**Zusammenfassung**

In der vorliegenden Arbeit werden Näherungswerte (mit strengen Fehlerschranken) hergeleitet für die Eigenfrequenzen der Schlingerbewegungen einer einen Halbraum mit kreis- oder streifenförmiger Öffnung ausfüllenden idealen Flüssigkeit. Das mathematische Modell kann als Grenzfall des klassischen Modells für endliche Behälter betrachtet werden. Wegen der bekannten Gebietsmonotonie der Eigenwerte sind die Werte für den Halbraum universelle obere Schranken für die entsprechenden Eigenwerte beliebiger beschränkter oder unbeschränkter Behälter mit gleichartigen Öffnungen.

Die Anlage der Arbeit ist wie folgt:

Die Formulierung des mathematischen Modells erfolgt in § 2, wobei insbesondere auf die Randbedingung im Unendlichen eingegangen wird. In § 3 wird die Aufgabe auf die Eigenwertaufgabe für einen gewissen selbstadjungierten, kompakten Operator in einem Hilbertraum zurückgeführt, wodurch der Weg zur Anwendung klassischer numerischer Verfahren der Eigenwertbestimmung (Ritz, Krylow-Bogoljubow, Temple) freigelegt wird. In § 4 werden geeignete Koordinatenfunktionen für das Ritzsche Verfahren eingeführt und die erforderlichen Skalarprodukte analytisch berechnet. Einige dabei auftretende Integrale mit Gegenbauerpolynomen sind im Anhang ausgewertet. Das Ritzsche Verfahren führt in der hier benützten Form auf die algebraische Aufgabe der Bestimmung der stationären Werte einer quadratischen Form unter einer linearen und einer quadratischen Nebenbedingung. Die numerische Lösung dieser Aufgabe wird in § 5 besprochen. Die sich daraus ergebenden, recht genauen Eigenwertschranken sind in § 6 tabelliert.

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# Stationary Values of the Ratio of Quadratic Forms Subject to Linear Constraints

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Dedicated to Professor L. Collatz on his sixtieth birthday

## 1. Introduction and Theoretical Background

Let $A$ be a real symmetric matrix of order $n$, $B$ a real symmetric positive definite matrix of order $n$, and $C$ an $n \times p$ matrix of rank $r$ with $r \leqslant p < n$. We wish to determine vectors $\boldsymbol{x}$ such that

$$\frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T B \boldsymbol{y}}$$

is stationary and $C^T \boldsymbol{x} = \theta$, the null vector.

By rearranging the columns of $C$, we may write

$$Q\,C = \begin{bmatrix} \tilde{R}_r & S \\ 0 & 0 \end{bmatrix}$$

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where $\tilde{R}_r$ is an upper triangular matrix of order $r$, $S$ is $r \times (p - r)$, and $Q^T Q = I$. The matrix $Q$ may be constructed as the product of $r$ Householder transformations (cf. [3]).

Let

$$\boldsymbol{x} = Q^T \boldsymbol{w} \equiv Q^T \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}$$

where $\boldsymbol{y}$ is a vector of the first $r$ components of $\boldsymbol{w}$ and $\boldsymbol{z}$ consists of the last $(n - r)$ components of $\boldsymbol{w}$. Thus

$$C^T \boldsymbol{x} = \left[\begin{array}{c|c} \tilde{R}_r^T & 0 \\ \hline S^T & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{y} \\ \hline \boldsymbol{z} \end{array}\right] = \theta ,$$

and hence

$$\boldsymbol{y} = \theta .$$

Therefore, the problem of determining the stationary values of

$$\frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T B \boldsymbol{x}} \quad \text{subject to} \quad C^T \boldsymbol{x} = \theta$$

is equivalent to determining the stationary values of

$$\frac{\boldsymbol{w}^T Q A Q^T \boldsymbol{w}}{\boldsymbol{w}^T Q B Q^T \boldsymbol{w}} \quad \text{subject to} \quad w_1 = w_2 = \cdots = w_r = 0 .$$

Let

$$G = Q A Q^T = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} , \quad H = Q B Q^T = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$$

where $G_{11}$, $H_{11}$ are $r \times r$ matrices, and $G_{22}$, $H_{22}$ are $(n - r) \times (n - r)$ matrices. The matrices $H$ and $G$ are symmetric; $H$ is positive definite, and $H_{22}$ is positive definite. Indeed,

$$0 < \lambda_{min}(H) \leqslant \lambda_{min}(H_{22}) \leqslant \lambda_{max}(H_{22}) \leqslant \lambda_{max}(H) .$$

Thus the stationary values we seek are the eigenvalues of the matrix equation

$$G_{22} \boldsymbol{z} = \lambda H_{22} \boldsymbol{z} \tag{1}$$

Since $G_{22}$ and $H_{22}$ are symmetric and $H_{22}$ is positive definite, we may solve (1) by using standard algorithms (cf. [7]). Finally, if

$$G_{22} \boldsymbol{z}_i = \lambda_i H_{22} \boldsymbol{z}_i \quad (i = 1, 2, \ldots , n - r) ,$$

then

$$\boldsymbol{x}_i = Q^T \begin{bmatrix} 0 \\ \cdots\cdots \\ I_{n-r} \end{bmatrix} \boldsymbol{z}_i . \tag{2}$$

When $p = 1$, and $B = I$, a slightly different algorithm may be used for computing the stationary values. We assume

$$\boldsymbol{c}^T \boldsymbol{c} = 1 . \tag{3}$$

Let

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} - \lambda \boldsymbol{x}^T \boldsymbol{x} + 2 \mu \boldsymbol{x}^T \boldsymbol{c} \tag{4}$$

where $(\lambda, \mu)$ are Lagrange multipliers. Differentiating (4), we are led to the equation

$$A \boldsymbol{x} - \lambda \boldsymbol{x} + \mu \boldsymbol{c} = \theta . \tag{5}$$

Multiplying (5) on the left by $\boldsymbol{c}^T$ and using (3), we have

$$\mu = - \boldsymbol{c}^T A \boldsymbol{x} . \tag{6}$$

Thus substituting (6) into (5), we have

$$P A \boldsymbol{x} = \lambda \boldsymbol{x}$$

where $P = I - \boldsymbol{c} \boldsymbol{c}^T$. Note $P^2 = P$ so that

$$\lambda(P A) = \lambda(P^2 A) = \lambda(P A P) .$$

The matrix $P A P$ is symmetric and consequently one of the standard methods may be used for computing its eigenvalues.

It is easy to construct the matrix $P A P$ using a device of Wilkinson [10]. Let

$$K = P A P = (I - \boldsymbol{c} \boldsymbol{c}^T) A (I - \boldsymbol{c} \boldsymbol{c}^T) = A - \boldsymbol{c} \boldsymbol{w}^T - \boldsymbol{w} \boldsymbol{c}^T + \alpha \boldsymbol{c} \boldsymbol{c}^T$$

where

$$\alpha = \boldsymbol{c}^T A \boldsymbol{c} \quad \text{and} \quad \boldsymbol{w} = A \boldsymbol{c} .$$

Then if

$$\boldsymbol{u} = \frac{\alpha}{2} \boldsymbol{c} - \boldsymbol{w} , \quad K = A + \boldsymbol{c} \boldsymbol{u}^T + \boldsymbol{u} \boldsymbol{c}^T .$$

Therefore if

$$K \boldsymbol{z}_i = \lambda_i \boldsymbol{z}_i ,$$

then

$$\boldsymbol{x}_i = P \boldsymbol{z}_i \quad (i = 1, 2, \ldots , n) .$$

At least one eigenvalue will be zero and the corresponding eigenvector will be proportional to $\boldsymbol{c}$.

## 2. Applicability

### 2.1 *Testing for Serial Correlations*

Let $X$ be a given $n \times p$ matrix of rank $r$ and $\boldsymbol{y}$ be a known vector. The vector $\boldsymbol{b}$ is the least squares estimate of regression vector so that

$$\| \boldsymbol{y} - X \boldsymbol{b} \|_2 = \min .$$

In many situations, it is desirable to consider the statistic

$$d = \frac{\mathbf{z}^T A \mathbf{z}}{\mathbf{z}^T \mathbf{z}}$$

where $\mathbf{z} = \mathbf{y} - X \mathbf{b}$, the residual vector, and $A$ is a given symmetric matrix. For

$$A = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & \text{O} & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ \text{O} & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

the statistic $d$ is the serial correlation of lag one. Note that $X^T \mathbf{z} = \theta$. We wish to consider the distribution of $d$ over all possible $\mathbf{z}$. Thus under a suitable transformation, we may write

$$d = \sum_{i=1}^{n-r} \lambda_i \xi_i^2 \Big/ \sum_{i=1}^{n-r} \xi_i^2$$

where $\{\lambda_i\}_{i=1}^{n-r}$ are the stationary values of $\mathbf{z}^T A \mathbf{z}$ over $\mathbf{z}^T \mathbf{z} = 1$ with $X^T \mathbf{z} = \theta$. The distribution of $d$ is discussed in special cases in [2].

### 2.2 *Exponential Fitting*

In many situations, we observe a sequence $\{z_k\}_{k=1}^m$, and we wish to determine parameters $\{\alpha_i\}_{i=0}^q$, $\{\lambda_i\}_{i=1}^q$ so that

$$z_k \approx \alpha_0 + \sum_{i=1}^{q} \alpha_i \lambda_i^k \quad (k = 1, 2, \ldots, m) . \tag{7}$$

From (7), we note that $\{z_k\}_{k=1}^m$ satisfies a difference equation of the form

$$a_0 z_k + a_1 z_{k-1} + \cdots + a_{q+1} z_{k-q-1} = \varepsilon_k \quad (k = q+1, \ldots, m)$$

where $\varepsilon_k$ is a random perturbation. The coefficients $\{a_i\}_{i=0}^{q+1}$ determine the *characteristic polynomial:*

$$p(\lambda) = a_0 \lambda^{q+1} + a_1 \lambda^q + \cdots + a_{q+1} .$$

Note $p(1) = 0$ by (7).

One procedure which may be used to estimate the coefficients of the characteristic polynomial is to determine $\{a_i\}_{i=0}^k$ so that

$$\sum_{k=q+1}^{m} \varepsilon_k^2 = \min.$$

subject to the constraints $\sum_{i=0}^{q+1} a_i^2 = 1$ and $\sum_{i=0}^{q+1} a_i = 0$. In matrix form, we have the problem of determining $\boldsymbol{a}$ so that

$$\boldsymbol{a}^T W^T W \boldsymbol{a} = \min.$$

with

$$\boldsymbol{a}^T \boldsymbol{a} = 1 \quad \text{and} \quad \boldsymbol{e}^T \boldsymbol{a} = 0$$

where

$$W = \begin{bmatrix} z_{q+1} & , \ldots, z_1, z_0 \\ z_{q+2} & , \ldots, z_2, z_1 \\ \vdots & \vdots \\ z_m, z_{m-1}, & \ldots, z_{m-q-1} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{q+1} \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Thus the procedure outlined in Section 1 may be used for determining $\boldsymbol{a}$. A more sophisticated statistical model for determining $\boldsymbol{a}$ is given in [9] by Osborne.

### 2.3 *Sloshing Frequencies*

In [6], Henrici et al. give a method for determining approximations (with rigorous error bounds) for the sloshing frequencies of an ideal fluid contained in a half-space with a circular or strip-like aperture. The stationary values may be obtained numerically by the method described in Section 1.

### 2.4 *Zeroes of a Rational Form*

From (5), we see that $\boldsymbol{x} = -\mu (A - \lambda I)^{-1} \boldsymbol{c}$ so that $\boldsymbol{c}^T \boldsymbol{x} = 0$ implies

$$-\boldsymbol{c}^T (A - \lambda I)^{-1} \boldsymbol{c} = 0 .$$

Thus if one wishes to find the zeros of

$$\psi(\lambda) = \sum_{i=1}^{n} \frac{c_i^2}{(\lambda - \lambda_i)},$$

one simply has to apply the algorithms of Section 1 to $A = \operatorname{diag}[\lambda_i]$, $\boldsymbol{c}^T = (c_1, c_2, \ldots, c_n)$.

## 3. Description of the Algorithm

The first step is to construct the matrix $Q$ as the product of $r$ Householder transformations. Let

$$C^{(1)} = C\,, \quad \overline{C}^{(k)} = C^{(k)} N_{k,k'}\,, \quad k \geq 1,\ k' \geq k\,, \quad C^{(k+1)} = P^{(k)} \overline{C}^{(k)}\,, \quad k \geq 1\,,$$

and

$$P^{(k)} = \left(I - \beta_k\, u^{(k)} u^{(k)^T}\right),$$

where

$$\beta_k = (S_k (S_k + |\overline{C}_{kk}^{(k)}|))^{-1}, \quad u_i^{(k)} = 0, \quad i < k,$$

$$u_k^{(k)} = \operatorname{sgn}(\overline{C}_{kk}^{(k)}) (S_k + |\overline{C}_{kk}^{(k)}|), \quad u_i^{(k)} = \overline{C}_{ik}^{(k)}, \quad i > k,$$

and

$$S_k^2 = \sum_{i=k}^{n} (\overline{C}_{ik}^{(k)})^2 .$$

The matrix $N_{k,k'}$ interchanges columns $k$ and $k'$ of $C^{(k)}$ so that

$$S_k^2 = \sum_{i=k}^{n} (\overline{C}_{ik}^{(k)})^2 = \sum_{i=k}^{n} (C_{ik'}^{(k)})^2 = \max_{j \geq k} \sum_{i=k}^{n} (C_{ij}^{(k)})^2 .$$

The effect of $P^{(k)}$ is to transform $\overline{C}^{(k)}$ so that

$$C_{ik}^{(k+1)} = 0, \quad i > k .$$

If, after $k$ such transformations, $k \leqslant p$, we have

$$|C_{ij}^{(k+1)}| < \varepsilon, \quad i > k, \ j > k,$$

where $\varepsilon$ is some given tolerance, then

$$r = \text{rank of } C = k,$$

and

$$Q = P^{(r)} P^{(r-1)} \ldots P^{(2)} P^{(1)} .$$

In practice, $Q$ will be retained and used in this form.

The problem of determining a good value for the tolerance $\varepsilon$ is rather difficult, (cf. [4]) and beyond the scope of this paper.

Once $Q$ has been determined, the next step is to form the matrix

$$G = Q A Q^T = P^{(r)} P^{(r-1)} \ldots P^{(1)} A P^{(1)} \ldots P^{(r)},$$

and

$$H = Q B Q^T = P^{(r)} P^{(r-1)} \ldots P^{(1)} B P^{(1)} \ldots P^{(r)},$$

since ${P^{(k)}}^T = P^{(k)}$. $G$ can be formed in $r$ steps as follows:

$$A^{(1)} = A, \quad A^{(k+1)} = P^{(k)} A^{(k)} P^{(k)}, \ k = 1, \ldots, r, \quad G = A^{(r+1)} .$$

The matrices $P^{(k)} A^{(k)} P^{(k)}$ can be determined using a device of Wilkinson as outlined previously. $H$ is formed in an entirely analogous manner using $B$.

The stationary values are the eigenvalues $\lambda_i$ of the generalized symmetric eigenproblem (1). If the point $\boldsymbol{x}_i$ at which the stationary value $\lambda_i$ occurs is desired, then it will be necessary to transform the eigenvector $\boldsymbol{z}_i$ according to (2).

## 4. Numerical Properties

The stability of the eigensystem of a matrix with respect to similarity transformations by elementary Hermitian matrices is discussed by Wilkinson in [11].

## 5. Test Results

This algorithm was programmed and tested on the IBM 360/67 at the Stanford Computation Center, Stanford, California.

Long floating point arithmetic was used (14 hexadecimal-digit fraction). Inner products were *not* accumulated in double precision.

To provide an example of the results produced by these procedures, the following matrices were used:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 8 & 5 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 8 & 5 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 8 & 5 \\ 1 & -1 & 2 & 1 \end{bmatrix}.$$

With $\varepsilon = 3_{10}-14$, it was correctly determined that the rank of $C$ was 2.

The following stationary values and vectors were then determined by finding the eigensystem of the resulting generalized eigenproblem (1):

Stationary values:

$1.70039264847579_{10}-01$
$1.23788202328080_{10}+00$
$4.91760119261002_{10}+00$
$9.27447751926161_{10}+00$

Vectors:

| | |
|---|---|
| $2.86085382484507_{10}-01$ | $-4.89644700766029_{10}-01$ |
| $2.82124288705312_{10}-01$ | $2.21020749102174_{10}-02$ |
| $1.55676307221979_{10}-02$ | $5.72549998363964_{10}-01$ |
| $-1.09686418150406_{10}-01$ | $4.49859712956573_{10}-01$ |
| $-3.01653013206705_{10}-01$ | $-8.29052975979350_{10}-02$ |
| $-1.72437870554907_{10}-01$ | $-4.71961787866790_{10}-01$ |
| | |
| $-4.95022659856411_{10}-01$ | $4.83069132908663_{10}-01$ |
| $3.95292112932390_{10}-01$ | $-9.81662635257467_{10}-01$ |
| $7.68429013103898_{10}-01$ | $5.30528981364161_{10}-01$ |
| $-8.92878392907869_{10}-01$ | $4.34008414446343_{10}-01$ |
| $-2.73406353247487_{10}-01$ | $-1.01359811427282_{10}+00$ |
| $4.97586279975478_{10}-01$ | $5.47654220811123_{10}-01$ |

In addition, for each vector $\boldsymbol{x}$ above, the vector $\boldsymbol{x}^T C$ was computed. In each case, the value of the maximum element in this vector was less in modulus than $1.1_{10}-15$.

The eigensystems of the generalized eigenproblems arising in our work were found using the procedures *reduc* 1 and *rebaka* [7], *tred* 2 [8], and *tql* 2 [1].

ALGOL 60 procedures describing this algorithm are included in [5].

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*Keywords*

Stationary values Eigenvalues Householder transformations Matrices

**Sommario**

Sia $A$ una matrice reale simmetrica di ordine $n$, $B$ una matrice reale simmetrica di ordine $n$ per cui $\boldsymbol{x}^T B \boldsymbol{x} > 0$ e $C$ una matrice $n \times p$ di rango $r$ con $r \leqslant p < n$. Si vogliono deteminare i vettori $\boldsymbol{x}$ per cui,

$$\boldsymbol{x}^T A \boldsymbol{x} / \boldsymbol{x}^T B \boldsymbol{x}$$

è stazionaria e $C^T \boldsymbol{x} = \boldsymbol{\theta}$ è il vettore nullo. È dato un algoritmo per generare un autosistema simmetrico i cui autovalori sono i valori stazionarì e per determinare i vettori $\boldsymbol{x}$. Sono altresì presentate parecchie applicazioni dell'algoritmo.

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# Upper and Lower Bounds for Eigenvalues of Torsion and Bending Problems by Finite Difference Methods[1])

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## Introduction

We wish to approximate the eigenvalues of the second and fourth order ordinary differential eigenvalue problems associated with the torsion and bending of free beams. We consider difference analogues of these problems and in Sections 1–4 employ variational methods to obtain explicit upper and lower bounds for the continuous eigenvalues in terms of the discrete eigenvalues. The technique is well known and has been employed by Weinberger [9] for fixed membranes, Hubbard [7] for free membranes, and the author [8] for clamped plates. Additional references may be found in [7] and [9].

The difference schemes have matrices which are three- or five-diagonal, and so their eigenvalues may be found very efficiently by the method of Sturm sequences [4]. In Section 5 we give the explicit equations for applications of the method to problems

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