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Bounds for cut elimination in intuitionistic propositional logic

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0 Introduction

The central theorem of Gentzen's theory of proofs states that every deduction d (in classical or intuitionistic, propositional or quantifier logic) can be transformed into a deduction $\mathbf{G}(d)$ which does not make use of the *cut rule*. Avoiding the use of a particular proof rule will, obviously, have the effect that $\mathbf{G}(d)$ becomes longer than d, and Gentzen's algorithm for cut elimination establishes an upper bound for the length $l(\mathbf{G}(d))$ of $\mathbf{G}(d)$. In this article, I shall construct a (different) cut free deduction $\mathbf{J}(d)$ for the case of intuitionistic propositional logic and derive considerably sharper upper bounds for $l(\mathbf{J}(d))$. Also, I shall use the methods developed for this purpose in order to set up an effective decision method.

Gentzen's upper bound for $l(\mathbf{G}(d))$ depends on both the length $l(d)$ and the *cut degree* $g(d)$ of d, viz. the maximum of the degrees, increased by 1, of cut formulas used in d; it has the form

$$l(\mathbf{G}(d)) \leqq 2^{2^{\cdot^{\cdot^{2^{g(d)}}}}} \quad (g(d)\,2\text{'s}) \quad \text{or} \quad l(\mathbf{G}(d)) \leqq 2^{l(d)}_{g(d)} \quad \text{with} \quad 2^n_0 = n,\ 2^n_{i+1} = 2^{(2^n_i)}.$$

The reason for the appearance of these iterated exponentiations lies in the nature of Gentzen's algorithm which proceeds by a double induction on both $n = l(d)$ and $g = g(d)$ with respect to the lexicographical order of pairs (n, g). It follows from results of [W-P] that in the case of quantifier logic the enormity of these bounds cannot be avoided: there is no constant c such that 2^n_c would be an upper bound independent of the cut degree of the particular d.

On the other hand, it is known that in the case of classical propositional logic a deduction d can be transformed into a cut free deduction $\mathbf{K}(d)$ such that already

$$l(\mathbf{K}(d)) \leqq 2^{2^{g(d) \cdot l(d)}} \cdot l(d)$$

will hold (cf. [Go]). Clearly, a proof of this fact will have to employ methods different from Gentzen's lexicographic induction, and what it actually does use is the technique of *inversion rules*. Inversion rules for sequent calculi are well known, and a technique to use them in order to eliminate cuts from a deduction d is described by the following two observations.

Local reduction. Given an instance of cut with a composite cut formula, the inversion rules referring to that type of formula may be applied to the premisses and will give deductions of sequents in which the cut formula has been replaced by formulas of lesser degree. In order to obtain the conclusion of the original cut, these replacing formulas will have to be removed by new cuts which, however, then have cut formulas of lesser degree.

Applying the local reduction sufficiently often will result in a cut free deduction. Doing this in an economical manner by the

Global process. Consider the instances of cut in d which have maximal position. If their cut formula is composite then apply the local reduction; if it is not then apply Gentzen's reduction,

will permit to computate bounds for the length of the resulting deduction. Run by itself, this algorithm of inversion rules does not yet provide such an upper bound; it will, therefore, be supplemented by the introduction of suitable new *cut degree functions* k and j which serve this purpose; in view of the limitation of exponentiations mentioned above, the decisive property of these functions is that their values $k(d), j(d)$ for a deduction d do only depend *linearly* on the length n of d.

In the preparatory Sect. 1, I shall consider the calculus LK for classical propositional logic, define an operator $\mathbf{K}$ which transforms deductions d into cut free deductions $\mathbf{K}(d)$, and prove, for a suitable cut degree function k, that

$$l(\mathbf{K}(d)) \leqq 2^{k(d)} \cdot l(d)$$

holds; this implies the result mentioned above since $k(d) \leqq 2^{g(d)} \cdot l(d)$.

In Sect. 2, I shall consider the calculus LJ for intuitionistic propositional logic, define an operator $\mathbf{J}$ which transforms deductions d into cut free deductions $\mathbf{J}(d)$, and prove, for a suitable cut degree function j, that

$$l(\mathbf{J}(d)) \leqq 2^{j(d)} \cdot l(d)$$

holds. Since $j(d) \leqq 2^{2^{2g(d)}} \cdot l(d)$, this implies

$$l(\mathbf{J}(d)) \leqq 2^{(2^{2^{2g(d)}} \cdot l(d))} \cdot l(d),$$

expressing the bound in terms of Gentzen's original parameters. Suffice it to say here that the intuitionistic case will be considerably more complex than the classical one since its inversion rules may be applied less liberally than those of the classical calculus.

In Sect. 3, I shall introduce a calculus LH and prove it to be equivalent to LJ. The decisive property of LH is this: for every sequent s, I can define a number $\deg(s)$, depending alone on the structure of formulas in s, which is an upper bound for the lengths of all (possibly existing) LH-deductions of s. Consequently, $\deg(s)$ is also an upper bound for the lengths of branches occurring in decision procedures with respect to derivability in LH. In addition, the rules of LH contain, essentially, the forward procedures which the inversion rules of LJ state as backward procedures. This has the effect that most of the rules of LH become invertible; thus the decision procedures for LH will require only a relatively small amount of backtracking.

(The proof of Theorem 3 as well as the preceding lemma is due to the anonymous referee whose advice I gratefully acknowledge.)

1 Classical propositional logic: an example

In this chapter, I shall illustrate the technique of inversion rules on the example of classical propositional logic. I consider formulas built from atoms (propositional variables) and the constant Δ (absurdity) with the connectives $\wedge$, $\vee$, $\rightarrow$; the *degree* of an atom is 0, and the degree of a composite formula is the maximum of the degrees of its immediate subformulas increased by 1. (Finite, possibly empty) *sequences* M, M' of propositional formulas will be taken as equal if they differ only by a permutation of their index set. *Sequents* are ordered pairs of such sequences, written in the form $M \Rightarrow N$. Deductions are trees, the nodes of which carry sequents such that (a) the maximal nodes carry sequents belonging to a specified set of *axioms*, and (b) the sequents on non-maximal nodes e are related to the sequents on the upper neighbours of e by instances of a specified set of *rules*. The calculus LK of classical propositional logic is defined by specifying as axioms all sequents of the form

$$M, v \Rightarrow v, N \quad \text{and} \quad M, \Delta \Rightarrow N,$$

where v is atomic, and specifying as rules

$$(I\wedge)\ \frac{M \Rightarrow u, u\wedge v, N \qquad M \Rightarrow v, u\wedge v, N}{M \Rightarrow u\wedge v, N} \qquad (E\wedge l)\ \frac{M, u\wedge v, u \Rightarrow N}{M, u\wedge v \Rightarrow N} \qquad (E\wedge r)\ \frac{M, u\wedge v, v \Rightarrow N}{M, u\wedge v \Rightarrow N}$$

$$(I\vee l)\ \frac{M \Rightarrow u, u\vee v, N}{M \Rightarrow u\vee v, N} \qquad (I\vee r)\ \frac{M \Rightarrow v, u\vee v, N}{M \Rightarrow u\vee v, N} \qquad (E\vee)\ \frac{M, u\vee v, u \Rightarrow N \qquad M, u\vee v, v \Rightarrow N}{M, u\vee v \Rightarrow N}$$

$$(I\rightarrow)\ \frac{M, u \Rightarrow v, u\rightarrow v, N}{M \Rightarrow u\rightarrow v, N} \qquad (E\rightarrow)\ \frac{M, u\rightarrow v \Rightarrow u, N \qquad M, u\rightarrow v, v \Rightarrow N}{M, u\rightarrow v \Rightarrow N}$$

$$(\mathrm{CUT})\ \frac{M \Rightarrow c, N \qquad M, c \Rightarrow N}{M \Rightarrow N}.$$

(CUT) is the *cut rule* and c is its *cut formula*; the rules different from cut will be called the *logical* rules of LK. In every instance of a logical rule there occurs a well defined composite formula in the conclusion, called the *principal* formula of this instance. A *branch* of a deduction is a (full) branch of its tree together with the sequents on its nodes; the length of a branch shall be the number of its nodes. The *length* $l(d)$ of a deduction is the maximum of the lengths of its branches.

It will be noticed that there is neither a weakening nor a contraction rule. They are not needed since their admissibility can be proven. Actually, applications of these two rules may be viewed as those of *operators* **W** and **M** acting on deductions (where I omit the parameters referring to the formulas weakening or being contracted):

W transforms a deduction of $M \Rightarrow N$ into a deduction of $M, A \Rightarrow N, B$

M transforms a deduction of $M, A, A \Rightarrow N, B, B$ into a deduction of $M, A \Rightarrow N, B$.

Moreover, both these operators preserve the structure of deductions and hence the property of being cut free as well as the length.

Given a sequent $M \Rightarrow N$ which occurs on a node of a deduction of *LK* which is neither maximal nor a conclusion of cut; then it will be conclusion of a well defined logical rule with a well defined principal formula. Singling out arbitrarily the one or the other of the composite formulas w in $M \Rightarrow N$, we may ask whether there is a (presumably different) deduction leading to $M \Rightarrow N$ with this particular formula w as the principal formula of the logical rule relating to its leading connective. For the calculus *LK*, the answer is always Yes, and actually we may, for each of the three connectives and for each of the possible premises leading to $M \Rightarrow N$ with w as the principal formula, define algorithms which construct a deduction of that premise out of the deduction of $M \Rightarrow N$. These algorithms I shall view as *inversion operators*, and the following table lists their names, the sequent the deduction of which they start from, and the sequent their results end with:

I∧L	$M \Rightarrow u \wedge v, N$	$M \Rightarrow u, N$
I∧R	$M \Rightarrow u \wedge v, N$	$M \Rightarrow v, N$
I∨	$M \Rightarrow u \vee v, N$	$M \Rightarrow u, v, N$
I→	$M \Rightarrow u \rightarrow v, N$	$M, u \Rightarrow v, N$
E∧	$M, u \wedge v \Rightarrow N$	$M, u, v \Rightarrow N$
E∨L	$M, u \vee v \Rightarrow N$	$M, u \Rightarrow N$
E∨R	$M, u \vee v \Rightarrow N$	$M, v \Rightarrow N$
E→L	$M, u \rightarrow v \Rightarrow N$	$M \Rightarrow u, N$
E→R	$M, u \rightarrow v \Rightarrow N$	$M, v \Rightarrow N$.

The construction of these algorithms is by a straightforward recursion on the lenghts of the given deductions. (cf. e.g. [C], p. 203ff.) They all preserve the property of being cut free and they all do not increase lengths.

Making use of the inversion operators, I shall now perform the transformation of a deduction d by the global process as explained in the introduction. For a formal description, I define an operator **RED** on deductions d by recursion on $l(d)$: if $l(d) = 1$, i.e. if d is 1-node tree carrying an axiom, then **RED**$(d) = d$; if $l(d) > 1$ and if d ends with an instance of a rule (R)

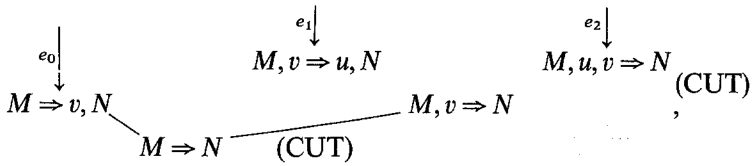
$$\dfrac{\overset{d_0\downarrow}{M_0 \Rightarrow N_0} \qquad \overset{d_1\downarrow}{M_1 \Rightarrow N_1}}{M \Rightarrow N}\ \text{(R)}$$

which is *not* a cut in maximal position, then **RED**(d) shall be the deduction obtained by prolonging **RED**(d_0), **RED**(d_1) with the same application of (R). If, however, (R) *is* a cut in maximal position

$$\dfrac{\overset{d_0\downarrow}{M \Rightarrow c, N} \qquad \overset{d_1\downarrow}{M, c \Rightarrow N}}{M \Rightarrow N}\ \text{(R)}$$

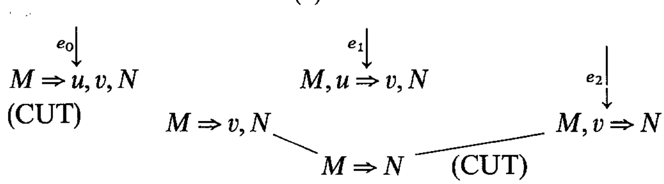
with the cut formula c then d_0 and d_1 both are cut free, and I distinguish four cases, depending on the form of c. (Case 1) that c is atomic, I handle in the traditional manner: put d_0, say, on top of the axioms of d_1 and perform the cut there, making use of the fact that the set of axioms is closed under cut. Observe that then $l(\mathbf{RED}(d)) \leqq 2 \cdot l(d)$.

Case 2. $c = u \wedge v$. In this case, I define $\mathbf{RED}(d)$ by



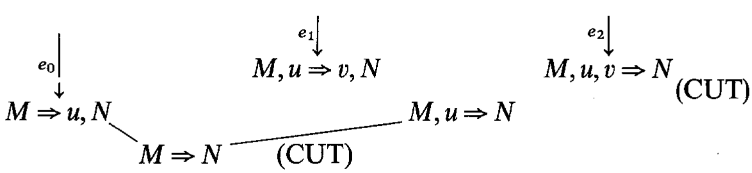
where e_0 is $\mathbf{I}\wedge\mathbf{R}(d_0)$, e_1 is $\mathbf{W}(\mathbf{I}\wedge\mathbf{L}(d_0))$ with the weakening formula v, and e_2 is $\mathbf{E}\wedge(d_1)$. Observe that $l(\mathbf{RED}(d)) \leqq 1 + l(d)$ since the operators producing the e_i do not increase lengths.

Case 3. $c = u \vee v$. Then I define $\mathbf{RED}(d)$ as



with $e_0 = \mathbf{I}\vee(d_0)$, $e_1 = \mathbf{W}(\mathbf{E}\vee\mathbf{L}(d_1))$, and $e_2 = \mathbf{E}\vee\mathbf{R}(d_1)$; observe again that $l(\mathbf{RED}(d)) \leqq 1 + l(d)$.

Case 4. $c = u \rightarrow v$. I define $\mathbf{RED}(d)$ to be



with $e_0 = \mathbf{E}{\rightarrow}\mathbf{L}(d_1)$, $e_1 = \mathbf{I}{\rightarrow}(d_0)$, and $e_2 = \mathbf{W}(\mathbf{E}{\rightarrow}\mathbf{R}(d_1))$ and, once more, $l(\mathbf{RED}(d)) \leqq 1 + l(d)$.

This concludes the definition of the operator **RED**. The following observation is immediate:

> If a deduction d contains a maximal cut which is atomic (i.e. which has an atomic cut formula) then in $\mathbf{RED}(d)$ it has disappeared and no new cuts are introduced during this removal. If d contains a maximal cut with a composite cut formula of degree m, then in $\mathbf{RED}(d)$ the number of cuts with cut formulas of degree larger than or equal to m has decreased.

It follows that iterating **RED** sufficiently often will result in a cut free deduction $\mathbf{K}(d)$. The question is: how often? In order to answer it, I will need to find a simple parameter which decreases under applications of d.

Let me examine the definition of **RED**, e.g. that of (Case 2) discussed above. The cut with $u \wedge v$ has disappeared; instead, the branch going through e_0 contains a new cut with v, and the branches going through e_1, e_2 contain new cuts with u as well as with v. Now the length of the formulas $u \wedge v$ is 1 plus the sum of the lengths of u and v. Consequently, the

sum of the lengths of cut formulas

on e_1, e_2 is less than the corresponding sum taken for d_1 – and the same, only much more so, is true for the sums with respect to e_0 and to d_0. The same situation exists in all other cases, and so I define for every branch V of d the number

$$N(V) = \text{sum of the lengths of cut formulas on } V$$

and conclude that for the branches V' of **RED**(d) arising from a branch V of d it holds that $N(V') < N(V)$. But every branch of **RED**(d) arises as a branch V' from a branch V of d. Consequently, for

$$k(d) = \text{maximum of the numbers } N(V) \text{ taken over all branches } V \text{ of } d$$

it follows that $k(\mathbf{RED}(d)) < k(d)$. This implies that at most $k(d)$ iterations of **RED** will result in $\mathbf{K}(d)$.

Finally: what about the increase of lengths? In all four cases in which **RED** was applied to deductions d ending with a maximal cut, I had observed that $l(\mathbf{RED}(d)) \leqq 2 \cdot l(d)$. It then follows from the recursive definition of **RED** that $l(\mathbf{RED}(d)) \leqq 2 \cdot l(d)$ holds generally. Since $\mathbf{K}(d)$ will be arrived at after at most $k(d)$ applications of **RED**, it follows that

$$l(\mathbf{K}(d)) \leqq 2^{k(d)} \cdot l(d).$$

A formula of degree 0 has length $1 = 2^1 - 1$; thus a formula of degree n will have at most the length $2^{n+1} - 1$. Gentzen's cut degree $g = g(d)$ is the maximum of degrees, increased by 1, of all cut formulas in d; hence 2^g is an upper bound for all these lengths. If V is a branch of length m, it will contain at most m cut formulas; hence $2^g \cdot m$ is an upper bound for $N(V)$. But m is bounded by $l(d)$, and thus

$$k(d) < 2^{g(d)} \cdot l(d).$$

2 Cut elimination in intuitionistic propositional logic

The calculus LJ of intuitionistic propositional logic differs from LK basically only in one respect: it uses sequents $M \Rightarrow N$ in which the right hand N consists of *one formula only*. So its axioms are of the form

$$M, v \Rightarrow v \quad \text{and} \quad M, \Delta \Rightarrow r,$$

where v is atomic, and its rules are the rules $(I \wedge)$, $(E \wedge L)$, $(E \wedge R)$, $(I \vee L)$, $(I \vee R)$, $(E \vee)$, $(I \rightarrow)$, $(E \rightarrow)$, (CUT) of LK, written now for the new, intuitionistic sequents. Thus in the I-rules the set N, as well as the parameters, disappears altogether, in $(E \wedge L)$, $(E \wedge R)$, $(E \vee)$ it becomes a single formula, in $(E \rightarrow)$ and (CUT) it disappears in the left premise and becomes a single formula in the right premise and the conclusion. That are all the differences already.

The operators **W** and **M** for weakening and contraction with the effect

$$\begin{array}{lll} \mathbf{W} & M \Rightarrow r & M, N \Rightarrow r \\ \mathbf{M} & M, M, N \Rightarrow r & M, N \Rightarrow r \end{array}$$

are defined as before and retain their properties. The same holds for the inversion operators

I$\wedge$**L**	$M\Rightarrow u\wedge v$	$M\Rightarrow u$
I$\wedge$**R**	$M\Rightarrow u\wedge v$	$M\Rightarrow v$
I$\rightarrow$	$M\Rightarrow u\rightarrow v$	$M,u\Rightarrow v$
E$\wedge$	$M,u\wedge v\Rightarrow r$	$M,u,v\Rightarrow r$
E$\vee$**L**	$M,u\vee v\Rightarrow r$	$M,u\Rightarrow r$
E$\vee$**R**	$M,u\vee v\Rightarrow r$	$M,v\Rightarrow r$
E$\rightarrow$**R**	$M,u\rightarrow v\Rightarrow r$	$M,v\Rightarrow r$.

But **I**$\vee$ and **E**$\rightarrow$**L** are not available since they would produce sequents with two formulas on the right. As for **I**$\vee$, used on the left premise of a cut with a disjunction, its missing is not serious, and the removal of the cut formula from the subdeduction leading to that premise can be managed by the familiar methods. The case of **E**$\rightarrow$**L** is a different matter, and for this operator I shall substitute three new ones which perform the transformations made in [H]:

if $M,(u\wedge v)\rightarrow w\Rightarrow r$ is derivable then so is $M,u\rightarrow(v\rightarrow w)\Rightarrow r$

if $M,(u\vee v)\rightarrow w\Rightarrow r$ is derivable then so is $M,u\rightarrow w,v\rightarrow w\Rightarrow r$

if $M,(u\rightarrow v)\rightarrow w\Rightarrow r$ is derivable then so is $M,u,v\rightarrow w\Rightarrow r$.

Clearly, these transformations act precisely on the sequents which **E**$\rightarrow$**L** would be applied to, and appropriate applications of (CUT) will show them to be correct. I shall now prove three lemmas which describe the construction of inversion operators producing the actual deductions *without* applications of (CUT).

In order to do so, I need a measure of the occurence of left premises of $(E\rightarrow)$ in a deduction d. Given a formula $u\rightarrow v$, I denote by

$m(u\rightarrow v|d)$ the maximal length of a subdeduction of d leading to a left premise of $(E\rightarrow)$ with principal formula $u\rightarrow v$.

Furthermore, given an occurrence $u\rightarrow v$ of a formula $u\rightarrow v$ in d I denote by $n(u\rightarrow v|d)$ the maximal length of a (full) branch of d containing $u\rightarrow v$ and a left premise of $(E\rightarrow)$ with principal formula $u\rightarrow v$.

It is easy to see that the inversion operators **F** already introduced do not increase these measures: it holds that $m(u\rightarrow v|d)\geqq m(u\rightarrow v|\mathbf{F}(d))$ and $n(u\rightarrow v|d)\geqq n(u\rightarrow v|\mathbf{F}(d))$.

Lemma 1. *There is an operator* **E**$\rightarrow\wedge$ *converting cut free deductions d of*

$$M,(u\wedge v)\rightarrow w\Rightarrow r$$

into cut free deductions **E**$\rightarrow\wedge(d)$ *of*

$$M,u\rightarrow(v\rightarrow w)\Rightarrow r$$

which at most doubles the lengths: $l(\mathbf{E}\rightarrow\wedge(d))\leqq 2\cdot l(d)$.

If d is an axiom $M,(u\wedge v)\rightarrow w\Rightarrow r$ then **E**$\rightarrow\wedge(d)$ shall be the axiom $M,u\rightarrow(v\rightarrow w)\Rightarrow r$; if $l(d)>1$ then I define **E**$\rightarrow\wedge(d)$ by recursion on $l(d)$. If d is of the form

$$\dfrac{\overset{d_0}{\downarrow}\; M_0,(u\wedge v)\to w \Rightarrow r_0 \qquad \overset{d_1}{\downarrow}\; M_1,(u\wedge v)\to w \Rightarrow r_1}{M,(u\wedge v)\to w \Rightarrow r}\;(R)$$

and (R) is not an application of $(E\to)$ with $(u\wedge v)\to w$ as principal formula, then I define $\mathbf{E}\!\to\wedge(d)$ as

$$\dfrac{\overset{e_0}{\downarrow}\; M_0,u\to(v\to w) \Rightarrow r_0 \qquad \overset{e_1}{\downarrow}\; M_1,u\to(v\to w) \Rightarrow r_1}{M,u\to(v\to w) \Rightarrow r}\;(R)$$

where e_i is $\mathbf{E}\!\to\wedge(d_i)$. If d is of the form

$$\dfrac{\overset{d_0}{\downarrow}\; M,(u\wedge v)\to w \Rightarrow u\wedge v \qquad \overset{d_1}{\downarrow}\; M,(u\wedge v)\to w, w \Rightarrow r}{M,(u\wedge v)\to w \Rightarrow r}\;(E\to)$$

then I define $\mathbf{E}\!\to\wedge(d)$ by

$$\dfrac{\overset{e_0}{\downarrow}\; M,u\to(v\to w) \Rightarrow u \qquad \dfrac{\overset{e_1}{\downarrow}\; \mathrm{M,u}\to(v\to w), v\to w \Rightarrow v \qquad \overset{e_2}{\downarrow}\; M,u\to(v\to w), v\to w, w \Rightarrow r}{M,u\to(v\to w), v\to w \Rightarrow r}\;(E\to)}{M,u\to(v\to w) \Rightarrow r}\;(E\to)$$

with $e_0=\mathbf{E}\!\to\wedge(\mathbf{I}\wedge\mathbf{L}(d_0))$, $e_1=\mathbf{W}(\mathbf{E}\!\to\wedge(\mathbf{I}\wedge\mathbf{R}(d_0)))$, and $e_2=\mathbf{W}(\mathbf{M}(\mathbf{E}\!\to\!\mathbf{R}(d_1)))$. The bound for $l(\mathbf{E}\!\to\wedge(d))$ is established by induction on $l(d)$.

Lemma 2. *There is an operator* $\mathbf{E}\!\to\wedge$ *converting cut free deductions d of*

$$M,(u\vee v)\to w \Rightarrow r$$

into cut free deductions $\mathbf{E}\!\to\vee(d)$ *of*

$$M,u\to w, v\to w \Rightarrow r$$

such that $l(\mathbf{E}\!\to\vee(d))\leqq 2\cdot l(d)$.

$\mathbf{E}\!\to\vee(d)$ will be defined by recursion on the number

$$p(d)=l(d)+m((u\vee v)\to w\,|\,d)$$

and it will be shown that $l(\mathbf{E}\!\to\vee(d))\leqq p(d)$: If $p(d)=1$ then d is an axiom of the form $M,(u\vee v)\to w\Rightarrow r$, and $\mathbf{E}\!\to\vee(d)$ shall be the axiom $M,u\to w, v\to w\Rightarrow r$. For $p(d)>1$ I will have to distinguish various cases.

Case 1. d is of the form

$$\dfrac{(I\vee L)\;\dfrac{\overset{d_0}{\downarrow}\; M,(u\vee v)\to w \Rightarrow u}{M,(u\vee v)\to w \Rightarrow u\vee v} \qquad \overset{d_1}{\downarrow}\; M,(u\vee v)\to w, w \Rightarrow r}{M,(u\vee v)\to w \Rightarrow r}\;(E\to).$$

Then I define $\mathbf{E}{\rightarrow}\vee(d)$ by

$$\frac{\overset{e_0\downarrow}{M, u\rightarrow w, v\rightarrow w \Rightarrow u} \qquad \overset{e_1\downarrow}{M, u\rightarrow w, v\rightarrow w, w \Rightarrow r}}{M, u\rightarrow w, v\rightarrow w \Rightarrow r}\ (E\rightarrow)$$

with $e_0=\mathbf{E}{\rightarrow}\vee(d_0)$ and $e_1=\mathbf{W}(\mathbf{M}(\mathbf{E}{\rightarrow}\mathbf{R}(d_1)))$. If $(I\vee R)$ was applied instead of $(I\vee L)$ then I proceed analogously.

Case 2. d is of the form

$$\frac{(E\rightarrow)\ \dfrac{\overset{d_0\downarrow}{M,(u\vee v)\rightarrow w \Rightarrow u\vee v} \qquad \overset{d_1\downarrow}{M,(u\vee v)\rightarrow w, w \Rightarrow u\vee v}}{M,(u\vee v)\rightarrow w \Rightarrow u\vee v} \qquad \overset{d_2\downarrow}{M,(u\vee v)\rightarrow w, w \Rightarrow r}}{M,(u\vee v)\rightarrow w \Rightarrow r}\ (E\rightarrow).$$

Then I define $\mathbf{E}{\rightarrow}\vee(d)$ as $\mathbf{E}{\rightarrow}\vee(e)$ where e is

$$\frac{\overset{d_0\downarrow}{M,(u\vee v)\rightarrow w \Rightarrow u\vee v} \qquad \overset{e_0\downarrow}{M,(u\vee v)\rightarrow w, w \Rightarrow r}}{M,(u\vee v)\rightarrow w \Rightarrow r}\ (E\rightarrow)$$

with $e_0=\mathbf{W}(\mathbf{M}(\mathbf{E}{\rightarrow}\mathbf{R}(d_2)))$; observe here that by definition of the operator $\mathbf{E}{\rightarrow}\mathbf{R}$ we have $m((u\vee v)\rightarrow w\,|\,e_0)=0$ and thus $p(e)<p(d)$.

Case 3. d is of the form

$$\frac{(E\rightarrow)\ \dfrac{\overset{d_0\downarrow}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w \Rightarrow r_0} \qquad \overset{d_1\downarrow}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w, r_1 \Rightarrow u\vee v}}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w \Rightarrow u\vee v} \qquad \overset{d_2\downarrow}{M,(u\vee v)\rightarrow w, w \Rightarrow r}}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w \Rightarrow r}\ (E\rightarrow)$$

then I define $\mathbf{E}{\rightarrow}\vee(d)$ to be

$$\frac{\overset{\mathbf{E}\rightarrow\vee(d_0)\downarrow}{M, r_0\rightarrow r_1, u\rightarrow w, v\rightarrow w \Rightarrow r_0} \qquad \overset{\mathbf{E}\rightarrow\vee(e)\downarrow}{M, r_0\rightarrow r_1, u\rightarrow w, v\rightarrow w, r_1 \Rightarrow r}}{M, r_0\rightarrow r_1, u\rightarrow w, v\rightarrow w \Rightarrow r}\ (E\rightarrow),$$

where e is defined by

$$\frac{\overset{d_1\downarrow}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w, r_1 \Rightarrow u\vee v} \qquad \overset{\mathbf{W}(\mathbf{M}(\mathbf{E}\rightarrow\mathbf{R}(d_2)))\downarrow}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w, w, r_1 \Rightarrow r}}{M, r_0\rightarrow r_1, (u\vee v)\rightarrow w, r_1 \Rightarrow r}\ (E\rightarrow).$$

Case 4. d is of the form

$$
\begin{array}{ccc}
\dfrac{\overset{d_0}{\downarrow}\quad\quad\overset{d_1}{\downarrow}}{\dfrac{M_0,(u\vee v)\to w\Rightarrow u\vee v \quad M_1,(u\vee v)\to w\Rightarrow u\vee v}{M,(u\vee v)\to w\Rightarrow u\vee v}\,(R)} & & \overset{d_2}{\downarrow} \\
 & & M,(u\vee v)\to w, w\Rightarrow r \\
\hline
 & M,(u\vee v)\to w\Rightarrow r & (E\to)
\end{array},
$$

where (R) is different from $(E\to)$ and $(I\vee)$. Then I introduce auxiliary deductions e_0, e_1 defined by

$$
\dfrac{\overset{d_i}{\downarrow}\qquad\qquad \overset{\mathbf{W}(\mathbf{M}(\mathbf{E}\to\mathbf{R}(d_2)))}{\downarrow}}{\dfrac{M_i,(u\vee v)\to w\Rightarrow u\vee v \quad M_i,(u\vee v)\to w, w\Rightarrow r}{M_i,(u\vee v)\to w\Rightarrow r}}\,(E\to).
$$

From e_0 and e_1 I obtain $\mathbf{E}\to\vee(d)$ as

$$
\dfrac{\overset{\mathbf{E}\to\vee(e_0)}{\downarrow}\qquad\qquad \overset{\mathbf{E}\to\vee(e_1)}{\downarrow}}{\dfrac{M_0,u\to w,v\to w\Rightarrow r \quad M_1,u\to w,v\to w\Rightarrow r}{M,u\to w,v\to w\Rightarrow r}}\,(R).
$$

Finally

Case 5. d is of the form

$$
\dfrac{\overset{d_0}{\downarrow}\qquad\qquad \overset{d_1}{\downarrow}}{\dfrac{M_0,(u\vee v)\to w\Rightarrow r_0 \quad M_1,(u\vee v)\to w\Rightarrow r_1}{M,(u\vee v)\to w\Rightarrow r}}\,(R),
$$

where (R) is not an application of $(E\to)$ with principal formula $(u\vee v)\to w$, then I define $\mathbf{E}\to\vee(d)$ by

$$
\dfrac{\overset{e_0}{\downarrow}\qquad\qquad \overset{e_1}{\downarrow}}{\dfrac{M_0,u\to w,v\to w\Rightarrow r_0 \quad M_1,u\to w,v\to w\Rightarrow r_1}{M,u\to w,v\to w\Rightarrow r}}\,(R)
$$

with $e_i := \mathbf{E}\to\vee(d_i)$. In all these cases I have $l(\mathbf{E}\to\vee(d))\leqq p(d)=l(d)+m((u\vee v)\to w|d)$, and this implies in particular $l(\mathbf{E}\to\vee(d))\leqq 2\cdot l(d)$.

Lemma 3. *There is an operator* $\mathbf{E}\twoheadrightarrow$ *converting every cut free deduction* d *of a sequent*

$$M,(u\to v)\to w\Rightarrow r$$

into a cut free deduction $\mathbf{E}\twoheadrightarrow(d)$ *of the sequent*

$$M,u,v\to w\Rightarrow r$$

such that $l(\mathbf{E}\twoheadrightarrow(d))\leqq l(d)$.

If d is an axiom $M,(u\to v)\to w \Rightarrow r$, then I define $\mathbf{E}\twoheadrightarrow(d)$ to be $M,u,v\to w\Rightarrow r$. Now the construction of $\mathbf{E}\twoheadrightarrow(d)$ proceeds by recursion on $l(d)$. If d is of the form

$$\frac{\overset{d_0\downarrow}{M_0,(u\to v)\to w\Rightarrow r_0} \qquad \overset{d_1\downarrow}{M_1,(u\to v)\to w\Rightarrow r_1}}{M,(u\vee v)\to w\Rightarrow r}\,(R),$$

where (R) is not an application of $(E\to)$ with principal formula $(u\to v)\to w$, then I define $\mathbf{E}\twoheadrightarrow(d)$ by

$$\frac{\overset{e_0\downarrow}{M_0,u,v\to w\Rightarrow r_0} \qquad \overset{e_1\downarrow}{M_1,u,v\to w\Rightarrow r_1}}{M,u,v\to w\Rightarrow r}\,(R)$$

with $e_i:=\mathbf{E}\twoheadrightarrow(d_i)$. If d is of the form

$$\frac{\overset{d_0\downarrow}{M,(u\to v)\to w\Rightarrow u\to v} \qquad \overset{d_1\downarrow}{M,(u\to v)\to w,w\Rightarrow r}}{M,(u\to v)\to w\Rightarrow r}\,(E\to)$$

then I define $\mathbf{E}\twoheadrightarrow(d)$ to be

$$\frac{\overset{e_0\downarrow}{M,u,v\to w\Rightarrow v} \qquad \overset{e_1\downarrow}{M,u,v\to w,w\Rightarrow r}}{M,u,v\to w\Rightarrow r}\,(E\to),$$

where $e_0:=\mathbf{M}(\mathbf{E}\twoheadrightarrow(\mathbf{I}\to(d_0)))$ and $e_1:=\mathbf{W}(\mathbf{M}(\mathbf{E}\to\mathbf{R}(d_1)))$.

These three lemmas having been proved, the program for the reduction procedure presented in the following theorem is as follows. I have constructed the operators $\mathbf{E}\to\wedge$, $\mathbf{E}\to\vee$, and $\mathbf{E}\twoheadrightarrow$ as substitutes for the missing operator $\mathbf{E}\to\mathbf{L}$. As for the first two, they may be applied in the appropriate situations and will work as in the classical case. As for $\mathbf{E}\twoheadrightarrow$, its application in a situation leading from

$$\frac{\overset{\downarrow}{M\Rightarrow(u\to v)\to w} \qquad \overset{\downarrow d}{M,(u\to v)\to w\Rightarrow r}}{M\Rightarrow r} \quad \text{to} \quad \overset{\downarrow}{M\Rightarrow(u\to v)\to w} \qquad \overset{\downarrow \mathbf{E}\twoheadrightarrow(d)}{M,u,v\to w\Rightarrow r}$$

will, in general, leave me without knowledge about how to proceed. In the particular case that $M,(u\to v)\to w\Rightarrow r$ in d happens to be the conclusion of $(E\to)$ with principal formula $(u\to v)\to w$, however, a suitable reduction will be possible (cf. Case 6 of the following proof); moreover, it will be shown that all other cases can be transformed in such a way that, in the end, only this particular case needs to be treated.

There remain the situations

$$\frac{\overset{d_0\downarrow}{M\Rightarrow a\to v} \qquad \overset{d\downarrow}{M,a\to v\Rightarrow r}}{M\Rightarrow r}$$

with a atomic in which the left side may be reduced with help of $\mathbf{I}\to$. Again, in general no reduction for d seems to be available. In the particular case, however, that d ends with an instance of $(E\rightarrow)$ with principal formula $a\rightarrow v$ *and the left* premise *being an axiom*, the reduction of

$$\dfrac{\overset{d_0}{\downarrow}\quad M,a\Rightarrow a\rightarrow v \qquad \dfrac{M,a,a\rightarrow v\Rightarrow a \qquad \overset{d_1}{\downarrow}\quad M,a,a\rightarrow v,v\Rightarrow r}{M,a,a\rightarrow v\Rightarrow r}\ (E\rightarrow)}{M,a\Rightarrow r}$$

may be performed (cf. Case 3.2 in the following proof). Moreover, it may be shown that all other cases can be transformed in such a manner that only this particular case need to be treated.

Carrying out all these transformations will, obviously, increase the lengths of deductions, and so I will need recursion parameters in order to measure them. I begin by defining a new degree function deg for formulas:

$$\begin{aligned}
\deg(v)&=2 \quad \text{if } v \text{ is atomic},\\
\deg(u\wedge v)&=\deg(u)\cdot(1+\deg(v)),\\
\deg(u\vee v)&=1+\deg(u)+\deg(v),\\
\deg(u\rightarrow v)&=1+\deg(u)\cdot\deg(v),
\end{aligned}$$

and in all the following *degree* shall always refer to this degree function. I shall be using the following properties of this function:

$$\begin{aligned}
&\deg((u\wedge v)\rightarrow w)>\deg(u\rightarrow(v\rightarrow w)),\\
&\deg((u\vee v)\rightarrow w)>\deg(u\rightarrow w)+\deg(v\rightarrow w),\\
&\deg((u\rightarrow v)\rightarrow w)>\deg(u)+\deg(v)+\deg(v\rightarrow w),\\
&\deg((u\rightarrow v)\rightarrow w)>\deg(u\rightarrow v)+\deg(v\rightarrow w).
\end{aligned}$$

If d is a deduction and V is a (full) branch of d then I set

$$j(V):=\textit{Sum} \text{ of the degrees of cut formulas on } V$$

and

$$j(d):=\textit{Maximum} \text{ of } j(V) \text{ for all branches of } d.$$

The reason behind these particular choices is simply the fact that the function j defined in this manner will decrease under application of the operator **RED** to be defined now. – If a deduction d ends with an instance of a 2-premise rule then I will write $ll(d)$ and $lr(d)$ for the lengths of the subdeductions leading to the left or the right premise respectively.

Theorem 1. *There is an operator* **RED** *converting every deduction d of a sequent s with*

$$0<j(d)$$

into a deduction **RED**(d) *of s such that*

$$j(\mathbf{RED}(d))<j(d)$$

which at most doubles the length of $d: l(\mathbf{RED}(d))\leq 2\cdot l(d)$.

I define **RED**(d) by recursion on $v(d)$, where $v(d)$ is defined as follows:

$$1.\ \text{If } d = \dfrac{\overset{d_0\downarrow}{M \Rightarrow c} \qquad \overset{d_1\downarrow}{M, c \Rightarrow r}}{M \Rightarrow r}\ (\text{Cut}),$$

where d_0 and d_1 are cutfree, then I set

$$v(d) := \begin{cases} l(d), & \text{if } c \text{ is atomic or a disjunction} \\ n(u\to v\,|\,d), & \text{if } c = u\to v \text{ and } u \text{ atomic} \\ l(d_1), & \text{if } c = (u\to v)\to w, \\ l(d) & \text{otherwise}. \end{cases}$$

2. In all other cases I set $v(d) := l(d)$.

Simultaneously I prove $l(\mathbf{RED}(d)) \leq l(d) + v(d)$. Since $v(d) \leq l(d)$ this implies the theorem. I distinguish cases according to whether the last rule applied in d is or is not a maximal cut. The simple case is the latter one in which d is of the form

$$\dfrac{\overset{d_0\downarrow}{M_0 \Rightarrow r_0} \qquad \overset{d_1\downarrow}{M_1 \Rightarrow r_1}}{M \Rightarrow r}\ (R);$$

here I define **RED**(d) as

$$\dfrac{\overset{e_0\downarrow}{M_0 \Rightarrow r_0} \qquad \overset{e_1\downarrow}{M_1 \Rightarrow r_1}}{M \Rightarrow r}\ (R)$$

with $e_i = \mathbf{RED}(d_i)$. If, however, the last rule of d is a maximal cut, I shall distinguish six cases according to the form of the cut formula c.

Case 1. c is atomic or a disjunction.

Case 1.0. The left premise of the cut inference is an axiom. Then d is of the form

$$\dfrac{M, c \Rightarrow c \qquad \overset{d_0\downarrow}{M, c, c \Rightarrow r}}{M, c \Rightarrow r}\ (\text{CUT})$$

and I set $\mathbf{RED}(d) = \mathbf{M}(d_0)$.

Case 1.1. The left premise is the conclusion of an $(I\vee L)$- or $(I\vee R)$-inference, $c = u \vee v$ for formulas u and v:

$$\dfrac{\dfrac{\overset{d_0\downarrow}{M \Rightarrow u}}{M \Rightarrow u \vee v}\ (I\vee L) \qquad \overset{d_1\downarrow}{M, u \vee v \Rightarrow r}}{M \Rightarrow r}\ (\text{CUT}).$$

Then I define **RED**(d) by

$$\dfrac{\overset{d_0}{\downarrow} \atop M \Rightarrow u \qquad \overset{e}{\downarrow} \atop M, u \Rightarrow r}{M \Rightarrow r}\ (\mathrm{CUT})$$

with $e = \mathbf{E} \vee \mathbf{L}(d_1)$, and similarly if $(I \vee L)$ is replaced by $(I \vee R)$.

Case 1.2.

$$\dfrac{\dfrac{\overset{d_0}{\downarrow} \atop M, r_0 \to r_1 \Rightarrow r_0 \qquad \overset{d_1}{\downarrow} \atop M, r_0 \to r_1, r_1 \Rightarrow c}{M, r_0 \to r_1 \Rightarrow c}\ (E\to) \qquad \overset{d_2}{\downarrow} \atop M, r_0 \to r_1, c \Rightarrow r.}{M, r_0 \to r_1 \Rightarrow r}\ (\mathrm{CUT})$$

Here I set **RED**(d)=

$$\dfrac{\overset{d_0}{\downarrow} \atop M, r_0 \to r_1 \Rightarrow r_0 \qquad \overset{\mathbf{RED}(e)}{\downarrow} \atop M, r_0 \to r_1, r_1 \Rightarrow r}{M, r_0 \to r_1 \Rightarrow r}\ (E\to),$$

where e is

$$\dfrac{\overset{d_1}{\downarrow} \atop M, r_0 \to r_1, r_1 \Rightarrow c \qquad \overset{\mathbf{W}(d_2)}{\downarrow} \atop M, r_0 \to r_1, r_1, c \Rightarrow r}{M, r_0 \to r_1, r_1 \Rightarrow r}\ (\mathrm{CUT}).$$

The remaining possibilities are covered by

Case 1.3.

$$\dfrac{\dfrac{\overset{d_0}{\downarrow} \atop M_0 \Rightarrow c \qquad \overset{d_1}{\downarrow} \atop M_1 \Rightarrow c}{M \Rightarrow c}\ (R) \qquad \overset{d_2}{\downarrow} \atop M, c \Rightarrow r,}{M \Rightarrow r}\ (\mathrm{CUT})$$

where (R) is different from $(I \vee)$ and $(E\to)$. In this case I consider the deductions e_i given as

$$\dfrac{\overset{d_i}{\downarrow} \atop M_i \Rightarrow c \qquad \overset{\mathbf{W}(d_2)}{\downarrow} \atop M_i, c \Rightarrow r}{M_i \Rightarrow r}\ (\mathrm{CUT})$$

and define **RED**(d) by

$$\dfrac{\overset{\mathbf{RED}(e_0)}{\downarrow} \atop M_0 \Rightarrow r \qquad \overset{\mathbf{RED}(e_1)}{\downarrow} \atop M_1 \Rightarrow r}{M \Rightarrow r}\ (R).$$

It then follows that

$$j(\mathbf{RED}(d)) \leq \max\{j(\mathbf{RED}(e_0)), j(\mathbf{RED}(e_1))\} < \max\{j(e_0), j(e_1)\} = \deg(u \vee v) = j(d)$$

and

$$l(\mathbf{RED}(d)) \leqq l(d) + v(d).$$

Case 2. $c = u \wedge v$.

If d has the form

$$\dfrac{\begin{array}{c} d_0 \downarrow \\ M \Rightarrow u \wedge v \end{array} \qquad \begin{array}{c} d_1 \downarrow \\ M, u \wedge v \Rightarrow r \end{array}}{M \Rightarrow r} \ \text{(CUT)}$$

then I define **RED**(d) to be

$$\dfrac{\dfrac{\begin{array}{c} e_0 \downarrow \\ M \Rightarrow v \end{array} \qquad \begin{array}{c} e_1 \downarrow \\ M, v \Rightarrow u \end{array} \qquad \begin{array}{c} e_2 \downarrow \\ M, u, v \Rightarrow r \end{array}}{M, v \Rightarrow r} \ \text{(CUT)},}{M \Rightarrow r} \ \text{(CUT)}$$

where $e_0 = \mathbf{I} \wedge \mathbf{R}(d_0)$, $e_1 = \mathbf{W}(\mathbf{I} \wedge \mathbf{L}(d_0))$, and $e_2 = \mathbf{E} \wedge (d_1)$. It follows that

$$j(\mathbf{RED}(d)) = \deg(u) + \deg(v) < \deg(u \wedge v) = j(d)$$

and

$$l(\mathbf{RED}(d)) \leqq l(d) + 1 \leqq l(d) + v(d).$$

Case 3. $c = u \to v$ and u is atomic.

Here I use recursion on the parameter $n(u \to v \mid d)$, where $u \to v$ is the occurrence of $u \to v$ in the antecedens of the right premise of the bottommost (CUT) of d. If this number is 0 then I set **RED**(d) : = the deduction resulting from the deduction of the right premise of the cut inference, by ommitting all occurences of $u \to v$.

There remain the cases that $n(u \to v \mid d) > 0$ in which it is easy to see that the right premise of the conclusion cannot be an axiom.

Case 3.1. This right premise is *not* conclusion of an instance of ($\mathbf{E}\to$) with principal formula $u \to v$.

$$\dfrac{\begin{array}{c} d_0 \downarrow \\ M \Rightarrow u \to v \end{array} \qquad \dfrac{\begin{array}{c} d_1 \downarrow \\ M_0, u \to v \Rightarrow r_0 \end{array} \qquad \begin{array}{c} d_2 \downarrow \\ M_1, u \to v \Rightarrow r_1 \end{array}}{M, u \to v \Rightarrow r} \ (R)}{M \Rightarrow r} \ \text{(CUT)}.$$

Then I set **RED**(d) =

$$\dfrac{\begin{array}{c} \mathbf{RED}(e_0) \downarrow \\ M_0 \Rightarrow r_0 \end{array} \qquad \begin{array}{c} \mathbf{RED}(e_1) \downarrow \\ M_1 \Rightarrow r_1 \end{array}}{M \Rightarrow r} \ (R),$$

where the e_i are

$$\dfrac{\overset{\mathbf{W}(d_0)}{\downarrow}\; M_i \Rightarrow u\to v \qquad \overset{d_{i+1}}{\downarrow}\; M_i, u\to v \Rightarrow r_i}{M_i \Rightarrow r_i}\ (\mathrm{CUT}).$$

The remaining cases cover the situation that the right premise *is* conclusion of an instance of $(E\to)$ with principal formula $u\to v$ and distinguish according to the form of its left premise.

Case 3.2.

$$\dfrac{\overset{d_0}{\downarrow}\; M,u \Rightarrow u\to v \qquad \dfrac{M,u,u\to v \Rightarrow u \qquad \overset{d_1}{\downarrow}\; M,u,u\to v, v \Rightarrow r}{M,u,u\to v \Rightarrow r}\,(E\to)}{M,u \Rightarrow r}\ (\mathrm{CUT}).$$

Then I define $\mathbf{RED}(d)$ by

$$\dfrac{\overset{e_0}{\downarrow}\; M,u \Rightarrow v \qquad \overset{e_1}{\downarrow}\; M,u,v \Rightarrow r}{M,u \Rightarrow r}\ (\mathrm{CUT}),$$

where $e_0 = \mathbf{M}(\mathbf{I}{\to}(d_0))$ and $e_1 = \mathbf{M}(\mathbf{E}{\to}\mathbf{R}(d_1))$.

Case 3.3.

$$\dfrac{\overset{d_0}{\downarrow}\; M \Rightarrow u\to v \qquad \dfrac{\dfrac{\overset{d_1}{\downarrow}\; M,u\to v \Rightarrow u \qquad \overset{d_2}{\downarrow}\; M,u\to v, v \Rightarrow u}{M,u\to v \Rightarrow u}\,(E\to) \qquad \overset{d_3}{\downarrow}\; M,u\to v, v \Rightarrow r}{M,u\to v \Rightarrow r}\,(E\to)}{M \Rightarrow r}\ (\mathrm{CUT})$$

with $u\to v$ as principal formula in the upper instance of $(E\to)$. I define $\mathbf{RED}(d)$ as $\mathbf{RED}(e)$, where e is

$$\dfrac{\overset{d_0}{\downarrow}\; M \Rightarrow u\to v \qquad \dfrac{\overset{d_1}{\downarrow}\; M,u\to v \Rightarrow u \qquad \overset{e_0}{\downarrow}\; M,u\to v, v \Rightarrow r}{M,u\to v \Rightarrow r}\,(E\to)}{M \Rightarrow r}\ (\mathrm{CUT})$$

and $e_0 = \mathbf{W}(\mathbf{M}(\mathbf{E}{\to}\mathbf{R}(d_3)))$.

Case 3.4.

$$\dfrac{\overset{d_0}{\downarrow}\; M \Rightarrow u\to v \qquad \dfrac{\dfrac{\overset{d_1}{\downarrow}\; M_0,u\to v \Rightarrow r_0 \qquad \overset{d_2}{\downarrow}\; M_1,u\to v \Rightarrow u}{M,u\to v \Rightarrow u}\,(E\to) \qquad \overset{d_3}{\downarrow}\; M,u\to v, v \Rightarrow r}{M,u\to v \Rightarrow r}\,(E\to)}{M \Rightarrow r}\ (\mathrm{CUT}),$$

where the principal formula of the upper instance of $(E\to)$ is different from $u\to v$. Then $\mathbf{RED}(d)=$

$$\frac{\overset{\mathbf{RED}(e_0)\downarrow}{M\Rightarrow r_0} \qquad \overset{\mathbf{RED}(e_1)\downarrow}{M,r_1\Rightarrow r}}{M\Rightarrow r}\ (E\to),$$

where e_0 is

$$\frac{\overset{d_0\downarrow}{M\Rightarrow u\to v} \qquad \overset{d_1\downarrow}{M,u\to v\Rightarrow r_0}}{M\Rightarrow r_0}\ (\mathrm{CUT})$$

and e_1 is

$$\frac{\overset{\mathbf{W}(d_0)\downarrow}{M,r_1\Rightarrow u\to v} \qquad \dfrac{\overset{d_2\downarrow}{M,r_1,u\to v\Rightarrow u} \qquad \overset{\downarrow\mathbf{W}(\mathbf{M}(\mathbf{E}\to\mathbf{R}(d_3)))}{M,r_1,u\to v,v\Rightarrow r}}{M,r_1,u\to v\Rightarrow r}\ (E\to)}{M,r_1\Rightarrow r}\ (\mathrm{CUT}).$$

Case 3.5.

$$\frac{\overset{d_0\downarrow}{M\Rightarrow u\to v} \qquad \dfrac{\dfrac{\overset{d_1\downarrow}{M_0,u\to v\Rightarrow u} \qquad \overset{d_2\downarrow}{M_1,u\to v\Rightarrow u}}{M,u\to v\Rightarrow u}\ (R) \qquad \overset{d_3\downarrow}{M,u\to v,v\Rightarrow r}}{M,u\to v\Rightarrow r}\ (E\to)}{M\Rightarrow r}\ (\mathrm{CUT})$$

with (R) different from $(E\to)$. I set $\mathbf{RED}(d)=$

$$\frac{\overset{\mathbf{RED}(e_0)\downarrow}{M_0\Rightarrow r} \qquad \overset{\mathbf{RED}(e_1)\downarrow}{M_1\Rightarrow r}}{M\Rightarrow r}\ (R),$$

where the e_i are

$$\frac{\overset{\mathbf{W}(d_0)\downarrow}{M_i\Rightarrow u\to v} \qquad \dfrac{\overset{d_{i+1}\downarrow}{M_i,u\to v\Rightarrow u} \qquad \overset{\downarrow\mathbf{W}(\mathbf{M}(\mathbf{E}\to\mathbf{R}(d_3)))}{M_i,u\to v,v\Rightarrow r}}{M_i,u\to v\Rightarrow r}\ (E\to)}{M_i\Rightarrow r}\ (\mathrm{CUT}).$$

The resulting deduction $\mathbf{RED}(d)$ has the length $l(\mathbf{RED}(d))\leqq l(d)+n(u\to v\,|\,d)=l(d)+v(d)$.

Case 4. $c=(u\wedge v)\to w$.

$$\frac{\overset{d_0\downarrow}{M\Rightarrow (u\wedge v)\to w} \qquad \overset{d_1\downarrow}{M,(u\wedge v)\to w\Rightarrow r}}{M\Rightarrow r}\ (\mathrm{CUT}).$$

I define **RED**(d) by

$$\dfrac{\dfrac{(I\to)\quad \dfrac{\dfrac{\overset{e_0}{\downarrow}}{M,u,v\Rightarrow w}}{M,u\Rightarrow v\to w}}{(I\to)\quad M\Rightarrow u\to(v\to w)} \qquad \dfrac{\overset{e_1}{\downarrow}}{M,u\to(v\to w)\Rightarrow r}}{M,u\to(v\to w)\Rightarrow r}\ \text{(CUT)},$$

where $e_0=\mathbf{E}\wedge(\mathbf{I}\to(d_0))$ and $e_1=\mathbf{E}\to\wedge(d_1)$. It follows that $l(\mathbf{RED}(d))\leqq l(d)+v(d)$ and

$$\begin{aligned}j(\mathbf{RED}(d))&=\deg(u\to(v\to w))\\&=1+\deg(u)\cdot(1+\deg(v)\deg(w))<1+\deg(w)\cdot\deg(u)\cdot(1+\deg(v))\\&=\deg((u\wedge v)\to w)=j(d).\end{aligned}$$

Case 5. $c=(u\vee v)\to w$.

$$\dfrac{\dfrac{\overset{d_0}{\downarrow}}{M\Rightarrow(u\vee v)\to w}\qquad \dfrac{\overset{d_1}{\downarrow}}{M,(u\vee v)\to w\Rightarrow r}}{M\Rightarrow r}\ \text{(CUT)}.$$

I define **RED**(d) by

$$\dfrac{(I\to)\ \dfrac{\dfrac{\overset{e_0}{\downarrow}}{M,v\Rightarrow w}}{M\Rightarrow v\to w}\qquad \dfrac{\dfrac{\overset{e_1}{\downarrow}}{M,u\to w,v\to w\Rightarrow r}\qquad \dfrac{\dfrac{\overset{e_2}{\downarrow}}{M,v\to w,u\Rightarrow w}}{M,v\to w\Rightarrow u\to w}\ (I\to)}{M,v\to w\Rightarrow r}\ \text{(CUT)}}{M\Rightarrow r}\ \text{(CUT)},$$

where $e_0=\mathbf{E}\vee\mathbf{R}(\mathbf{I}\to(d_0))$, $e_1=\mathbf{E}\to\vee(d_1)$, and $e_2=\mathbf{W}(\mathbf{E}\vee\mathbf{L}(\mathbf{I}\to(d_0)))$. It follows that $l(\mathbf{RED}(d))\leqq l(d)\ v(d)$ and

$$\begin{aligned}j(\mathbf{RED}(d))&=\deg(u\to w)+\deg(v\to w)\\&=1+\deg(u)\cdot\deg(w)+1+\deg(v)\cdot\deg(w)\\&<1+\deg(w)\cdot(1+\deg(u)+\deg(v))=\deg((u\vee v)\to w)=j(d).\\&=\deg((u\vee v)\to w)=j(d).\end{aligned}$$

Case 6. $c=(u\to v)\to w$.

If $M,(u\to v)\to w\Rightarrow r$ is an axiom then so is $M\Rightarrow r$. If $M,(u\to v)\to w\Rightarrow r$ is not an axiom then I distinguish two subcases:

Case 6.1.

$$\dfrac{\dfrac{\overset{d_0}{\downarrow}}{M\Rightarrow(u\to v)\to w}\qquad \dfrac{\dfrac{\overset{d_1}{\downarrow}}{M,(u\to v)\to w\Rightarrow u\to v}\qquad \dfrac{\overset{d_2}{\downarrow}}{M,(u\to v)\to w,w\Rightarrow r}}{M,(u\to v)\to w\Rightarrow r}\ (E\to)}{M\Rightarrow r}\ \text{(CUT)}.$$

I define **RED**(d) to be

$$\dfrac{\dfrac{\dfrac{\overset{e_0}{\downarrow}}{M,v\Rightarrow w}\ (I\to)\quad \dfrac{\overset{e_1}{\downarrow}}{M,u,v\to w\Rightarrow v}}{\dfrac{M\Rightarrow v\to w \qquad M,v\to w\Rightarrow u\to v}{M\Rightarrow u\to v}\ \text{(CUT)}} \qquad \dfrac{\overset{e_2}{\downarrow}\ M,u\to v\Rightarrow w \qquad \overset{e_3}{\downarrow}\ M,u\to v,w\Rightarrow r}{M,u\to v\Rightarrow r}\ \text{(CUT)}}{M\Rightarrow r}\ \text{(CUT)},$$

where $e_0=\mathbf{E}{\to}\mathbf{R}(\mathbf{I}{\to}(d_0))$, $e_1=\mathbf{M}(\mathbf{E}{\to\to}(\mathbf{I}{\to}(d_1)))$, $e_2=\mathbf{I}{\to}(d_0)$, and $e_3=\mathbf{W}(\mathbf{M}(\mathbf{E}{\to}\mathbf{R}(d_2)))$.

Case 6.2.

$$\dfrac{\dfrac{\overset{d_0}{\downarrow}}{M\Rightarrow(u\to v)\to w} \qquad \dfrac{\overset{d_1}{\downarrow}\ M_0,(u\to v)\to w\Rightarrow r_0 \qquad \overset{d_2}{\downarrow}\ M_1,(u\to v)\to w\Rightarrow r_1}{M,(u\to v)\to w\Rightarrow r}\ (R)}{M\Rightarrow r}\ \text{(CUT)}$$

where (R) is not an instance of $(E\to)$ with principal formula $(u\to v)\to w$. Then I define **RED**(d) =

$$\dfrac{\overset{\mathbf{RED}(e_0)}{\downarrow}\ M_0\Rightarrow r_0 \qquad \overset{\mathbf{RED}(e_1)}{\downarrow}\ M_1\Rightarrow r_1}{M\Rightarrow r}\ (R),$$

where the e_i are

$$\dfrac{\overset{\mathbf{W}(d_0)}{\downarrow}\ M_i\Rightarrow(u\to v)\to w \qquad \overset{d_{i+1}}{\downarrow}\ M_i,(u\to v)\to w\Rightarrow r_i}{M_i\Rightarrow r_i}\ \text{(CUT)}.$$

It follows that $l(\mathbf{RED}(d))\leqq l(d)+v(d)$ and

$$\begin{aligned} j(\mathbf{RED}(d)) &\leqq \deg(u\to v)+\deg(v\to w)\\ &=1+\deg(u)\deg(v)+1+\deg(v)\deg(w)<1+\deg(w)\cdot(1+\deg(u)\deg(v))\\ &=\deg((u\to v)\to w)=j(d)\,. \end{aligned}$$

This concludes the proof of Theorem 1. Iterating the operator **RED**, I immediately obtain the

Theorem 2. *There is an operator* **J** *converting every deduction d of a sequent s into a deduction* **J**(d) *of this same sequent such that*

$$\mathbf{J}(d) \text{ is cut free and } l(\mathbf{J}(d))\leqq 2^{j(d)}\cdot l(d)\,.$$

If v is a formula of (traditional) degree g, a straightforward induction will show

$$\deg(v)\leqq 2^{2^{2g}}\,.$$

Since the maximal number of cut formulas on a branch of d is bounded by $l(d)$, it follows for Gentzen's cut degree that

$$l(\mathbf{J}(d)) \leq 2^{(2^{2^{2\cdot g(d)}}\cdot l(d))} \cdot l(d).$$

3 A decision procedure for intuitionistic propositional logic

Consider the sequent calculus LJ and a sequent s which is not an axiom. If we want to know whether s can be derived, we may restrict ourselves to search for cut free deductions, and our first attempt will be to choose a composite formula $v=\varphi(s)$ in s and to look for rules which produce s as their conclusion with v as principal formula. Unfortunately, the E-rules of LJ are such that v then also occurs in the one or two premises $t(s,\varphi(s)), r(s,\varphi(s))$ of that rule, meaning that our task may have to be repeated for them with *that same* formula v. There now are two strategies to pursue:

Search by Depth First: iterate such choices of always one formula and form the sequents

$$s, t(s,\varphi(s)), t(t(s,\varphi(s)), \varphi(t(s,\varphi(s)))), \ldots,$$

hoping to arrive finally at some axiom; in that case, return to the branching nodes still left (i.e. to $r(s,\varphi(s))$ as the first such node) and repeat the process until all branches close with axioms.

Search by Breadth First: if $t(s,\varphi(s))$, $r(s,\varphi(s))$ are not both axioms, then inspect all other possible choices $\psi(s)$, $\chi(s)$, ... of composite formulas, collect all the corresponding premises

$$t(s,\varphi(s)), r(s,\varphi(s)), t(s,\psi(s)), r(s,\psi(s)), t(s,\chi(s)), r(s,\chi(s))$$

as the result of a first stage, and then repeat the construction of stage $n+1$ from all the sequents of stage n which are not axioms.

Unfortunately, search by depth first may lead to not terminating branches even if s happens to have a deduction. On the other hand, search by breadth first will discover an existing deduction, but at the price of considerable storage space and time, and it will not terminate at all if no deduction happens to exist.

The literature contains descriptions of two ways leading out of this dilemma. The first one was stated by [G] and improved by [D]; it relies on the observation that search by depth first may be broken off if sequents start to repeat themselves in a certain manner. The other method was found by [F] and makes use of the fact that the number of instances of E-rules, all with the same principal formula, on a branch of a deduction may be bounded (viz. by 1 in case of $(E\wedge)$ and $(E\vee)$, and in case of $(E\rightarrow)$ by expressions depending on the form of that principal formula). Both these methods suffer the disadvantage that they often have to keep formulas $u\rightarrow v$ on the left sides of premises of $(E\rightarrow)$, leaving them again as possible conclusions of this rule. This will rise the number of possibilities considerably, since instances of the non invertible rules can, in general, not be permuted.

In this chapter, I shall present a calculus LH which is equivalent to LJ. It has the property that every chain of sequents, constructed during a depth first search, breaks off, either with an axiom or with a sequent not a conclusion of any of its rules. Moreover, with the exception of two rules corresponding to $(I\vee)$ and a subcase of $(E\rightarrow)$, all the rules of LH are invertible, meaning that a much smaller number of possible deductions can arise as result of the search.

Let me begin by defining LH. It shall work with the same sequents as LJ. Axioms of LH are the axioms of LJ; its rules are

$$(HI\wedge)\quad \frac{M\Rightarrow u \quad M\Rightarrow v}{M\Rightarrow u\wedge v} \qquad\qquad (HE\wedge)\quad \frac{M,u,v\Rightarrow r}{M,u\wedge v\Rightarrow r}$$

$$(HI\vee L)\quad \frac{M\Rightarrow u}{M\Rightarrow u\vee v} \qquad (HI\vee R)\quad \frac{M\Rightarrow v}{M\Rightarrow u\vee v} \qquad\qquad (HE\vee)\quad \frac{M,u\Rightarrow r \quad M,v\Rightarrow r}{M,u\vee v\Rightarrow r}$$

$$(HI\rightarrow)\quad \frac{M,u\Rightarrow v}{M\Rightarrow u\rightarrow v}$$

$$(HE\rightarrow a)\quad \frac{M,a,v\Rightarrow r}{M,a,a\rightarrow v\Rightarrow r}\ [a \text{ atomic}] \qquad\qquad (HE\rightarrow\wedge)\quad \frac{M,u\rightarrow(v\rightarrow w)\Rightarrow r}{M,(u\wedge v)\rightarrow w\Rightarrow r}$$

$$(HE\rightarrow\vee)\quad \frac{M,u\rightarrow w,v\rightarrow w\Rightarrow r}{M,(u\vee v)\rightarrow w\Rightarrow r} \qquad\qquad (HE\rightarrow\rightarrow)\quad \frac{M,u,v\rightarrow w\Rightarrow v \quad M,w\Rightarrow r}{M,(u\rightarrow v)\rightarrow w\Rightarrow r}.$$

Here $(HI\vee L)$, $(HI\vee R)$ coincide with $(I\vee L)$, $(I\vee R)$, and the meaning of the rules $(HI\wedge)$, $(HE\wedge)$, $(HE\vee)$, $(HI\rightarrow)$ becomes evident if they are read upwards: they then express the effect of the inversion operators $\mathbf{I}\wedge\mathbf{L}, \mathbf{I}\wedge\mathbf{R}, \mathbf{E}\wedge, \mathbf{E}\vee\mathbf{L}, \mathbf{E}\vee\mathbf{R}, \mathbf{I}\rightarrow$. The last four rules of LH replace $(E\rightarrow)$, and $(HE\rightarrow\wedge)$, $(HE\rightarrow\vee)$ again express the effect of $\mathbf{E}\rightarrow\wedge$ and $\mathbf{E}\rightarrow\vee$. Obviously with the exception of $(HI\vee L)$, $(HI\vee R)$, $(HE\rightarrow\rightarrow)$ the rules of LH are invertible. Consider now a search process relying on the successive application of such inversions; in order to state that it terminates, I will use a measure of complexity of sequents which decreases under such inversions: the premises of the LH-rules should, with respect to this measure, be simpler than the conclusions. In contrast to the rules of LJ, those of LH are not cumulative in the sense that the principal formula also appears in the premises, so that at least in this respect inversion may lead to simpler sequents. Measuring complexity with help of the usual degree of formulas, however, will not suffice in view of the situation in $(HE\rightarrow\wedge)$, $(HE\rightarrow\vee)$, $(HE\rightarrow\rightarrow)$. But if I employ the degree deg introduced before Theorem 1, then it is easily verified that the function

$$\deg(s) = \mathit{Sum} \text{ of the numbers } \deg(v) \text{ for formulas in } s$$

has the property that, for every rule of LH, $\deg(s') < \deg(s)$ for the conclusion s and for each premise s'. In particular, it follows that *every* deduction d of a sequent s will satisfy $l(d) \leqq \deg(s)$!

Thus I obtain the following decision procedure for LH. Perform a search by depth first, considering only the invertible rules as long as possible. It will terminate with sequents which, if not axioms, are conclusions either of $(HI\vee L)$, $(HI\vee R)$ or $(HE\rightarrow\rightarrow)$. In such a sequent s, I collect the candidates for principal formulas of these rules (provided there are any) and test, one by one, whether the respective premises s' have deductions – since $\deg(s') < \deg(s)$ these tests will terminate. In the case of $(HE\rightarrow\rightarrow)$ it is advisable first to test the deducibility of the right premise s'' since $(HE\rightarrow\rightarrow)$ is *semi-invertible* in that together with s also s'' must be derivable.

It remains to prove the equivalence of LH and LJ. I begin with the simple observation that all the rules of LH are admissible for LJ. This is obvious for all but the four $(HE\to)$-rules. For the first three of them, the admissibility is seen by applying cuts to their premises and to the LJ-provable sequents

$$\begin{array}{lll} a, a\to v \Rightarrow v & & \text{for } (HE\to a), \\ (u\wedge v)\to w \Rightarrow u\to(v\to w) & & \text{for } (HE\to\wedge), \\ (u\vee v)\to w \Rightarrow u\to w \quad \text{and} & (u\vee v)\to w \Rightarrow v\to w & \text{for } (HE\to v) \end{array}$$

and simplifying with help of the operator $\mathbf{M}$. Concerning $(HE\twoheadrightarrow)$, an LJ-deduction d_0 of $M, u, v\to w \Rightarrow v$ together with an LJ-deduction d_1 of $M, w \Rightarrow r$ gives an LJ-deduction d of $M, (u\to v)\to w \Rightarrow r$ by

$$\begin{array}{llll}
 & \downarrow d_0 & & \\
 & M, u, v\to w \Rightarrow v & \downarrow e & \\
(I\to) & M, v\to w \Rightarrow u\to v & (u\to v)\to w \Rightarrow v\to w & \downarrow \mathbf{W}(d_1) \\
(\mathrm{CUT}) & \quad M, (u\to v)\to w \Rightarrow u\to v & & M, (u\to v)\to w, w \Rightarrow r \\
 & & M, (u\to v)\to w \Rightarrow r & (E\to)
\end{array},$$

where e is an LJ-deduction of $(u\to v)\to w \Rightarrow v\to w$.

The remainder of this chapter will be devoted to the proof that every LJ-deduction d can be transformed into an LH-deduction $\mathbf{D}(d)$. In view of Theorem 2, d may be assumed as cut free.

For LH we have the same set of inversion operators as for LJ, vic. the operators $\mathbf{I}\wedge\mathbf{L}$, $\mathbf{I}\wedge\mathbf{R}$, $\mathbf{I}\to$, $\mathbf{E}\wedge$, $\mathbf{E}\vee\mathbf{L}$, $\mathbf{E}\vee\mathbf{R}$, $\mathbf{E}\to\mathbf{R}$, $\mathbf{E}\to\wedge$, $\mathbf{E}\to\vee$, and $\mathbf{E}\twoheadrightarrow$. Moreover we have an inversion operator $\mathbf{E}\to$ converting any deduction of a sequent $M, a, a\to v \Rightarrow r$ into a deduction of $M, a, v \Rightarrow r$. The definition of all these operators is straightforward for LH, and it is obvious that none of these operators lengthens deductions. Now we show:

Lemma 4. *Given an LH-deduction d of a sequent $M, u\to v \Rightarrow u$ and an LH-deduction e of $M, v \Rightarrow r$, one obtains an LH-deduction of $M, u\to v \Rightarrow r$.*

This is proved by induction on the length of the d: If it is an axiom, then u is in M, and from $M, v \Rightarrow r$ by an application of $(HE\to)$ I obtain the required sequent.

If the last inference of d is an application of $(HE\twoheadrightarrow)$, then its premises are of the form $N, u_0, u_1\to w, u\to v \Rightarrow u_1$ and $N, w, u\to v \Rightarrow u$. Then from the deduction of the second premise and from $\mathbf{E}\twoheadrightarrow(e)$ by the induction hypothesis I obtain a deduction of $N, w, u\to v \Rightarrow r$, and from this deduction and the deduction of the first premise by an application of $(HE\twoheadrightarrow)$ I obtain the required deduction of $N, (u_0\to u_1)\to w, u\to v \Rightarrow r$.

If this last inference is an application of $(HE\to\vee)$ then its premise is of the form N, $u_0\to w$, $u_1\to w$, $u\to v \Rightarrow u$. Thus from this premise and from $\mathbf{E}\to\vee(e)$ by the induction hypothesis I obtain a deduction of N, $u_0\to w$, $u_1\to w$, $u\to v \Rightarrow r$, and thus by an application of $(HE\to\vee)$ I obtain the required sequent.

The cases that the last inference is by $(HE\to\wedge)$ or $(HE\to)$ are handled analogously.

If the last inference is by $(HE\vee)$, then its premises are of the form $N, u_i, u\to v \Rightarrow u$ for $i=0$ resp. $i=1$ and from these premises and from $\mathbf{E}\vee\mathbf{L}(e)$ resp. $\mathbf{E}\vee\mathbf{R}(e)$ one obtains deductions of $N, u_i, u \to v \Rightarrow r$ from which an application of $(HE \vee)$ deduces the required sequent.

The case that the final inference is by $(HE \wedge)$ is handled in the same way.

Theorem 3. *LH deduces every sequent s which LJ deduces.*

For the proof I set for a sequent s $\deg(s) :=$ the sum of all $\deg(v)$ for the formulas v of s. Now I use induction on the number $3^{\deg(s)} l$, where l is the length of the shortest cutfree LJ-deduction d of s: If the last inference of d is by an application of $(E\to)$ with principal formula $u \to v$, where u is atomic, then the two premises are of the form $N, u \to v \Rightarrow u$ and $N, u \to v, v \Rightarrow r$. Thus by the induction hypothesis both $N, u \to v \Rightarrow u$ and $N, v \Rightarrow r$ are provable by LH. Hence by the lemma $N, u \to v \Rightarrow r$ is also provable by LH.

If the last inference is an application of $(E\to)$ with principal formula $(u_0 \to u_1) \to v$, then the two premises are of the form $N, (u_0 \to u_1) \to v \Rightarrow u_0 \to u_1$ and $N, (u_0 \to u_1) \to v, vn \Rightarrow r$. From the deduction of the first premise one obtains a deduction of $N, u_0, u_1 \to v \Rightarrow u_1$ and from the deduction of the second premise one obtains a deduction of $N, v \Rightarrow r$. Since to these deductions the induction hypothesis applies, an application of $(HE \to\!\!\to)$ gives the required LH-deduction.

If the last inference is an application of $(E\to)$ with principal formula $(u_0 \wedge u_1) \to v$, then, since the degree of the last sequent of $\mathbf{E} \to \wedge (d)$ is smaller than the degree of s, but the length of $\mathbf{E} \to \wedge (d)$ is at most twice the length of d, the sequent s with $(u_0 \wedge u_1) \to v$ replaced by $u_0 \to (u_1 \to v)$ is provable by LH by the induction hypothesis. Thus an application of $(HE \to \wedge)$ yields an LH-deduction of the original sequent.

All the other cases with the exception of the trivial case, where the final inference is an $(I \vee)$, are treated similarly.

(The proof of this theorem as well as the preceding lemma is due to the anonymous referee whose advice I gratefully acknowledge.)

References

[C] Curry, H.B.: Foundations of mathematical logic. New York: McGraw-Hill 1963

[D] Dosen, K.: A note on Gentzen's decision procedure for intuitionistic propositional logic. Z. Math. Logik Grundlagen Math. **33**, 453–456 (1987)

[F] Franzen, T.: Algorithmic aspects of intuitionistic propositional logic, SICS Research Report R87010B

[G] Gentzen, G.: Untersuchungen über das logische Schliessen. Math. Z. **39**, 176–210, 405–431 (1935)

[Go] Gordeev, L.: On cut elimination in the presence of peirce rule. Arch. Math. Logik Grundlagenforschung **26**, 147–164 (1987)

[H] Hudelmaier, J.: A prolog program for intuitionistic logic. SNS-Bericht 88-28, 1988

[W-P] Wilkie, A.J., Paris, J.B.: On the scheme of induction for bounded arithmetic formulas. Ann. Pure Appl. Logic **35**, 261–302 (1987)