PARTIAL DERIVATIVES OF TRAVEL-TIME CURVES OF REFLECTED WAVES IN A LAYERED MEDIUM

OLDŘICH NOVOTNÝ

Institute of Geophysics, The Charles University, Prague)*

Резюме: *Выведены простые формулы для расчета частных производных годографов отраженных волн по параметрам слоистой среды.*

1. INTRODUCTION

In interpreting travel-time curves it is very valuable if one knows how the travel-time curve varies under small changes of the parameters of the medium. These changes can be characterized by means of the partial derivatives of the travel-time curves with respect to the parameters of the medium. Besides this, the calculation of the partial derivatives is a necessary constituent of some methods of solving the inverse problem.

The partial derivatives of the travel-time curves can be determined by numerical differentiation. However, this method is time consuming and not very accurate, because, in the case of a layered medium, the travel-time curve cannot be expressed by a single formula, but only in so-called parametric form. It is much more convenient to calculate the derivatives of travel-time curves of a reflected wave by means of Eqs (8), given below.

Section 2 gives the well-known formulae for the travel-time curve of a reflected wave. The formulae for the partial derivatives of travel-time curves with respect to the parameters of the medium are given in Section 3 and the way in which they were derived is described in Section 4. Some of the properties of these partial derivatives are described in Section 5 and in Section 6 a numerical example is presented.

Equations (8) for calculating the partial derivatives of travel-time curves of reflected waves were already published in [3], and the use of these partial derivatives for interpretations was described in [5, 6]. However, it took a long time before these formulae were derived accurately. The proof in [3] was founded on the use of Taylor's developments (ref. to Proof 2 below), the proof in [6] was already founded on differentiating the parametric form of the travel-time curve. However, both proofs contained some steps which were not quite accurate. A new contribution of this paper is Proof 3 which is based on the parametric form (2). It is only this proof that can be considered as accurate from a mathematical point of view. Besides this, also some results from papers [3, 5, 6] are given, because these papers were not published in journals.

2. MODEL OF THE MEDIUM AND THE TRAVEL-TIME CURVE OF A REFLECTED WAVE

We shall consider a medium composed of n plane-parallel, homogeneous and isotropic layers, lying on a substratum (Fig. 1). Assume that the source of the seismic waves and the observer are located on the surface of the medium. Assume the

*) Address: Ke Karlovu 3, 121 16 Praha 2.

propagation of seismic waves which were reflected from the interface between the n-th layer and the substratum. Also assume that multiple reflections and transformation of waves do not occur (i.e. the wave propagates along the whole ray either as a P-wave or as an S-wave).

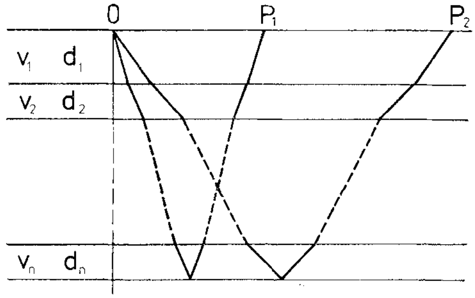

Fig. 1. Model of the medium and rays of the reflected wave.

The equation of the travel-time curve expresses the dependence of the travel time t on the epicentral distance r and on the parameters of the medium:

$$t = t(r, v_1, v_2, \ldots, v_n, d_1, d_2, \ldots, d_n)\,, \tag{1}$$

where v_m and d_m are the velocity of the seismic waves and the thickness of the m-th layer, respectively. Apart from the simplest case of one layer ($n = 1$), Eq. (1) cannot be expressed explicitly. Therefore, the equation of the travel-time curve is usually written in parametric form

$$t = \tau(v_1, \ldots, v_n, d_1, \ldots, d_n, p)\,, \quad r = \xi(v_1, \ldots, v_n, d_1, \ldots, d_n, p)\,, \tag{2a, b}$$

where

$$p = \sin i_1/v_1 = \sin i_m/v_m \tag{3}$$

is the parameter of the ray and i_m is the angle of incidence in the m-th layer. Therefore,

$$\sin i_m = v_m p = v_m(\sin i_1)/v_1\,. \tag{4}$$

We have introduced two symbols for time, t and τ, because we consider both quantities to be functions of different variables. Similarly, for the epicentral distance we use the symbols r and ξ; r is considered to be the independent variable, whereas ξ is a function of the parameters of the medium and of the parameter of the ray.

It follows from Fig. 1 that the r.h.s. of the equation of the travel-time curve (2) can be altered to read

$$\tau = 2 \sum_{m=1}^{n} \tau_m\,, \quad \xi = 2 \sum_{m=1}^{n} \xi_m\,, \tag{5a, b}$$

where

$$\tau_m = d_m(v_m \cos i_m)^{-1} = d_m v_m^{-1}(1 - v_m^2 p^2)^{-1/2}, \tag{6a}$$

$$\xi_m = d_m \tan i_m = d_m v_m p(1 - v_m^2 p^2)^{-1/2}. \tag{6b}$$

If we adopt various values of the parameter p (or of the angle i_1 or of the quantity $\sin i_1$), Eqs (5) and (6) can be used to calculate the travel-time curve of a reflected wave for a given model of the layered medium.

3. PARTIAL DERIVATIVES OF THE TRAVEL-TIME CURVE

In section 4 we shall derive the following formulae for the partial derivatives of the travel-time curve of a reflected wave with respect to the parameters of the medium:

$$\partial t/\partial v_j = \partial\tau/\partial v_j - p(\partial\xi/\partial v_j), \quad \partial t/\partial d_j = \partial\tau/\partial d_j - p(\partial\xi/\partial d_j), \quad j = 1, \ldots, n. \tag{7a, b}$$

By substituting Eqs (5) and (6) into the above we arrive at the final very simple formulae

$$\partial t/\partial v_j = -2\tau_j/v_j, \quad \partial t/\partial d_j = 2(1 - v_j^2 p^2)^{1/2}/v_j. \tag{8}$$

It should be noted that the partial derivatives (7) and (8) are partial derivatives of function (1), i.e. partial derivatives at a fixed epicentral distance r. These are indeed that partial derivatives we require in interpreting the experimental data.

We shall now describe the calculation of the travel time and of the partial derivatives for a given epicentral distance r. The computation is carried out in two steps:

a) Using an iteration method we determine a parameter p such that for it the epicentral distance ξ is equal to the given value r within the limits of the required accuracy [1]. (This procedure in fact represents the numerical solution of Eq. (2b) in terms of the unknown p).

b) Once the corresponding parameter p has been determined, the travel time can be computed by substituting into Eqs (5a) and (6a), and all partial derivatives are then obtained by substituting into Eqs (8).

Clearly, the largest amount of computer time (for a given epicentral distance) will usually be required to determine the parameter p, i.e. for the numerical solution of Eq. (2b). The computation of the travel time and of the partial derivatives is then very fast.

4. DERIVATION OF THE FORMULAE FOR THE PARTIAL DERIVATIVES

We shall now describe three different proofs of Eqs (7). The first two proofs are partly intuitive and cannot be considered as quite accurate. The third is more accurate. We shall give three proofs to provide a deeper insight into the parametric expression of the travel-time curve and into the meaning of Eqs (7).

Proof 1, (geometric). This proof has been elaborated on the basis of the procedure, described in [2]. We shall describe the proof with the aid of Fig. 2. For the sake of simplicity we have chosen a two-layered medium, however, the procedure described below has a general validity. In the upper part of the figure the full line represents the travel-time curve for some initial model, the dashed line the travel-time curve for the new model in which one of the parameters of the medium has been changed (in our case we have increased the thickness d_2 by the value Δd_2).

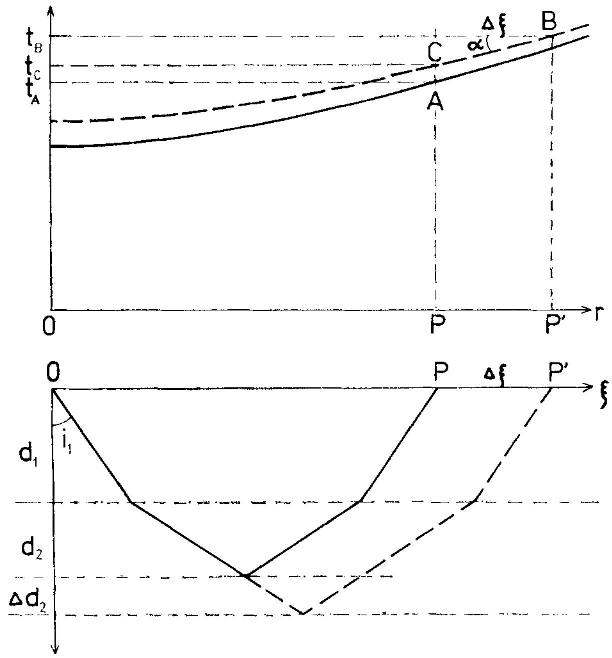

Fig. 2. The seismic ray and travel-time curve for the original model (full lines) and for the model with a thicker second layer (dashed line).

We are interested in the extent to which the time of propagation of the reflected wave will change at the point where the observer is located, P, in changing from the original model to the new. Denote this change by $\Delta t = t_C - t_A$. We now want to express the quantity Δt with the aid of the parametric expression of the travel-time curve, see Eq. (2).

Assume the parameter of the seismic ray p to be fixed and calculate to what extent the time τ and the epicentral distance ξ will change if the thickness d_2 changes by the value Δd_2. The new ray, represented in Fig. 2 by the dashed line, will no longer be incident at point P, but at point P'. The appropriate change of the epicentral distance will be denoted by $\Delta\xi$. The new time of propagation is denoted by point B on the dashed travel-time curve. Time τ has thus changed by the value $\Delta\tau = t_B - t_A$. It now holds that

(9) $$\Delta t = t_C - t_A = (t_B - t_A) - (t_B - t_C) = \Delta\tau - (t_B - t_C)\,.$$

As regards the last term in (9) it is approximately true that

(10) $$t_B - t_C = \Delta\xi \tan\alpha\,,$$

where α is the slope of the travel-time curve at point B. In virtue of Benndorf's equation, $p = \mathrm{d}t/\mathrm{d}r$, we arrive at $\tan \alpha = p$. (On the original travel-time curve point A is appropriate to the parameter p). It then follows that

(11) $$\Delta t = \Delta \tau - p\, \Delta \xi\,.$$

Divide both sides of this equation by the quantity Δd_2. If we decrease Δd_2 to zero, in the limit we shall find that

(12) $$\partial t/\partial d_2 = \partial \tau/\partial d_2 - p(\partial \xi/\partial d_2)\,.$$

And this is already one of the formulae (7). The others can be obtained quite analogously, because Eqs (9) to (11) remain valid even if other parameters of the medium change.

Proof 2 (with the aid of Taylor's development). In the foregoing proof we retained parameter p constant, but we had to "correct" the quantity $\Delta\tau$ for the effect due to the change of the epicentral distance. We shall now proceed in a different way. We shall accompany each change of the parameter of the medium by a change of parameter p such that the new ray is incident at the same epicentral distance. This proof has been adopted from [3].

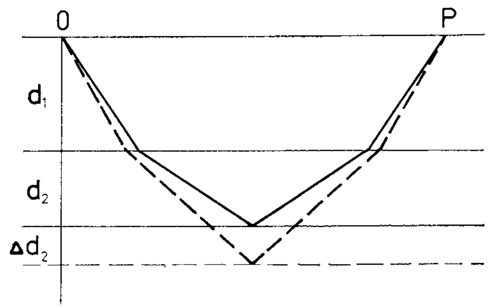

Fig. 3. Seismic ray for the original model (full line) and for the model with a thicker second layer (dashed line).

For the sake of illustration we shall again use a two-layered model (Fig. 3). The seismic ray for the original model is marked with a full line. Increase the thickness of the second layer by Δd_2. For the new ray (dashed line) to be incident at the same epicentral distance, it must leave source O under a smaller angle. Therefore, we must change the parameter of the seismic ray by some, hitherto unknown value Δp.

We shall use Taylor's development in Eqs (2). If we only change d_2 and p, we shall arrive at

(13a) $$\Delta t = (\partial \tau/\partial d_2)\, \Delta d_2 + (\partial \tau/\partial p)\, \Delta p\,,$$

(13b) $$\Delta r = (\partial \xi/\partial d_2)\, \Delta d_2 + (\partial \xi/\partial p)\, \Delta p\,.$$

In Eqs (13) we have neglected the terms containing the second and higher derivatives. We now require the partial derivatives of the travel time at a fixed epicentral distance. Therefore, we require that the epicentral distance does not change, i.e. $\Delta r = 0$. Equation (13b) then yields

(14) $$\Delta p = -\Delta d_2 (\partial \xi/\partial d_2)\, (\partial \xi/\partial p)^{-1}\,.$$

Substitute (14) into (13a) and execute the limiting procedure as in Proof 1. We then have

(15) $$\frac{\partial t}{\partial d_2} = \frac{\partial \tau}{\partial d_2} - \frac{\partial \xi}{\partial d_2}\, \frac{\partial \tau}{\partial p} \bigg/ \frac{\partial \xi}{\partial p}\,.$$

Let us modify the last term. By differentiating (6) with respect to p we obtain

$$\partial\tau_m/\partial p = p(\partial\xi_m/\partial p)\,. \tag{16}$$

It then follows that

$$(\partial\tau/\partial p)\,(\partial\xi/\partial p)^{-1} = p\,. \tag{17}$$

Equations (15) now take the form of Eqs (12). The other formulae in (7) can be derived analogously. This constitutes the proof.

A certain drawback of the given proofs is in that the formulae (10) and (13) used are only approximate, because the terms of higher orders in them have been neglected. These proofs do not indicate quite clearly whether Eqs (7) are accurate or whether they should be complemented, e.g., by the second derivatives of the functions τ and ξ, etc. In order to clarify this question, we would have to consider higher terms in the proofs as well. The following proof will show that Eqs (7) are accurate and that no terms in them are missing.

Proof 3 (analytical). We attempted a proof of this type in [5, 6], however, some vaguenesses remained, particularly as a result of unsuitable notations in the parametric expression of the travel-time curve.

Let us revert to the computation of the travel-time curve for given epicentral distances, as described in Section 3. In step a) we solved Eq. (2b) for the unknown p. The solution yields parameter p as a function of the remaining variables:

$$p = p(r, v_1, \ldots, v_n,\ d_1, \ldots, d_n)\,. \tag{18}$$

Substitute (18) into the parametric expression (2) and mark all the functional dependences:

$$t(r, v_1, \ldots, v_n,\ d_1, \ldots, d_n) = \tag{19a}$$

$$= \tau[v_1, \ldots, v_n,\ d_1, \ldots, d_n,\ p(r, v_1, \ldots, d_n)]\,,$$

$$r = \xi[v_1, \ldots, v_n,\ d_1, \ldots, d_n,\ p(r, v_1, \ldots, d_n)]\,. \tag{19b}$$

Here, r is considered to be the independent variable and we select the values of r. The parametric form (19) is the basis of this proof, the other relations being obtained formally by mathematical procedure.

By differentiating (19) with respect to v_j we arrive at

$$\partial t/\partial v_j = \partial\tau/\partial v_j + (\partial\tau/\partial p)\,(\partial p/\partial v_j)\,, \tag{20a}$$

$$0 = \partial\xi/\partial v_j + (\partial\xi/\partial p)\,(\partial p/\partial v_j)\,. \tag{20b}$$

It should be noted that two similar explanations why we had to put $\partial r/\partial v_j = 0$ in Eq. (20b), are possible: a) The epicentral distance r is an independent variable, therefore independent of v_j; b) In computing the partial derivatives at a given epicentral distance we consider r to be a constant, see procedure in Proof 2.

Equation (20b) yields

$$\partial p/\partial v_j = -(\partial\xi/\partial v_j)\,(\partial\xi/\partial p)^{-1}\,. \tag{21}$$

By substituting into (20a) we arrive at

$$\partial t/\partial v_j = \partial \tau/\partial v_j - (\partial \tau/\partial p)(\partial \xi/\partial p)^{-1}(\partial \xi/\partial v_j) . \tag{22}$$

Now, on the r.h.s. of this equation, only derivatives of the functions τ and ξ occur and we may, therefore, substitute Eqs (5) and (6). By using (17) we can simplify Eq. (22) to read

$$\partial t/\partial v_j = \partial \tau/\partial v_j - p(\partial \xi/\partial v_j) . \tag{23}$$

And this is Eq. (7a). Equation (7b) can be proved in a similar way.

We shall add a few remarks concerning the mathematical terminology. The described computation of the partial derivatives of the travel-time curve of a reflected wave resembles the computation of the derivatives of an implicit function. We developed the procedure on the basis of the analogy with the computation of the partial derivatives of dispersion curves [4, 7] in which the theorem of implicit functions was used. Equations (2b) and (19b) really have the character of implicit equations, because they can be rewritten in the following form:

$$\xi(v_1, \ldots, v_n, d_1, \ldots, d_n, p) - r = 0 . \tag{24}$$

Introduce the function $f = \xi - r$. Equation (24) can then be expressed as

$$f(r, v_1, \ldots, v_n, d_1, \ldots, d_n, p) = 0 . \tag{25}$$

Function p is given by Eq. (25) in implicit form. Some numerical method must be used to determine p as a function of the other variables. However, if the value of p is known for a given value of r and a given model of the medium, the derivatives of function p can be determined with the aid of the theorem of implicit functions, i.e., for example,

$$\partial p/\partial v_j = -(\partial f/\partial v_j)(\partial f/\partial p)^{-1} = -(\partial \xi/\partial v_j)(\partial \xi/\partial p)^{-1} , \tag{26}$$

see Eq. (21). Formulae (2a) and (19a) are not equations, but formulae for computing the travel time with the aid of the composite function $\tau[v_1, \ldots, d_n, p(r, v_1, \ldots, d_n)]$. If we consider r to be an independent variable, the computation of the travel-time curve then rests in solving the equation $f = 0$ (determination of the implicit function p) and in computing the composite function τ. Thus, determining the partial derivatives of the travel-time curve consisted of computing the derivatives of the implicit function and of the derivatives of the composite function.

5. SOME PROPERTIES OF THE PARTIAL DERIVATIVES

Equations (8) clearly indicate that the partial derivatives $\partial t/\partial v_j$ are always negative and the partial derivatives $\partial t/\partial d_j$ always positive. This agrees with the physical concept: if the velocity increases in any layer, the travel time of the reflected wave decreases, and if the thickness of any layer increases, the travel time increases.

Like the travel time t also the partial derivatives $\partial t/\partial v_j$ and $\partial t/\partial d_j$ are functions of the parameters of the medium and of the epicentral distance. The dependence of the partial derivatives on the parameters of the medium is responsible for the inverse problems for the travel-time curves of reflected waves being non-linear problems (the second and the higher partial derivatives will in general be non-zero as follows from

Eq. (8)). We shall now only consider the dependence of the partial derivatives on the epicentral distance. First determine the values of the partial derivatives at zero epicentral distance and at infinite epicentral distance.

If the epicentral distance r is equal to zero also the parameter of the seismic ray p is equal to zero. If we substitute $p = 0$ into Eq. (8) and (6a), we find that

$$\lim_{r\to 0} (\partial t/\partial v_j) = -2d_j v_j^{-2}, \quad \lim_{r\to 0} (\partial t/\partial d_j) = 2v_j^{-1}. \tag{27}$$

Assume the velocities of the seismic waves to attain their maximum values in the k-th layer, i.e.

$$v_k = \max_{i=1,\dots,n} v_i. \tag{28}$$

If $p \to 1/v_k$, $r \to \infty$, and Eqs (8) yield

$$\lim_{r\to\infty} (\partial t/\partial v_j) = -2d_j v_j^{-2}[1 - (v_j^2/v_k^2)]^{-1/2}, \tag{29}$$

$$\lim_{r\to\infty} (\partial t/\partial d_j) = 2[1 - (v_j^2/v_k^2)]^{1/2} v_j^{-1}.$$

In particular for the derivatives with respect to the parameters in the fastest layer this yields

$$\lim_{r\to\infty} (\partial t/\partial v_k) = -\infty, \quad \lim_{r\to\infty} (\partial t/\partial d_k) = 0. \tag{30}$$

The parameter of the seismic ray p is an increasing function of the epicentral distance r (p increases from zero to $1/v_k$ when r increases from zero to infinite). It follows that the partial derivatives (8) are monotonous and decreasing functions of the epicentral distance r. Moreover, $|\partial t/\partial v_j|$ are increasing functions of the variable r and $|\partial t/\partial d_j|$ are decreasing functions of the variable r. These properties of the partial derivatives can also be seen in Fig. 4 which is described below.

6. NUMERICAL EXAMPLE

Consider a two-layered model of the medium with the parameters

$$v_1 = 6{\cdot}0\ \text{km/s}, \quad d_1 = 20\ \text{km}, \quad v_2 = 7{\cdot}0\ \text{km/s}, \quad d_2 = 15\ \text{km}. \tag{31}$$

The travel times of the reflected wave and the appropriate partial derivatives for several epicentral distances are given in Tab. 1. The partial derivatives for an infinite epicentral distance were determined with the aid of Eqs (29). The partial derivatives of the travel-time curve are also depicted in Fig. 4.

The analysis in Section 5, Fig. 4 and other computations which we shall not go into here, indicate that the travel-time curve of the reflected wave carries information about the thicknesses of the layers particularly at small epicentral distances. At large

Table 1. Travel times and partial derivatives for model (31). The letter r represents the epicentral distance, t is travel time of the reflected wave. The partial derivatives $\partial t/\partial v_i$ are given in units of $km^{-1}s^2$, $\partial t/\partial d_i$ in km^{-1} s.

r(km)	t(s)	$\partial t/\partial v_1$	$\partial t/\partial v_2$	$\partial t/\partial d_1$	$\partial t/\partial d_2$
0	10·952	—1·1111	—0·6122	0·3333	0·2857
100	18·984	—1·6826	—1·2699	0·2201	0·1378
200	32·472	—2·0348	—2·8948	0·1820	0·0604
300	46·564	—2·1110	—4·8426	0·1755	0·0361
∞	∞	—2·1572	—∞	0·1717	0·0000

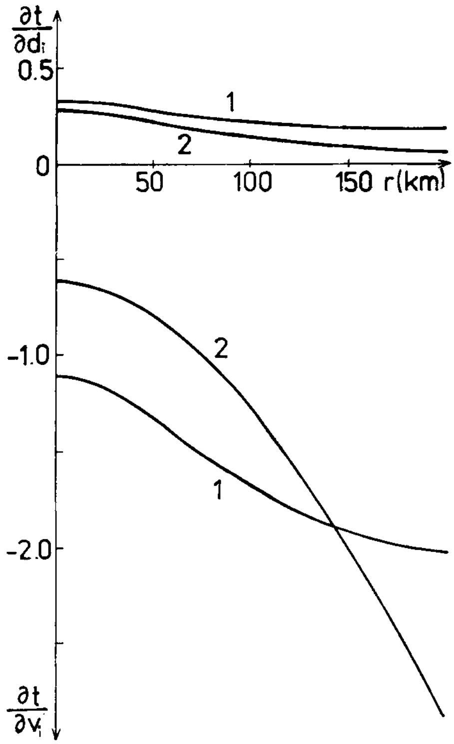

Fig. 4. Partial derivatives of the travel time of the reflected wave with respect to the thicknesses of the layers (positive values) and with respect to the velocities in the layers (negative values). The numbers marking the curves represent the ordinal number of the layer.

epicentral distances the travel time curve is mostly affected by the velocities in the fastest layers.

Compare the partial derivatives of the travel-time curves of reflected waves with respect to the parameters of the medium with the partial derivatives of the dispersion curves of seismic surface waves [4, 7]. The graphs of these partial derivatives have different shapes. Whereas the partial derivatives of the travel-time curves are monotonous functions of the epicentral distance, the partial derivatives of the dispersion curves as functions of the period have typical local extremes. Therefore, a certain parameter of the medium has a marked effect on the dispersion curve only in a certain range of periods. This is favourable for solving the inverse problem if we want to determine simultaneously a larger number of parameters of the medium. In a certain sense this implies that one dispersion curve carries more information about the structure of the medium than one travel-time curve of the reflected wave. In order to obtain more detailed information about the structure of the medium with the aid of reflected waves, we would have to use, e.g. the system of travel-time curves for waves reflected from interfaces at different depths.

7. CONCLUSION

Equations (8), which enable the partial derivatives of the travel-time curve of reflected waves in a layered medium to be computed quickly and accurately, have been derived. Only for a given epicentral distance need the parameter of the seismic ray be computed using some iteration method, but the travel time and its derivatives can be determined by substituting into simple formulae.

We have demonstrated that it is very easy to derive the formulae for the partial derivatives, if a suitable notation is used in the parametric expression of the travel-time curve. It is evident that the procedure, described in Proof 3, could also be used to compute the higher partial derivatives or to compute the partial derivatives of the travel-time curve for other models of the medium.

Received 4. 9. 1979 *Reviewer: I. Pšenčík*

References

[1] V. Červený: Universální podprogram ATIME pro kinematiku seismických vln v horizontálně vrstevnatém prostředí. Program ATIME. Res. Rep. No 2, Fac. of Math. and Physics, Charles Univ., Prague 1973 (not published).

[2] B. R. Julian, D. L. Anderson: Travel Times, Apparent Velocities and Amplitudes of Body Waves. Bull. Seism. Soc. Am., 58 (1968), 339.

[3] O. Novotný: Parciální derivace hodochron odražených vln pro prostředí složené z homogenních vrstev. Program W6. Res. Rep. No 16, Fac. of Math. and Physics, Charles Univ., Prague 1969 (not published).

[4] O. Novotný: Partial Derivatives of Dispersion Curves of Love Waves in a Layered Medium. Studia geoph. et geod., 14 (1970), 36.

[5] O. Novotný: Výpočet rychlostí a mocností vrstev horizontálně vrstevnatého prostředí z hodochron odražených vln. Program RTO1. Res. Rep. No 12, Fac. of Math. and Physics, Charles Univ., Prague 1974 (not published).

[6] O. Novotný: Výpočet parametrů horizontálně vrstevnatého prostředí z hodochron odražených vln. In: Teorie a počítače v geofyzice, Geofyzika n. p., Brno 1974.

[7] O. Novotný: Methods of Computing the Partial Derivatives of Dispersion Curves. Pageoph, 114 (1976), 765.