \ldots, a_k(\omega) = r_k\}$ for $r_1, \ldots, r_k \geqslant 1$ fixed are called the fundamental intervals of rank $k$. The $\sigma$-algebra generated by the fundamental intervals of rank $\leqslant k$ is denoted by $\mathcal{M}_1^k$. By $\mathcal{M}_n^\infty$ we denote the $\sigma$-algebra generated by the sets $\{\omega \in (0, 1) : a_{n+m}(\omega) = r_m\}$, $m \geqslant 1$, $r_m \geqslant 1$ integer. It consists of all sets of the form $T^{-n} E$ where $E$ is a Borel set.

For the proof of the theorems it is more convenient to work with the Gauss measure $P$, defined on the Lebesgue measurable sets $A \subset (0,1)$ by

$$P(A) = \frac{1}{\log 2} \int_A \frac{d\omega}{1+\omega}.$$

For any such set $A$ we have $P(T^{-1}A) = P(A)$.

**Lemma 2.1.** *For all sets* $A \in \mathscr{M}_1^k$ *and* $B \in \mathscr{M}_{n+k}^{\infty}$

$$|P(A \cap B) - P(A)P(B)| \leqslant P(A)P(B)\varrho^n \tag{2.1}$$

*with* $\varrho < 0.8$. *Thus* $\{a_j, j \geqslant 1\}$ *is a stationary* $\psi$*-mixing sequence with mixing coefficient* $\psi(n) = \varrho^n$. *In particular, this implies* $\psi(1) < 0.8 < 1$.

With the right-hand side of (2.1) replaced by $C P(A) P(B) \varrho^n$ with $C > 0$ and $0 < \varrho < 1$ Lemma 2.1 is well-known [1, p. 50]. In [16] Szüsz claims that by his refinement [15] of the Gauss-Kuzmin theorem he can obtain $\varrho < \frac{1}{2}$. I was unable to reconstruct the indicated argument because the function $g_0$ defined in (2.3) below does not satisfy either condition (1.7) or (1.8) of Satz 1 in [15]. However, using the remark on p. 452 of Szüsz's paper [15] one can at least get the bound for $\varrho$ as claimed in Lemma 2.1.

A sequence of random variables is called $\varphi$-mixing if instead of (2.1)

$$|P(A \cap B) - P(A)P(B)| \leqslant \varphi(n) P(A) \tag{2.1*}$$

holds for some $\varphi(n) \downarrow 0$. This is, of course, a weaker condition than (2.1). In his lecture Samur [14] announced a proof of $\varphi(1) < 1$. As is easy to see Lemma 2.1 yields $\varphi(1) < 0.4$.

For the proof of Lemma 2.1 we use the following lemma, implicitly contained in [15].

**Lemma 2.2.** *Let* $\{f_n(\alpha), n \geqslant 1\}$ *be a sequence of twice continuously differentiable functions on* $[0,1]$ *with* $f_0(0) = 0$, $f_0(1) = 1$ *and defined recursively by*

$$f_{n+1}(\alpha) = \sum_{k=1}^{\infty} \left( f_n\left(\frac{1}{k}\right) - f_n\left(\frac{1}{k+\alpha}\right)\right), \quad n = 0, 1, 2, \ldots . \tag{2.2}$$

*Set*

$$g_n(\alpha) = (1+\alpha) f_n'(\alpha) . \tag{2.3}$$

*Then*

$$\|g'_n\| \leqslant \varrho^n \|g'_0\| \quad n = 1, 2, \ldots \tag{2.4}$$

*with* $\varrho < 0.8$. *Here* $\|\cdot\|$ *stands for the supremum norm.*

Indeed, from [15, (1.14)] and by a brute force estimate of the two series one obtains $\|g'_{n+1}\| \leqslant \varrho \|g'_n\|$ and this yields (2.4).

*Proof of Lemma 2.1.* Let $E_k$ be a fundamental interval of rank $k$ with endpoints $p_k/q_k$ and $(p_k + p_{k-1})/(q_k + q_{k-1})$, say. Let

$$f_n(\alpha) = P(T^{n+k}\omega < \alpha \mid E_k) := \frac{1}{P(E_k)} P((T^{n+k}\omega < \alpha) \cap E_k)$$

be the conditional distribution function of $T^{n+k}\omega$ given $E_k$. Now $(T^k\omega < \alpha) \cap E_k$ is an interval with endpoints $p_k/q_k$ and $(p_k + \alpha p_{k-1})/(q_k + \alpha q_{k-1})$. Thus by the definition of $P$

$$f_0(\alpha) = \frac{1}{P(E_k)} \cdot \frac{(-1)^k}{\log 2}\left(\log\left(1 + \frac{p_k + \alpha p_{k-1}}{q_k + \alpha q_{k-1}}\right) - \log\left(1 + \frac{p_k}{q_k}\right)\right).$$

If $g_n$ is defined as in (2.3) then by a straightforward but lengthy calculation $g''_0(\alpha) < 0$ for $0 \leqslant \alpha \leqslant 1$. Hence $\|g'_0\| \leqslant \max(|g'_0(0)|, |g'_0(1)|) < 1$ as is easily checked. (But in general $g'_0$ is not of constant sign as is required for the application [15, Satz 1].) Thus by (2.3) and (2.4)

$$\left|f'_n(\alpha) - \frac{1}{(1+\alpha)\log 2}\right| \leqslant \frac{|g_n(\alpha) - g_n(0)|}{1+\alpha} + \frac{|g_n(0) - 1/\log 2|}{1+\alpha} \tag{2.5}$$

and

$$|g_n(\alpha) - g_n(0)| = \left|\int_0^\alpha g'_n(t)\,dt\right| \leqslant \|g'_n\|\,\alpha < \varrho^n \alpha\,.$$

Also, for some $0 < \theta < 1$

$$1 = f_n(1) = \int_0^1 f'_n(t)\,dt = \int_0^1 \frac{g_n(t)}{1+t}\,dt =$$

$$= g_n(0)\log 2 + \theta \int_0^1 \frac{|g_n(t) - g_n(0)|}{1+t}\,dt =$$

$$= g_n(0)\log 2 + \theta\varrho^n(1 - \log 2)$$

and so

$$g_n(0) = 1/\log 2 + \theta(1/\log 2 - 1)\varrho^n .$$

Hence from (2.5)

$$\left| f_n'(\alpha) - \frac{1}{(1+\alpha)\log 2} \right| \leqslant \frac{1}{(1+\alpha)\log 2} \varrho^n .$$

We integrate this relation over $F$ with respect to $d\alpha$ and obtain

$$| P(T^{-(n+k)} F \mid E_k) - P(F) | \leqslant \varrho^n P(F) .$$

Since each $A \in \mathcal{M}_1^k$ is a countable union of disjoint fundamental intervals of rank $k$ we obtain the result.

**3. Proof of Theorem 1.** Assume first that $\sum 1/\sigma(n) = \infty$. Let $M > 0$. According to a theorem of BERNSTEIN [9, p. 60] for almost all $\omega$ the system of inequalities $a_N(\omega) \geqslant M\sigma(N)$ has infinitely many solutions. Hence

$$\limsup_{N\to\infty} S_N(\omega)/\sigma(N) \geqslant \limsup_{N\to\infty} a_N(\omega)/\sigma(N) \geqslant M \quad \text{a.e.}$$

and so

$$\limsup_{N\to\infty} S_N(\omega)/\sigma(N) = \infty \quad \text{a.e.}$$

If on the other hand $\sum 1/\sigma(n) < \infty$ then for almost all $\omega$ the system of inequalities $a_N(\omega) > \sigma(N)$ has only finitely many solutions. Put

$$a_N^*(\omega) = a_N(\omega)\, 1\{a_N(\omega) \leqslant \sigma(N)\} . \tag{3.1}$$

Then

$$\int_0^1 a_N^{*2}(\omega)\, P(d\omega) = \int_0^1 a_1^2(\omega)\, 1\{a_1(\omega) \leqslant \sigma(N)\}\, P(d\omega) \leqslant$$

$$\leqslant 2\int_0^1 a_1^2(\omega)\, 1\{a_1(\omega) \leqslant \sigma(N)\}\, d\omega \ll \sum_{j \leqslant \sigma(N)} j^2 \frac{1}{j(j+1)} \ll \sigma(N)$$

and so

$$\sum_{N\geqslant 1} \sigma(N)^{-2} \int_0^1 a_N^{*2}(\omega)\, P(d\omega) \ll \sum 1/\sigma(N) < \infty .$$

Hence by [7, Theorem 1.1.15]

$$\frac{1}{\sigma(N)} \sum_{k \leqslant N} \left(a_k^* - \int_0^1 a_k^*(\omega)\, P(d\omega)\right) \to 0 \quad \text{a.e.} \tag{3.2}$$

Similarly,

$$\int_0^1 a_k^*(\omega)\, P(d\omega) \ll \log \sigma(k)\ . \tag{3.3}$$

By (1.2) $\sigma(n)$ is nondecreasing and thus

$$\sum_{k \geqslant 1} 2^k / \sigma(2^k) < \infty\ .$$

By (1.2) again and a well-known theorem $k\, 2^k/\sigma(2^k) \to 0$. Hence

$$N \log N / \sigma(N) \to 0, \quad N \to \infty$$

and so by (3.3)

$$\begin{aligned} \frac{1}{\sigma(N)} \sum_{k \leqslant N} \int_0^1 a_k^*(\omega)\, P(d\omega) &\ll \frac{1}{\sigma(N)} \sum_{k \leqslant N} \log \sigma(k) \ll \\ &\ll N \log \sigma(N)/\sigma(N) \to 0\ . \end{aligned} \tag{3.4}$$

By (3.1) and the remark preceding it for almost all $\omega$ we have $a_N(\omega) \neq a_N^*(\omega)$ for only finitely many $N$. Hence by (3.2) and (3.4)

$$S_N/\sigma(N) \to 0 \quad \text{a.e.}$$

**4. Proof of Theorem 3.** For fixed $N \geqslant 1$, set

$$\begin{aligned} u_j &= u_{Nj} := a_j 1\,\{a_j \leqslant N/r_N\},\ 1 \leqslant j \leqslant N. \\ v_j &= v_{Nj} = a_j 1\,\{a_j > N/r_N\},\ 1 \leqslant j \leqslant N\ . \\ U_N &= \sum_{j \leqslant N} u_j,\ V_N = \sum_{j \leqslant N} v_j. \end{aligned} \tag{4.1}$$

Here and throughout the rest of the paper we drop the $\omega$, whenever this is convenient.

The idea for the proof of Theorem 3 if the measure $\mu$ is replaced by the Gauss measure $P$ is taken directly from [5]. It consists of proving the following two lemmas.

**Lemma 4.1.** *The Gauss measure P of the set of all ω, with*

$$U_N - \frac{1}{\log 2} N \log (N/r_N) \leqslant \alpha N (r_N \log 2)^{-1/2}$$

*converges to* $G(\alpha)$.

**Lemma 4.2.**

$$V_N - \sum_{j \leqslant r_N \xi_N} a_N^{(j)} 1\{a_N^{(j)} > N/r_N\} \to 0 \quad \textit{in P-measure.}$$

In view of (1.4) Theorem 3 follows immediately from these two lemmas provided that the underlying probability measure is $P$. From this fact one obtains the general case of Theorem 3 in the same way as in the proof of [12, Theorem 4]. Hence it remains to prove these two lemmas.

For the proof of Lemma 4.2 we note that the $P$-measure of the left-hand side in Lemma 4.2 being positive equals

$$\sum_{k = r_N \xi_N + 1}^{N} P\,(\text{exactly } k \text{ of the } a_j\text{'s, } 1 \leqslant j \leqslant N, \text{ exceed } N/r_N) \leqslant$$

$$\leqslant \sum_{k = r_N \xi_N}^{\infty} \binom{N}{k} \left(\frac{r_N}{N}\right)^k (1+\varrho)^k \leqslant \sum_{k \geqslant r_N \xi_N} \frac{r_N^k}{k!} 1.8^k \to 0$$

by Lemma 2.1, Stirling's formula and the tail estimate for the power series of $\exp(x)$. □

For the proof of Lemma 4.1 we need some preliminary calculations. We have by (4.1)

$$E u_j = \frac{1}{\log 2} \sum_{k \leqslant N/r_N} k \log\left(1 + \frac{1}{k(k+2)}\right) = \frac{1}{\log 2} \log (N/r_N) + O(1) \tag{4.2}$$

$$E u_j^2 = \frac{1}{\log 2} \sum_{k \leqslant N/r_N} k^2 \log\left(1 + \frac{1}{k(k+2)}\right) = \frac{1}{\log 2} N/r_N + O(1)\,. \tag{4.3}$$

Put

$$x_j = u_j - E u_j\,.$$

Then

$$E x_j = 0,\ \operatorname{Var} x_j = E u_j^2 - (E u_j)^2 = \frac{1 + o(1)}{\log 2} N/r_N \tag{4.4}$$

and

$$E|x_j| \leqslant 2 E u_j \ll (\log N)^2\,. \tag{4.5}$$

Now

$$\operatorname{Var} U_N = \sum_{j \leqslant N} E x_j^2 + 2 \sum_{1 \leqslant i < j \leqslant N} E x_i x_j . \tag{4.6}$$

By Lemma 2.1, [11, Lemma 3] and (4.4)

$$E x_i x_j \ll E|x_i| E|x_j| \varrho^{j-i} \quad \text{for} \quad j \geqslant i .$$

Hence by (4.3)—(4.5)

$$\begin{aligned} \operatorname{Var} U_N &= \frac{1+o(1)}{\log 2} N^2/r_N + O(\log^4 N \sum_{1 \leqslant i < j \leqslant N} \varrho^{j-i}) = \\ &= \frac{1+o(1)}{\log 2} N^2/r_N . \end{aligned} \tag{4.7}$$

We now apply [13, Corollary 4.6] to

$$x_{Nn} = x_n (\operatorname{Var} U_N)^{-1/2}, \quad 1 \leqslant n \leqslant N . \tag{4.8}$$

By Lemma 2.1 $\{x_{Nn}, 1 \leqslant n \leqslant N, N \geqslant 1\}$ is a $\psi$-mixing, stationary, triangular array with $\psi(n) = \varrho^n$ and $\psi(1) < 1$. By (4.4) $E x_{Nn} = 0$ and by (4.8), (4.1) and (4.7)

$$|x_{Nn}| \ll \frac{N}{r_n} \frac{r_N 1/2}{N} = r_N^{-1/2} \to 0 . \tag{4.9}$$

Condition (1) of [13, (4.6)] is satisfied because of (4.3) and (4.7), condition (2) holds because of (4.9). Condition (3) holds because of (4.3) and (4.8), since the series is $\ll r_N \log^2 N/N \to 0$ and since the main terms tend to 1. Condition (4) finally is vacuous since it needs to be applied only if the random variables are infinite-dimensional. Hence Lemma 4.1 follows.

This concludes the proof of Theorem 3.

*Remark.* Lemma 4.2 also can be obtained from [11, Theorem 3]. The calculations are longer, they do not require $\varphi(1) < 1$ and were carried out in an earlier version of this paper.

**5. Proof of Theorem 2.** Again as in the proof of [12, Theorem 4] one can obtain the general case from the case when the underlying probability measure is $P$. But this last case follows directly from Corollary 5.10 of SAMUR [13]. Conditions (1) and (2) of [13, (5.10)] hold with $l_1 = 0$ and $l_2 = 1$ since for $x \to \infty$

$$P(a_1(\omega) > x) = \frac{1}{\log 2}\log\left(1 + \frac{1}{x}\right) + o(1) = \frac{1}{\log 2}\frac{1}{x} + o(1) .$$

Moreover, this last relation implies

$$\lim_{N\to\infty} N P(a_1(\omega) > x N/\log 2) = 1/x .$$

Hence by the proof and the conclusion of [13, (5.10)] we obtain the result with $\log N$ replaced by

$$\log 2\, E\{a_1 1\{a_1 \leqslant N/\log 2\}\} =$$

$$= \log 2 \sum_{k \leqslant N/\log 2} k \frac{1}{\log 2}\log\left(1 + \frac{1}{k(k+2)}\right) .$$

This also proves the statement about $\gamma$. The expression for the characteristic function $f(t)$ itself is perhaps most easily obtained from [4, p. 167].

### Note Added in Proof

In a recent paper "Rates of convergence in stable limit theorems for sums of exponentially $\psi$-mixing random variables with applications to metric theory of continued fractions", Math. Nachr. **131**, 149—165 (1987) LOTHAR HEINRICH obtains bounds on the rates of convergence in Theorem 2 in case that $\mu$ is the Gauss measure $P$. I am grateful to Professor M. IOSIFESCU for drawing my attention to Heinrich's paper and for pointing out an annoying misprint in the statement of Theorem 2.

Also, SAMUR recently has circulated in preprint form ("On some limit theorems for continued fractions") a complete version of his results announced in [14].

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