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Extremal Decomposition of Wightman Functions and of States on Nuclear *-Algebras by Choquet Theory

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Abstract. We give a short proof for the decomposability of states on nuclear *-algebras into extremal states by using the integral decompositions of Choquet and the nuclear spectral theorem, recovering a recent result by Borchers and Yngvason. The decomposition of Wightman fields into irreducible fields is a special case of this. We also indicate a quick solution of the moment problem on nuclear spaces.

Recently, Borchers and Yngvason [1] developed an extension theory for *-algebras of unbounded operators and applied it to the extremal decomposition of states[1] on nuclear *-algebras. The Choquet theory of extremal decompositions on cones [3] seemed to be not applicable; for the Borchers algebra $\mathscr{S}$ of test functions this has been discussed in [1] and [13]. In this paper we bypass the difficulty in a very simple way by going over to a larger cone to which Choquet theory can be applied[2] and then use nuclearity via the nuclear spectral theorem [5, 8].

Theorem [1]. *Let $\mathfrak{A}$ be a nuclear *-algebra with unit element, and let T be a state on $\mathfrak{A}$ such that $x \mapsto T(x^*x)$ is continuous*[3]. *Then there is a standard measure space*[4] *Z, a weakly measurable map $\zeta \to T_\zeta$ of Z to extremal states on $\mathfrak{A}$ and a positive measure ϱ on Z with $\varrho(Z)=1$ such that*

$$T = \int_Z T_\zeta d\varrho(\zeta)\,. \tag{1}$$

The main idea is to use the following observation.

Lemma. *Let $\mathfrak{A}_0$, a *-algebra with unit, be the finite linear span of a countable set of elements. Then $\mathfrak{A}^*_{0+}$, the positive cone in the algebraic dual $\mathfrak{A}^*_0$ equipped with the weak topology, is proper, metrizable and weakly complete.*

Proof. The topology of $\mathfrak{A}^*_0$ is given by a countable family of semi-norms and thus metrizable. Since $\mathfrak{A}^*_0$ contains *all* linear functionals on $\mathfrak{A}_0$, $\mathfrak{A}^*_0$ and $\mathfrak{A}^*_{0+}$ are weakly complete. $\mathfrak{A}^*_{0+}$ is proper since $\mathfrak{A}_{0+}$ contains the unit. QED.

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[1] States are positive continuous linear functionals which are 1 on the unit element.

[2] A similar idea has been used before by the author in [6, 7].

[3] This means that the associated representation is strongly continuous. In most cases this is automatically implied by the continuity of T, e.g. for the Borchers algebra $\mathscr{S}$ of test functions.

[4] One can assume $Z=[0, 1]$; cf. [4], B 20.

Proof of Theorem. (i) If q is a continuous semi-norm on $\mathfrak{A}$, let the Banach space $\hat{\mathfrak{A}}_q$ be the completion of $\mathfrak{A}/q^{-1}\{0\}$ in the norm obtained from q. The semi-norm $p(x) \equiv T(x^*x)^{1/2}$ is continuous, and $|T(x)| \leq p(x)$. Hence, by nuclearity, there is a continuous semi-norm $q \geq p$ such that $\mathfrak{A}$ equipped with q is separable and the canonical map from $\hat{\mathfrak{A}}_q$ onto $\hat{\mathfrak{A}}_p$ is nuclear [9]. Let $\mathfrak{A}_0$ be a *-subalgebra of $\mathfrak{A}$ spanned by a countable set of elements and containing the unit such that $\mathfrak{A}_{0+}$ is dense in $\mathfrak{A}_+$ with respect to q. Let T^0 be the restriction of T to $\mathfrak{A}_0$. Then, by the lemma, Choquet theory can be applied to T^0 and the cone $\mathfrak{A}^*_{0+}$ yielding[5]

$$T^0 = \int_{Z_0} T^0_\zeta d\varrho_0(\zeta) \tag{2}$$

where Z_0 is standard, $\varrho_0(Z_0) = 1$, and where T^0_ζ, $\zeta \in Z_0$, are extremal states on $\mathfrak{A}_0$.

(ii) Let π, π^0_ζ, $\mathfrak{H}$, $\mathfrak{H}_\zeta$, Ω, Ω_ζ be the representations, Hilbert spaces and cyclic vectors associated by the *GNS* construction with the states T and T^0_ζ, $\zeta \in Z_0$. Then Eq. (2) implies

$$\mathfrak{H} = \int^{\oplus}_{Z_0} \mathfrak{H}_\zeta d\varrho_0 \tag{3}$$

$$\pi^0 = \int^{\oplus}_{Z_0} \pi^0_\zeta d\varrho_0 \tag{4}$$

where $\pi^0 = \pi | \mathfrak{A}_0$. The map $x \mapsto \varphi(x) \equiv \pi(x)\Omega$ of $\mathfrak{A}$ into $\mathfrak{H}$ is strongly continuous[2] with respect to q. Let $\zeta \mapsto \varphi_\zeta(x)$ be a representative of $\varphi(x)$ in the direct integral decomposition of $\mathfrak{H}$. The nuclear spectral theorem then states that $x \mapsto \varphi_\zeta(x)$ can be chosen to be linear and strongly continuous with respect to q for all $\zeta \in Z_0$. We put $T_\zeta(x) \equiv \langle \Omega_\zeta, \varphi_\zeta(x) \rangle$. If $\{h_n; n = 1, \ldots\}$ spans $\mathfrak{A}_0$ one has $\pi^0_\zeta(h_n)\Omega_\zeta = \varphi_\zeta(h_n)$ for ζ not in some n-dependent null set. Taking the union of these we find that, for $x \in \mathfrak{A}_0$ and almost all ζ, $\zeta \in Z$ say,

$$T^0_\zeta(x) = \langle \Omega_\zeta, \pi^0_\zeta(x)\Omega_\zeta \rangle = \langle \Omega_\zeta, \varphi_\zeta(x) \rangle \equiv T_\zeta(x)\,. \tag{5}$$

Hence, for $\zeta \in Z$, T^0_ζ is the restriction to $\mathfrak{A}_0$ of the q-continuous linear functional T_ζ. Since $\mathfrak{A}_{0+}$ is dense in $\mathfrak{A}_+$ with respect to q, T_ζ is positive, $\zeta \in Z$. Since T^0_ζ is extremal so is, a fortiori, T_ζ. Denoting the restriction of ϱ_0 to Z by ϱ one obtains Eq. (1) from $T(x) = \langle \Omega, \varphi(x) \rangle$ and from Eq. (5). QED.

Application of Field Theory [1]. The n-point functions of a Wightman field[6] define a state on the Borchers algebra $\underline{\mathscr{S}}$ of test functions which is a nuclear *-algebra. By footnote 2, the theorem applies for any state on $\underline{\mathscr{S}}$ so that any Wightman state is a superposition of extremal states; almost all of the latter are then automatically again Wightman states [1], and they are characterized by the fact that the weak commutant of the associated field operators is trivial, i.e., the field is irreducible.

Remark. The nuclear spectral theorem can also be used for a quick alternative proof of the solution of the moment problem on real nuclear spaces due to Borchers and Yngvason [2][7]. A set of (continuous) moments on a nuclear space V defines

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[5] Theorem 30.22 of [3] applies. One can identify Z_0 with the nonzero extreme points of a cap containing T^0. Note that this is a G_δ-set, by Corollary 27.10 of [3]. Thus $T^0 = \int_{Z_0} S d\Theta(S)$ for some Radon measure Θ on Z_0. Putting $T^0_S = S/S(1)$ and $d\varrho_0 = S(1)d\Theta$ one obtains Eq. (2).

[6] With or without unique vacuum. If the vacuum vector is unique then the state is extremal, but the converse is not true as shown by an example in [1].

[7] Note added in proof: After completion of this paper I learned that a solution of the moment problem was given by Challifour and Slinker [14] simultaneous to [2]. Their proof is based on the Bochner-Minlos theorem.

a state T on the symmetric tensor algebra $S(V)$ over V. If $P(x_1, \ldots, x_n) \geqq 0$ is a positive polynomial on $\mathbb{R}^n$ we call $P(f_1, \ldots, f_n)$, $f_i \in V$, a positive polynomial in $S(V)$. Then the moments are the moments of a probability measure on V' if and only if T is positive on positive polynomials [2]. Sketch of proof: T defines a positive linear functional E on functions on V^* of the form $P(\omega(f_1), \ldots, \omega(f_n))$, P a polynomial, $\omega \in V^*, f_i \in V$. In a standard way[8] it can be extended to a positive linear functional on all polynomially bounded functions on V^*, in particular to $\exp\{i\omega(f)\}$. Hence [11, 12] there is a measure μ^* on V^* such that

$$E(f) \equiv E(\exp\{i\omega(f)\}) = \int_{V^*} e^{i\omega(f)} d\mu^*(\omega)$$

and the moments of μ^* are the given moments. The nuclear spectral theorem applied to $T(f \cdot g) = \int \omega(f)\omega(g) d\mu^*(\omega)$ then shows that $f \to \omega(f)$ is continuous for almost all ω, i.e., $\omega \in V'$. Necessity of the positivity condition is evident.

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[8] See [3], Theorems 34.2 and 35.4. Cf. also the treatment in [10].