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ROTATIONALLY SYMMETRIC HARMONIC MAPS FROM A BALL INTO A WARPED PRODUCT MANIFOLD

Atsushi TACHIKAWA

This paper deals with the existence problem for rotationally symmetric harmonic maps from an Euclidean unit ball $B \subset \mathbb{R}^n$ or $\mathbb{R}^n$ into a warped product manifold $N_f = [0,r_0)\times_f S^{n-1}$.

1. Introduction

Let $S^{n-1} = \{\theta \in \mathbb{R}^n : |\theta|_n = 1\}$ $(n \geqq 3)$, where $|\cdot|_n$ denotes the Euclidean norm of $\mathbb{R}^n$. We always use the n-dimensional representation $\theta = (\theta^1,\dots,\theta^n)$ with $|\theta|_n = 1$ for $\theta \in S^{n-1}$. For $r_0 \in \mathbb{R}_+ \cup \{+\infty\}$ let f be a function of class $C^1([0,r_0),\mathbb{R}_+)$ with $f(0) = 0$, $f'(0) = 1$ and N a product manifold $[0,r_0) \times S^{n-1}$ with projection π^1 and π^2 to $[0,r_0)$ and S^{n-1} respectively. We consider a manifold $N \equiv N_f$ furnished with the Riemannian structure such that

$$(1.1) \qquad ||X||_f^2 = |\pi_*^1(X)|_1^2 + f^2(\pi^1(P))\cdot|\pi_*^2(X)|_n^2$$

for every $X \in T_PN$. A product manifold of this type is called as a warped product manifold (see [2]).

For N_f we use the "polar" coordinate $P = (r,\theta)$ where $r = \pi^1(P)$ and $\theta = \pi^2(P)$. We call the point P with $\pi^1(P) = 0$ the origin of N_f.

We can introduce the normal coordinate system $P = u = (u^1,\dots,u^n)$ by putting $u^i = r\cdot\theta^i$. The relation between the original "polar" coordinate (r,θ) and this normal coordinate can be represented by

$$(r,\theta) = (|u|_n, u/|u|_n).$$

Using this normal coordinate we can identify N_f with $B_{r_0} \subset \mathbb{R}^n$, where we denote

$$B_r = \begin{cases} \{x \in \mathbb{R}^n : |x|_n \leqq r\} & \text{for } r \in \mathbb{R}_+, \\ \mathbb{R}^n & \text{for } r = +\infty . \end{cases}$$

Therefore the Sobolev spaces $H^{1,2}(B_R,N_f)$ and $H_0^{1,2}(B_R,N_f)$ $(R < +\infty)$ can be defined as in [4]. Furtheremore we can define $H^{1,2}_{loc}(\mathbb{R}^n, N_f)$. We denote for $R \in \mathbb{R}_+ \cup \{+\infty\}$

$$\mathcal{S}^n = \mathcal{S}^n(B_R,N_f) = \begin{cases} H^{1,2}(B_R,N_f) & \text{if } R \in \mathbb{R}_+ , \\ H^{1,2}_{loc}(\mathbb{R}^n,N_f) & \text{if } R = +\infty . \end{cases}$$

For $u \in \mathcal{S}^n(B_R,N_f)$ and a bounded open domain $\Omega \subset B_R$, we define the energy of u on Ω by

$$(1.2) \qquad E(u,\Omega) = \frac{1}{2}\int_\Omega \sum_{\alpha=1}^n \|D_\alpha u\|_f^2 \, dx$$

$$= \frac{1}{2}\int_\Omega (|Du^r|_n^2 + f^2(u^r)|Du^\theta|_{n^2}^2)dx.$$

where $u^r = \pi^1 \circ u$, $u^\theta = \pi^2 \circ u$, $|Du^r|_n^2 = \sum_{\alpha=1}^n |\partial u^r/\partial x^\alpha|^2$ and $|Du^\theta|_{n^2}^2 = \sum_{1\leqq\alpha,i\leqq n} |\partial u^{\theta i}/\partial x^\alpha|^2$.

We are going to study the critical point of E . The term "critical" is defined as follows. Denote the space of vector fields along $u \in \mathcal{S}^n(B_R,N_f)$ by

$$\delta_u\mathcal{S}^n(B_R) = \begin{cases} \{v \in H^{1,2}(B_R,\mathbb{R}^n) : (\pi^2 u,v)_n = 0\} & \text{if } R \in \mathbb{R}_+, \\ \{v \in H^{1,2}_{loc}(\mathbb{R}^n,\mathbb{R}^n) : (\pi^2 u,v)_n = 0\} & \text{if } R = +\infty. \end{cases}$$

where $(\cdot,\cdot)_n$ is the Euclidean scalar product on $\mathbb{R}^n$. Then every $v \in \delta_u \mathcal{S}^n(B_R)$ is a tangential vector field along u. Set

$$\delta_u \mathcal{S}^n_0 = \delta_u \mathcal{S}^n(B_R) \cap H^{1,2}_0(B_R, \mathbb{R}^n).$$

For $v \in \delta_u \mathcal{S}^n_0(B_R)$ we put

$$u_{t,v} = (\pi^1 \circ u + t|v|_n\ ,\ (\pi^2 \circ u + tv^\theta)/|\pi^2 \circ u + tv^\theta|_n),$$

denoting $v^\theta = v/|v|_n$, and define the first and second variation of E by

$$(1.3) \qquad \delta_u E(v) = \frac{d}{dt} E(u_{t,v}\ ,\ \operatorname{spt} v)|_{t=0}$$

and

$$(1.4) \qquad \delta^2_u E(v) = \frac{d^2}{dt^2} E(u_{t,v}\ ,\ \operatorname{spt} v)|_{t=0}\ ,$$

where $\operatorname{spt} v$ is the support of v.

u is a "critical" point of E iff

$$\delta_u E(v) = 0 \qquad \text{for } v \in \delta_u \mathcal{S}^n_0(B_R)$$

and u is stable iff

$$\delta^2_u E(v) \geqq 0 \qquad \text{for } v \in \delta_u \mathcal{S}^n_0(B_R)\ .$$

A critical point $u \in \mathcal{S}^n$ is called a weakly harmonic map. Moreover if a weakly harmonic map is of class C^2 we call it a harmonic map.

We consider the following

Problem 1 (Dirichlet problem) Given a boundary value $\psi : \partial B_1 \to N_f$ of class C^2, find a harmonic map $u : B_1 \to N_f$ satisfying the boundary condition $u|_{\partial B_1} = \psi$.

Problem 2 In case of $r_0 = +\infty$, find a harmonic map u from $\mathbb{R}^n$ onto N_f.

In this paper we treat only the case that u is "rotationally symmetric".

We call a map $u : B_R \to N$ rotationally symmetric iff

$$\pi^1 \circ u(x) = \eta(\rho) \quad \text{and} \quad \pi^2 \circ u(x) = \omega \quad ,$$

where $\rho = |x|_n$, $\omega = x/|x|_n$ and η is some function $\mathbb{R}_+ \to \mathbb{R}_+$.

About harmonic maps from B_1 into S^n, some results on existence and the number of rotationally symmetric solutions for the Dirichlet problem have been proved by Jäger and Kaul [5]. It is also shown in [5] that the equator map

$$u_* : B_1 \to S^n, \quad x \mapsto (|x|_n, x/|x|_n),$$

provides an absolute minimum of the energy functional if $n \geqq 7$ with respect to fixed boundary data, but is unstable if $3 \leqq n \leqq 6$.

In case that the target manifold is an ellipsoid

$$N = \{u = (u,z) \in \mathbb{R}^{n+1} : |u|_n^2 + z^2/a^2 = 1\} ,$$

Baldes [1] has shown that the equator map is strictly stable iff $a^2 \geqq 4(n-1)/(n-2)^2$; if not the equator map is unstable.

In this paper we would like to show, how these results change for our case $N = N_f$. Moreover we will get an existence result for Problem 2 .

2. Rotationally symmetric harmonic maps

Let $u \in \mathcal{S}^n(B_R, N_f)$ be a rotationally symmetric map and write

$$(2.1) \qquad u(x) = (\eta(\rho), \omega) \quad ,$$

where $\rho = |x|_n$ and $\omega = x/|x|_n$. Then the energy of u on $B_s \subset\subset B_R$ becomes

$$E(u,B_s) = \frac{1}{2}\int_{B_s} \{|D\eta(\rho)|_n^2 + f^2(\eta(x))|D\omega(\rho)|_{n^2}^2\}dx$$

$$= \frac{1}{2}\int_{B_s} \{\sum_{\alpha=1}^{n} |\eta'(\rho)\ \frac{\partial\rho}{\partial x^\alpha}|^2 + f^2(\eta) \sum_{1\le\alpha, i\le n} |\frac{\partial\omega^i}{\partial x^\alpha}|^2\}dx,$$

and using the equalities

$$\frac{\partial\rho}{\partial x^\alpha} = \frac{\partial|x|_n}{\partial x^\alpha} = \frac{x^\alpha}{|x|_n}$$

and

$$\frac{\partial\omega^i}{\partial x^\alpha} = \frac{\partial(x^i/|x|_n)}{\partial x^\alpha} \quad \frac{\delta^{\alpha i}|x|_n^2 - x^i x^\alpha}{|x|_n^3} \quad ,$$

we get

$$(2.2) \qquad E(u,B_s) = \frac{\omega_n}{2}\int_0^s \{(\eta'(\rho)^2 + \frac{n-1}{\rho^2} f^2(\eta(\rho))\}\rho^{n-1}\ d\rho \quad ,$$

where ω_n denotes the volume of the n-dimensional unit ball.

The first variation becomes

$$\delta_u E(v)$$

$$= \int_{\mathrm{spt}\ v} \{(D\eta, Dv)_n + f\cdot f'(\eta)v|D\omega|_{n^2}^2 + f^2(\eta)(D\zeta, D\omega)_{n^2}\}dx,$$

for $v \in \delta_u \mathcal{S}_0^n$ where $\nu = |v|_n$, $\zeta = v/|v|_n$ and $f\cdot f'(\eta) = f(\eta) \times f'(\eta)$.

Using the fact that $\Delta\omega = 0$ for $x \neq 0$, we can see that the Euler equation reduces to the ordinary differential equation

$$(2.3) \qquad \eta''(\rho) + \frac{(n-1)}{\rho}\eta'(\rho) - \frac{(n-1)f\cdot f'(\eta(\rho))}{\rho^2} = 0 .$$

Setting

$$(2.4) \qquad \phi(t) = \eta(e^t), \qquad \phi : (-\infty, \ln R) \to [0, r_0),$$

we transform (2.3) into an autonomous equation

$$(2.5) \qquad \phi''(t) + (n-2)\phi'(t) - (n-1)f\cdot f'(\phi(t)) = 0.$$

The energy is expressed in terms of ϕ by

$$(2.6) \qquad E(u, B_s) = \frac{\omega_n}{2}\int_{-\infty}^{\ln s}\{(\phi'(t))^2 + (n-1)f^2(\phi(t))\}e^{(n-2)t}dt.$$

Set

$$(2.7) \qquad V(t) = (\phi'(t))^2 - (n-1)f^2(\phi(t)).$$

Then we get

$$(2.8) \qquad V'(t) = -2(n-2)(\phi'(t))^2$$

for every solution of equation (2.5); therefore V is a Lyapunov function for that differential equation.

(2.5) is essentially the damped pendulum equation. Put

$$q(t) = \phi(t), \qquad p(t) = q'(t).$$

Then (2.5) goes over into the plane system

$$(2.9)\quad \begin{pmatrix} q'(t) \\ \\ p'(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ \\ -(n-2)p(t) + (n-1)f\cdot f'(q(t)) \end{pmatrix} .$$

From now on we assume always that

$$(2.10)\qquad f \in C^{2+\alpha}((0,r_0), \mathbb{R}_+) \quad , \; f(0) = 0, \quad f'(0) > 0$$

and

(2.11) $f\cdot f'(r)$ is at most of linear growth as $r \to +\infty$.

Under the condition (2.11) the right hand side of (2.9) is at most of linear growth; therefore the solution ϕ of (2.5) is extendable to a solution on $\mathbb{R}$.

From (2.8) and well known results for Lyapunov functions, α- and ω-limit sets (see [6]), we obtain

Lemma 2.1. *Let* $A = \{a \in \mathbb{R}_+ : f\cdot f'(a) = 0\}$. *For every solution* ϕ *of* (2.5), $(\phi(t),\phi'(t))$ *tends to infinity or converges to one of the critical points* $(a,0)$ *with* $a \in A$, *as* $t \to \pm\infty$.

We consider the following four cases separately for f.

$(i)_f$ There exists a number $b_0 \in (0,r_0)$ with $f(b_0) = 0$ and $f(t) > 0$ on $(0,b_0)$. Moreover there exists a unique $a_0 \in (0,b_0)$ such that $f'(a_0) = 0$,

$(ii)_f$ $f > 0$ on $(0,r_0)$ and there exists exactly one $a_0 \in (0,r_0)$ with $f'(a_0) = 0$, and $f'(t) < 0$ for $t \in (a_0,r_0)$,

$(\text{iii})_f$ $f > 0$ and $f' > 0$ on $(0,+\infty)$,

$(\text{iv})_f$ the remaining cases.

In this paper we treat the cases $(\text{i})_f$, $(\text{ii})_f$ and $(\text{iii})_f$. Therefore there will be at most three critical points $(0,0)$, $(a_0,0)$ and $(b_0,0)$.

The behavior of the system (2.9) in the neighborhood of the critical points $(0,0)$ and $(a_0,0)$ is determined by the linearized system

$$(2.12) \qquad \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (n-1)(f'(0))^2 & -(n-2) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

for (0.0) and

$$(2.13) \qquad \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (n-1)f \cdot f''(a_0) & -(n-2) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

for $(a_0,0)$. The eigenvalues of the matrix of the system (2.12) are

$$(2.14) \qquad \Lambda_{\pm} = -\frac{n-2}{2} \pm \frac{1}{2}\sqrt{(n-2)^2 + 4(n-1)(f'(0))^2}\ ,$$

and those of (2.13) are

$$(2.15) \qquad \lambda_{\pm} = -\frac{n-2}{2} \pm \frac{1}{2}\sqrt{(n-2)^2 + 4(n-1)f \cdot f''(a_0)}\ .$$

Since we are assuming that $f'(0) \neq 0$ (see (2.10)), $\Lambda_+ > 0$ and $\Lambda_- < 0$ for every $n \geqq 2$. Therefore (0.0) is always a saddle point. But $(a_0,0)$ changes its character with n and $f \cdot f''(a_0)$. In case of

$$f\cdot f''(a_0) < -\frac{(n-2)^2}{4(n-1)},$$

$(a_0,0)$ is a focus. In case of

$$0 > f\cdot f''(a_0) > -\frac{(n-2)^2}{4(n-1)},$$

$(a_0,0)$ is an improper node.

Remark: It is easy to see that $f\cdot f''(a_0) \leqq 0$ for our cases $(i)_f$ and $(ii)_f$, because $f' \in C^1$, $f'(t) > 0$ for $t \in [0,a_0)$ and $f'(a_0) = 0$.

By the well-known results about plane autonomous systems (see [3] Ch.VIII), we obtain

Lemma 2.2. *There exists exactly one invariant curve* γ_0 *of system* (2.9) *in the* (q,p)*-plane on which every trajectory* $(q(t),p(t))$ *satisfies*

$$\lim_{t\to-\infty}(q(t),p(t)) = (0,0), \tag{2.16}$$

and, for some $k \in R$, *we have*

$$q(t) > 0 \text{ and } p(t) > 0 \text{ for all } t \in (0,k), \tag{2.17}$$

Moreover the argument of γ_0 *is* $\arctan \Lambda_+$.

Using the properties of a Lyapunov function, we can study the behavior of the solution $(q(t),p(t))$ of (2.9) on the invariant curve γ_0.

Lemma 2.3. *Let* f *satisfy* $(i)_f$ *or* $(ii)_f$. *Then the solution* $(q(t),p(t))$ *of* (2.9) *on* γ_0 *satisfies*

(2.18) $$\lim_{t\to+\infty}(q(t),p(t)) = (a_0,0) .$$

Proof For the case $(i)_f$, the assertion can be shown as in [5].

For the case $(ii)_f$, by Lemma 2.1 , there are only three possiblities:

(a) $$\lim_{t\to+\infty}(q(t),p(t)) = (0,0) ,$$

(b) $$\lim_{t\to+\infty}(q(t),p(t)) = (a_0,0) ,$$

(c) $(q(t),p(t))$ tends to infinity as $t \to +\infty$.

First, suppose that (a) holds. Then $\lim_{t\to+\infty} V(t) = 0$. But we get from (2.16) $\lim_{t\to-\infty} V(t) = 0$ and from (2.8) $V'(t) \leqq 0$. Therefore $V(t)$ and $V'(t)$ must be identically zero. This means $(q(t),p(t)) \equiv (0,0)$ and contradicts our choice of $(q(t),p(t))$.

Now we are going to show that the case (c) also can not happen.

Suppose that $(q(t),p(t))$ tends to infinity. We derive a contradiction as follows.

Step 1. We show that $\limsup_{t\to+\infty} |q(t)| = \infty$.

As mentioned above, $\lim_{t\to-\infty} V(t) = 0$ and $V'(t) \leq 0$, and therefore $V(t) \leqq 0$ for all $t \in R$, i.e.,

(2.19) $$p^2(t) \leqq (n-1)f^2(q(t)).$$

This means that if $q(t)$ remains finite then also $p(t)$ remains finite. Thus we see that $(q(t),p(t))$ can not tend to infinity without $\limsup_{t\to+\infty} |q(t)| = \infty$.

Step 2. We show that $p(t) < 0$ for some $t \in \mathbb{R}$.

Suppose that $p(t) \geqq 0$ for all $t \in \mathbb{R}$. Then by the assertion of Step 1 and the fact that $q'(t) = p(t)$, we can take some $t_0 \in \mathbb{R}$ with $q(t_0) = a_0$. Integrating the equation

$$p'(t) = -(n-2)p(t) + (n-1)f \cdot f'(q(t))$$

from t_0 to $t > t_0$, we get

$$(2.20) \quad p(t) - p(t_0)$$
$$= -(n-2)(q(t) - q(t_0)) + (n-1) \int_{s=t_0}^{t} f \cdot f'(q(s))ds .$$

Since we are supposing that $p(t) > 0$ for all $t \in \mathbb{R}$, $q(t)$ is monotone nondecreasing. On the other hand for the case $(\text{ii})_f$, $f'(q) \leqq 0$ for all $q \geqq a_0$. Therefore

$$f \cdot f'(q(s)) \leqq 0 \quad \text{for all} \quad s \geqq t_0 ,$$

and we get

$$(2.21) \quad p(t) \leqq p(t_0) + (n-2)q(t_0) - (n-2)q(t).$$

Since $p(t_0)$ and $q(t_0)$ are finite and $\lim_{q \to +\infty} q(t) = +\infty$, we see from (2.21) that $p(t) < 0$ for some $t > t_0$. This is a contradiction. Thus the assertion of Step 2 is proved.

Step 3. Using the assertions of Step 1 and Step 2 , we show that the case (c) can not occur.

Because of the assertion of Step 2 , we can take $t_1 \in \mathbb{R}$ with $p(t_1) = 0$ and $p(t) < 0$ for $t \in (t_1, t_1+\delta)$ for some $\delta > 0$. Let γ_1 be the curve $\{(q(t),p(t)): -\infty < t \leqq t_1\}$, $\gamma_2 = \{(q,0): q(t_1) < q < q(t_1) + \varepsilon\}$ for some $\varepsilon > 0$, and $\gamma_3 = \{(q(t_1)+\varepsilon, p): p \leqq 0\}$. Then after $t = t_1$, $(q(t),p(t))$ can not cross the the curve $\gamma_1 \cup \gamma_2 \cup \gamma_3$. In fact, since the vector

$$X(q,p) = \begin{pmatrix} p \\ -(n-2)p + (n-1)f \cdot f'(q) \end{pmatrix}$$

is directed downward on γ_2 and leftward on γ_3, $(q(t), p(t))$ can not cross $\gamma_2 \cup \gamma_3$ after t_1. Moreover, by the uniqueness of the solution, $(q(t),p(t))$ can not cross γ_1. On the other hand, by the assertion of Step 1., $\limsup_{t \to +\infty} |q(t)| = +\infty$. Therefore the curve $(q(t),p(t))$ must cross the half line $\{(0,p): p \leqq 0\}$. This means that $q(t)$ must be zero at some $t = t_2 > t_1$. Taking this $t_2 > t_1$, we see that

$$(2.22) \qquad V(t_2) = 0$$

by (2.19) and (2.7). On the other hand $\lim_{t \to -\infty} V(t) = 0$ (by(2.16)) and $V'(t) \leqq 0$ (by(2.8)). Therefore (2.22) implies $V(t) \equiv 0$ for $t \in (-\infty, t_2]$ and contradicts our choice of $(q(t),p(t))$. Thus the case (c) can not happen.

q.e.d.

Lemma 2.4. *For the case* $(\text{iii})_f$, *the solution* $(q(t),p(t))$ *of* (2.9) *on* γ_0 *satisfies*

$$(2.23) \qquad p(t) \geqq 0 \quad \text{for all} \quad t \in R,$$

and

$$(2.24) \qquad \limsup_{t \to +\infty} q(t) = +\infty$$

Proof Since (2.19) holds for this case, we obtain

$$(2.25) \qquad \limsup_{t \to +\infty} |q(t)| = \infty.$$

Furthermore, proceeding as in Step 3. of the proof of Lemma 2.3., we can see that $p(t)$ can not become negative.

This proves (2.23) and that $q(t)$ is monotone non-decreasing. Therefore, by (2.25), we get (2.24). q.e.d.

Since the system (2.9) is autonomous, the trajectories on γ_0 are determined up to a translation in the parameter t. Moreover for every trajectory $(q(t),p(t))$ on γ_0,

$$\lim_{t\to-\infty} e^{-\Lambda_+ t}P(t) = c \tag{2.26}$$

exists and c uniquely determines $(q(t),p(t))$ (see [3] Ch.VIII. 3.). We define $(q_0(t),p_0(t))$ by choosing $c = 1$.

For the cases $(i)_f$ and $(ii)_f$, using the stability theory, we will study $(q_0(t),p_0(t))$ for $t \to +\infty$ in more detail as in [5] lemma (2.18).

Lemma 2.5. *Suppose that* f *satisfies* $(i)_f$ *or* $(ii)_f$. *Then, writing the curve* $(q_0(t),p_0(t))$ *in the form*

$$q_0(t) + ip_0(t) = s_0(t)e^{i\xi(t)}, \tag{2.27}$$

we obtain

$$0 < s_0(t) < Ce^{\mu t} \quad \textit{for all} \quad t \in \mathbb{R}$$

with a constant $C = C(n,\mu,f)$ *for every* $\mu > \mathrm{Re}\ \lambda_+$. *Moreover,*

(i) *in case* $f\cdot f''(a_0) < -(n-2)^2/4(n-1)$ *the curve* $(q_0(t),p_0(t))$ *is a spiral with center* $(a_0,0)$ *satisfying*

$$\lim_{t\to+\infty} \frac{\xi(t)}{t} = -\frac{1}{2}\sqrt{-(n-2)^2 - 4(n-1)f\cdot f''(a_0)} \ .$$

(ii) *for* $0 > f\cdot f''(a_0) > -(n-2)^2/4(n-1)$

$$\pi + \arctan \lambda_+ < \xi(t) < \pi$$

and

$$\lim_{t\to+\infty} \xi(t) = \pi + \arctan \lambda_+$$

Proof The assertions can be derived as [5] lemma (2.18) by a slight modification of the proof. q.e.d.

These lemmas enable us to study the existence question for the following Dirichlet problem.

Dir(ζ). Find a roationally symmetric harmonic map $u : B_1 \to N_f$ satisfying the boundary condition

$$u|_{\partial B_1} = \psi_\zeta : x \to (\zeta, x) \in N_f .$$

In the cases $(i)_f$ and $(ii)_f$, Lemma 2.5 shows that $q_0(t)$ oscillates about a_0 for $f\cdot f''(a_0)<-(n-2)^2/4(n-1)$ whereas for $0 > f\cdot f''(a_0) > -(n-2)^2/4(n-1)$ it is monotone and tends to a_0 as $t \to +\infty$. In case $f\cdot f''(a_0) < -(n-2)^2/4(n-1)$ we denote the maximum value of q_0 by Σ_0 and the smallest local minimum by σ_0.

Theorem 2.1. For the cases $(i)_f$ and $(ii)_f$, the number of solutions to Dir(ζ) covering the origin $(r=0)$ of N_f is

in case $f\cdot f''(a_0) < -(n-2)^2/4(n-1)$: one for $\zeta \in [0,\sigma_0)$, two for $\zeta = \sigma_0$, an odd number in (σ_0, a_0), countably infinite for $\zeta = a_0$, an even number for $\zeta \in (a_0, \Sigma_0)$, one for $\zeta = \Sigma_0$, zero for $\zeta > \Sigma_0$,

in case $0 > f\cdot f''(a_0) > -(n-2)^2/4(n-1)$: one for $\zeta \in [0,a_0)$, zero for $\zeta \geqq a_0$.

Proof It is enough to check the proof of [5] lemma(2.13) and [5] Theorem 1.. In fact we can constract the solutions of Dir(ζ) from $q_0(t)$ as follows. Take $\tau \in \mathbb{R}$ with $q_0(\tau) = \zeta$ and let $\phi(\rho) = q_0(\tau + \ln \rho)$. Then

$$u(\rho,\omega) = (\phi(\rho),\omega)$$

is a solution of Dir(ζ). Since ϕ is continuous, it follows from the general regularity theory of [4] that u is of class $C^{2+\alpha}$. Moreover we can show that if u is a solution of Dir(ζ) covering the origin , then for its rescaled radius function ϕ, $(\phi(t),\phi'(t))$ must be one of the trajectories on the invariant curve γ_0 . q.e.d.

For case $(\mathrm{iii})_f$, from Lemma 2.4 , we have the following theorem.

Theorem 2.2. *In case $(\mathrm{iii})_f$, for every $\zeta \in \mathbb{R}$ we have a $C^{2+\alpha}$-solution of Dir(ζ). Moreover there exist uncountably many rotationally symmetric harmonic maps from $\mathbb{R}^n$ to N_f with locally finite energy.*

Proof For this case by Lemma 2.4 we see that

$$\lim_{t\to+\infty} q_0(t) = \infty .$$

Therefore we can take $\tau \in R$ with $q_0(\tau) = \zeta$ for any ζ R, and u defined by

$$(2.28)\qquad u(\rho,\omega) = (q_0(\tau + \ln\rho),\omega)$$

solves Dir(ζ). About regularity we can proceed as in Theorem 2.1 . Thus the first part of Theorem 2.2 has been shown.

Now we are going to prove the latter part of this theorem. Let $\phi(t)$ be a solution of (2.5). Then by (2.5) and (2.7) we get

$$(V(t)e^{(n-2)t})'$$

$$= -(n-2)((\phi'(t))^2 + (n-1)f^2(\phi(t))e^{(n-2)t} ,$$

and, writing the energy on $B_R(0)$ as $E_R(u)$,

$$E_R(u) = \frac{-\omega_n}{2(n-2)} \int_{-\infty}^{\ln R} (V(t)e^{(n-2)t})'dt.$$

Moreover if

$$\lim_{t \to -\infty} V(t)e^{(n-2)t} = 0 \tag{2.29}$$

then

$$E_R(u) = \frac{\omega_n}{2(n-2)} [(n-1)f^2(\phi(\ln R)) - (\phi'(\ln R)^2)] < \infty. \tag{2.30}$$

Now for any $\tau \in \mathbb{R}$ let $\phi(t) = q_0(t + \tau)$. Then, since $\lim_{t\to-\infty} q_0(t) = 0$, (2.29) is satisfied. Therefore we obtain $E_R(u_\tau) < \infty$ for

$$u_\tau(\rho,\omega) = (q_0(t + \tau),\omega),$$

for every $\tau \in \mathbb{R}$ and $R \in \mathbb{R}_+$. Thus we obtain a rotationally symmetric harmonic map $u_\tau : \mathbb{R}^n \to N_f$ with locally finite energy for every $\tau \in \mathbb{R}$. Moreover it is clear that $u_\tau \not\equiv u_{\tau'}$ if $\tau \neq \tau'$, and therefore $\#\{u_\tau\} = \#\mathbb{R}$. q.e.d.

3. Stability properties of the equator maps

For the cases $(i)_f$ and $(ii)_f$, the function $\phi(t) \equiv a_0$ is a solution of (2.5) and therefore the map u_* defined by

$$u_*(\rho,\omega) = (a_0,\omega) \tag{3.1}$$

is a singular weakly harmonic map. When $f(t) = \sin t$, N_f is a sphere, $a_0 = \frac{\pi}{2}$ and u_* is the equator map. From now on we call u_* the "equator map" also for any cases.

In this section we investigate the stability properties of the "equator map" u_* for the cases $(i)_f$ and $(ii)_f$. (See [5] and [1] for the cases that the target manifold N is a sphere or an ellipsoid respectively.)

<u>Lemma 3.1.</u> <u>For</u> $v = (\eta,\nu) \in \delta_{u_*}\mathcal{S}_0^n$ <u>we get</u>

$$\delta^2_{u_*}E(v) = \int_{B_1} \{|D\eta|^2_n + \frac{n-1}{\rho^2} f\cdot f''(a_0)\eta^2\}dx + \int_{B_1} f^2(a_0)\{|D\nu|^2_{n^2} - \frac{n-1}{\rho^2}\cdot|\nu|^2_{n^2}\}dx, \tag{3.2}$$

<u>where</u> $\rho = |x|_n$.

<u>Proof</u> Writing $\omega_t = (\omega + t\nu)/|\omega + t\nu|_n$ and $u_t = (a_0 + t\eta, \omega_t)$, we get

$$\frac{d}{dt} E(u_t) = \int_{B_1} \{t|D\eta|_n + \eta f\cdot f'(a_0 + t\eta)|D\omega_t|^2_{n^2} + f^2(a_0 + t\eta)(D(\frac{d}{dt}\omega_t), D\omega_t)_{n^2}\}dx,$$

where $(D\theta, D\nu)_{n^2} = \sum_{1\le\alpha,i\le n} D_\alpha\theta^i D_\alpha\nu^i$ for $\theta,\nu : B \to \mathbb{R}^n$. From this we obtain

$$\frac{d^2}{dt^2} E(u_t)|_{t=0} = \int_B \{|D\eta|^2_n + ((f'(a_0))^2 + f\cdot f''(a_0))\eta^2|D\omega|^2_{n^2} + f^2(a_0)|D\nu|^2_{n^2} - f^2(a_0)(D(\omega|\nu|^2_n), D\omega)_{n^2}\}dx . \tag{3.3}$$

Since $|\omega|_n = 1$, we have

$$(D(\omega|\nu|^2_n), D\omega)_{n^2} = |\nu|^2_n|D\omega|^2_{n^2} + \sum_{\alpha=1}^{n}(\omega, D_\alpha\omega)_n \cdot 2(\nu, D_\alpha\nu)_n = |\nu|^2_n|D\omega|^2_{n^2} + \sum_{\alpha=1}^{n} D_\alpha|\omega|^2_n\cdot(\nu, D_\alpha\nu)_n \tag{3,4}$$

$$= |\nu|_n^2 |D\omega|_{n^2}^2 .$$

Using (3.4) and remarking that $f'(a_0) = 0$, we obtain

$$(3.5)\qquad \delta^2_{u_*} E(v) = \int_{B_1} \{|D\eta|_n^2 + f\cdot f''(a_0)|D\omega|_{n^2}^2 \eta^2\}dx + \\ + f^2(a_0)\int_{B_1} \{|D\nu|_{n^2}^2 - |D\omega|_{n^2}^2 |\nu|_n^2\}dx .$$

On the other hand,

$$D_\alpha \omega^i = \frac{\delta^{\alpha i}|x|_n^2 - x^\alpha x^i}{|x|_n^3} ,$$

and therefore

$$|D\omega|_{n^2}^2 = \frac{n-1}{\rho^2} .$$

Thus, from (3.5) and the above equality we get (3.2). q.e.d.

<u>Lemma 3.1.</u> corresponds lemma 1 in [1]. Proceeding as in [1], we arrive at

<u>Thorem 3.1.</u> <u>The equator map</u> $u_* \in \mathcal{S}^n(B_1, N_f)$ <u>is strictly stable if</u> $0 > f\cdot f''(a_0) > -(n-2)^2/4(n-1)$. <u>If</u> $f\cdot f''(a_0) < -(n-2)^2/4(n-1)$, u_* <u>is unstable</u>.

<u>Proof</u> Since it is enough to check the proof of [1] theorem 1 , we will only give the outline of the proof.

From [1] lemma 3 and 4 , we get

$$(3.6)\qquad \int_{B_1} \{|D\nu|_{n^2}^2 - \frac{n-1}{\rho^2}|\nu|_n^2\}dx \geq 0.$$

From [1] lemma 2,

$$(3.7)\qquad \int_{B_1} \eta^2 \rho^{-2} dx \leq 4(n-2)^{-2} \int_{B_1} |\frac{\partial \eta}{\partial \rho}|^2 dx$$

and therefore

$$(3.8) \qquad \int_{B_1} \{ |D\eta|_n^2 + \frac{n-1}{\rho^2} f \cdot f''(a_0)\eta^2 \} dx \geq 0$$

for $f \cdot f''(a_0) > -(n-2)^2/4(n-1)$.

Combining (3.2), (3.6) and (3.8), we see that if $f \cdot f''(a_0) > -(n-2)^2/4(n-1)$, then

$$\delta^2_{u_*} E(v) \geq 0 ,$$

and as mentioned in [1], equality holds iff $v \equiv 0$.

In case $f \cdot f''(a_0) < -(n-2)^2/4(n-1)$, we choose $\varepsilon > 0$ small enough to ensure

$$h := \frac{1}{4}\{(n-2)^2 + 4(n-1)f \cdot f''(a_0)\} + \varepsilon < 0$$

and $\rho_0 \in (0,1)$ such that $\sqrt{-h} \ln \rho_0$ is a multiple of π. Then we define

$$\eta_0(x) = \begin{cases} \rho^{\frac{n-2}{2}} \sin(\sqrt{-h} \ln \rho) & \rho_0 \leq \rho \leq 1, \\ 0 & \rho \leq \rho_0 , \end{cases}$$

so that η_0 satisfies

$$\Delta\eta_0 - (n-1)f \cdot f''(a_0)\eta_0 = \frac{\varepsilon}{\rho^2}\eta_0 \quad \text{for} \quad \rho_0 \leq \rho \leq 1.$$

Consequently, for $v_0 = (\eta_0, \omega)$

$$\delta^2_{u_*} E(v) = - \int_{\rho_0 \leq |x| \leq 1} \frac{\varepsilon}{\rho^2} |v_0|_n^2 dx < 0 .$$

q.e.d.

References

[1] Baldes,A.: Stability and uniqueness properties of the equator map from a ball into an ellipsoid. Math. Z. 185, 505-516(1984)

[2] Bishop,R.L., O'Neill,B.: Manifolds of negative curvature. Trans. Amer. Math. Soc. 145, 1-49(1969)

[3] Hartman,P.: Ordinary differential equations. New York (1964)

[4] Hildebrandt,S., Kaul,H., Widmann,K.O.: An existence theorem for harmonic mappings of Riemannian manifolds. Acta Math. 138, 1-16(1977)

[5] Jäger,W., Kaul,H.: Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. J. Reine Angew. Math. 343, 146-161(1983)

[6] LaSalle,J.P.: Stability theory for ordinary differenti differential equations. J. differential equations 4, 57-65(1968)

Atsushi Tachikawa

Department of Mathematics
Faculty of Science and Technology
Keio University
Hiyishi 3-14-1,
Kohoku-ku
Yokohama 223, Japan

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