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Simple C^*-Algebras Generated by Isometries

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Abstract. We consider the C^*-algebra $\mathcal{O}_n$ generated by $n \geqq 2$ isometries $S_1, \ldots, S_n$ on an infinite-dimensional Hilbert space, with the property that $S_1 S_1^* + \ldots + S_n S_n^* = \mathbf{1}$. It turns out that $\mathcal{O}_n$ has the structure of a crossed product of a finite simple C^*-algebra $\mathcal{F}$ by a single endomorphism scaling the trace of $\mathcal{F}$ by $1/n$. Thus, $\mathcal{O}_n$ is a separable C^*-algebra sharing many of the properties of a factor of type III_λ with $\lambda = 1/n$. As a consequence we show that $\mathcal{O}_n$ is simple and that its isomorphism type does not depend on the choice of $S_1, \ldots, S_n$.

A C^*-algebra is simple if it contains no non-trivial closed two-sided ideals. We call a simple C^*-algebra with unit infinite if it contains an element X such that $X^*X = \mathbf{1}$ and $XX^* \neq \mathbf{1}$. While non-separable algebras of this type are well known (e.g. the Calkin algebra or type III factors on a separable Hilbert space) there is to my knowledge no explicit example of a separable simple infinite C^*-algebra. The existence of such algebras was proved by Dixmier in [9, 2.1] by the following argument. Let S_1, S_2 be two isometries ($S_i^* S_i = \mathbf{1}, i = 1, 2$) on an infinite-dimensional Hilbert space $\mathcal{H}$ such that $S_1 S_1^* + S_2 S_2^* = \mathbf{1}$. Since the C^*-algebra $C^*(S_1, S_2)$ generated by S_1 and S_2 has a unit, it contains a maximal proper two-sided ideal $\mathcal{J}$. The quotient $C^*(S_1, S_2)/\mathcal{J}$ is separable, simple and infinite. One of the results of the present paper is that $C^*(S_1, S_2)$ itself is already simple (thus answering the question of Dixmier to this effect). More generally, we study the C^*-algebra generated by $n \geqq 2$ isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^{n} S_i S_i^* = \mathbf{1}$ (this condition implies in particular that the range projections $S_i S_i^*$ are pairwise orthogonal). We include the case $n = \infty$. We note incidentally that J. Roberts, motivated by investigations on superselection sectors, has studied closed linear spaces generated by isometries with this property [15]. These spaces are in fact Hilbert spaces and $C^*(S_1, \ldots, S_n)$ is from this point of view the C^*-algebra generated by a Hilbert space.

We construct a faithful conditional expectation of $C^*(S_1, \ldots, S_n)$ onto a C^*-subalgebra $\mathcal{F}$ and show that $C^*(S_1, \ldots, S_n)$ is the crossed product of $\mathcal{F}$ by a single endomorphism Φ (in a sense to be made precise in Section 2). If n is finite, then $\mathcal{F}$ is a

UHF-algebra in the sense of Glimm [12] of type n^∞ and Φ scales the trace of $\mathscr{F}$ by $1/n$. Thus we have here the C^*-analogue of a factor of type III_λ with $\lambda = 1/n$ (cf. [6]). We use this description of $C^*(S_1, \ldots, S_n)$ to show that the isomorphism class of $C^*(S_1, \ldots, S_n)$ does not depend on the choice of $S_1, \ldots, S_n$—that is, if $\hat{S}_1, \ldots, \hat{S}_n$ is a second family of isometries satisfying $\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^* = \mathbf{1}$ then $C^*(\hat{S}_1, \ldots, \hat{S}_n)$ is canonically isomorphic to $C^*(S_1, \ldots, S_n)$. We denote in the following (the isomorphism class of) $C^*(S_1, \ldots, S_n)$ by $\mathcal{O}_n$.

It is then easy to see that $\mathcal{O}_n$ is simple. What is more, $\mathcal{O}_n$ is simple in a very strong sense—for every $0 \neq X \in \mathcal{O}_n$ there are $A, B \in \mathcal{O}_n$ such that $AXB = \mathbf{1}$. Among infinite simple C^*-algebras the algebras $\mathcal{O}_n$ play a universal role comparable to that which UHF-algebras play among antiliminary C^*-algebras. Any simple infinite C^*-algebra $\mathscr{A}$ with unit $\mathbf{1}$ contains, given $n = 2, 3, \ldots, \infty$, a C^*-subalgebra $\mathscr{A}_n$ with $\mathbf{1} \in \mathscr{A}_n$ such that a quotient of $\mathscr{A}_n$ is isomorphic to $\mathcal{O}_n$. For $n = \infty$ the subalgebra $\mathscr{A}_\infty$ can even be chosen in such a way that $\mathscr{A}_\infty$ itself is isomorphic to $\mathcal{O}_\infty$.

Since the algebras $\mathcal{O}_n$ represent quite a new type of C^*-algebras they give rise to a number of counterexamples. From the representation as a crossed product it becomes clear by the recent results in [7], [4] that $\mathcal{O}_n$ is nuclear. On the other hand we show that $\mathcal{O}_n$ can not be an inductive limit of C^*-algebras of type I. This answers to the negative a question which arose naturally in the recent development of the theory of nuclear C^*-algebras (cf. [3]). J. Rosenberg after reading this article showed that $\mathcal{O}_n$ is even amenable [16]. Since $\mathcal{O}_n$ is clearly not strongly amenable this solves a problem of Johnson [13, 10.2].

C^*-algebras generated by isometries have been studied before by various authors. Curiously enough, it usually turns out that the isomorphism class of these C^*-algebras does not depend on the choice of the isometries—but only on their algebraic relations. The difference between the present paper and investigations such as [2, 5, 11] lies in the fact that the isometries considered here are in every respect non-commutative.

We remark further that O. Bratteli has recently shown that the crossed product of the CAR-algebra by a gauge automorphism is simple [1]. However, these automorphisms do not scale the trace, so the algebras obtained are finite.

1. The Algebras $\mathcal{O}_n$

In the following we fix $n = 2, 3, \ldots, \infty$ and a (finite or infinite) sequence $\{S_i\}_{i=1}^n$ of isometries (i.e. $S_i^* S_i = \mathbf{1}$) on a Hilbert space $\mathscr{H}$. If n is finite we assume that $\sum_{i=1}^{n} S_i S_i^* = \mathbf{1}$. If n is infinite we assume that $\sum_{i=1}^{r} S_i S_i^* \leq \mathbf{1}$ for every $r \in \mathbb{N}$. We are going to determine the structure of the C^*-algebra $C^*(S_1, \ldots, S_n)$ (we use this notation also if n is infinite) generated by $\{S_i\}_{i=1}^n$.

1.1. Given $k \in \mathbb{N}$, let W_k^n be the set of all k-tuples $(j_1, \ldots, j_k)$, with $j_i \in \{1, \ldots, n\}$ $(i = 1, \ldots, k)$ if n is finite, or $j_i \in \mathbb{N}$ if n is infinite. Further let $W_0^n = \{0\}$ and $W_\infty^n = \bigcup_{k=0}^{\infty} W_k^n$. We write $S_0 = 1$ and, given $\alpha = (j_1, \ldots, j_k) \in W_k^n$, we denote by S_α the isometry $S_\alpha = S_{j_1} S_{j_2} \ldots S_{j_k}$. Let $\ell(\alpha) = k$ be the length of α and $\ell(0) = 0$.

1.2. With this notation we have the following lemma.

Lemma. a) *Let* $\mu, \nu \in W_\infty^n$ *and* $\ell(\mu)=\ell(\nu)$. *Then* $S_\mu^* S_\nu = \delta_{\mu\nu}\mathbf{1}$.

b) *Let* $\mu, \nu \in W_\infty^n$ *and let* P, Q *be the range projections of* S_μ, S_ν *respectively. Suppose* $S_\mu^* S_\nu \neq 0$.

If $\ell(\mu)=\ell(\nu)$ *then* $S_\mu = S_\nu$ *and* $P=Q$.

If $\ell(\mu)<\ell(\nu)$ *then* $S_\nu = S_\mu S_{\mu'}$ *with* $\mu' \in W_{\ell(\nu)-\ell(\mu)}^n$ *and* $P>Q$.

If $\ell(\mu)>\ell(\nu)$ *then* $S_\mu = S_\nu S_{\nu'}$ *with* $\nu' \in W_{\ell(\mu)-\ell(\nu)}^n$ *and* $P<Q$.

Proof. a) follows easily from the relation $S_i^* S_j = \delta_{ij}\mathbf{1}$.

b) The first assertion follows immediately from a). To prove the second assertion write $S_\nu = S_\alpha S_{\mu'}$ where $\ell(\alpha)=\ell(\mu)$ and $\ell(\mu')=\ell(\nu)-\ell(\mu)$. By a) we have $S_\mu^* S_\alpha S_{\mu'} = \delta_{\mu\alpha} S_{\mu'}$, whence $\alpha=\mu$. Finally $Q = S_\nu S_\nu^* = S_\alpha (S_{\mu'} S_{\mu'}^*) S_\alpha^* < S_\alpha S_\alpha^* = P$.

1.3. Lemma. *Let* $M \neq 0$ *be a word in* $\{S_i\} \cup \{S_i^*\}$. *Then there are two unique elements* μ, $\nu \in W_\infty^n$ *such that* $M = S_\mu S_\nu^*$.

Proof. Let $M = X_1 \dots X_r$ where $X_j \in \{S_i\} \cup \{S_i^*\}$ $(j=1,\dots,r)$. In this expression we may cancel out every term of the form $X_i X_{i+1}$ with $X_i X_{i+1} = \mathbf{1}$. After finitely many such eliminations we get an expression for M in lowest terms $M = Y_1 \dots Y_s$ where $Y_i Y_{i+1} \neq \mathbf{1}$ $(i=1,\dots,s-1)$. Since $S_i^* S_j = \delta_{ij}\mathbf{1}$ and $M \neq 0$, the Y_i must satisfy the following

$$Y_j \in \{S_i\} \Rightarrow Y_{j-1} \in \{S_i\} \quad (j=2,\dots,s).$$

Thus, if j_0 is the largest number between 0 and s such that $Y_{j_0} \in \{S_i\}$, we have $Y_j \in \{S_i\}$ for $1 \leqq j \leqq j_0$ and $Y_j \in \{S_i^*\}$ for $j_0+1 \leqq j \leqq s$. This shows that there are μ, $\nu \in W_\infty^n$ such that $M = S_\mu S_\nu^*$. Assume that $\alpha, \beta \in W_\infty^n$ are such that $M = S_\alpha S_\beta^*$. Then obviously $S_\mu^* S_\alpha \neq 0$ (since $M^* M \neq 0$) and $S_\mu S_\mu^* = MM^* = S_\alpha S_\alpha^*$. Thus the range projections of S_μ and S_α coincide and according to Lemma 1.2b) we get $S_\mu = S_\alpha$. The same argument applied to M^* shows $S_\nu = S_\beta$.

1.4. Let $\mathscr{F}_0^n = \mathbb{C}\mathbf{1}$ and let $\mathscr{F}_k^n$ be the C^*-algebra generated by the set $\{S_\mu S_\nu^* \mid \mu, \nu \in W_k^n\}$. We denote by $\mathscr{M}_r$ the star algebra of $r \times r$ complex matrices and by $\mathscr{K}$ the algebra of compact operators on an infinite dimensional separable Hilbert space.

Proposition. *If* n *is finite then* $\mathscr{F}_k^n$ *is star isomorphic to* $\mathscr{M}_{n^k}$ *and* $\mathscr{F}_k^n \subset \mathscr{F}_{k+1}^n$ $(k=0,1,2,\dots)$. *If* n *is infinite then* $\mathscr{F}_k^n$ *is star isomorphic to* $\mathscr{K}$ *for all* $k>0$.

Proof. According to 1.2a), for $\mu, \mu', \nu, \nu' \in W_k^n$, we have

$$(S_\mu S_\nu^*)(S_{\mu'} S_{\nu'}^*) = \delta_{\nu\mu'} S_\mu S_{\nu'}^*.$$

Since also $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$ this shows that $\{S_\mu S_\nu^* \mid \mu, \nu \in W_k^n\}$ is a self-adjoint system of matrix units generating $\mathscr{F}_k^n$. If n is finite, then

$$S_\mu S_\nu^* = \sum_{i=1}^{n} S_\mu S_i S_i^* S_\nu^*$$

is in $\mathscr{F}_{k+1}^n$ since each summand on the right hand side is in $\mathscr{F}_{k+1}^n$.

1.5. Let $\mathscr{F}^n$ be the C^*-algebra generated by the union of all $\mathscr{F}_k^n$ $(k=0,1,2,\dots)$. Proposition 1.4 shows that $\mathscr{F}^n$ is a UHF-algebra of type n^∞, if n is finite. If n is infinite $\mathscr{F}^\infty$ is not a UHF-algebra but an AF-algebra.

1.6. We are now going to describe the algebra $\mathscr{P}$ generated algebraically by $\{S_i\}_{i=1}^n$ and $\{S_i^*\}_{i=1}^n$. We take and fix one of the S_i, say S_1. To emphasize the special role of

S_1, we will write V for S_1 and V^{-1} for S_1^*. Let $M = S_\mu S_\nu^*$ be a word in $\{S_i\}$ and $\{S_i^*\}$. Let $r = \ell(\mu)$, $s = \ell(\nu)$ and $k = r - s$.

If $k > 0$ set $\hat{M} = S_\mu S_\nu^* S_1^{*k}$. Then $\hat{M} \in \mathscr{F}_r^n$ and $M = \hat{M} V^k$.
If $k < 0$ set $\tilde{M} = S_1^{-k} S_\mu S_\nu^*$. Then $\tilde{M} \in \mathscr{F}_s^n$ and $M = V^k \tilde{M}$.
If $k = 0$ then $M \in \mathscr{F}_r^n = \mathscr{F}_s^n$.

Since any $A \in \mathscr{P}$ is a linear combination of words, A can be written in the form

$$A = \sum_{i=-N}^{-1} V^i A_i + A_0 + \sum_{i=1}^{N} A_i V^i$$

where the A_i are in $\mathscr{F}^n$. We write $A_i = F_i(A)$.

1.7. Proposition. *The elements $A_i = F_i(A)$ are uniquely determined by the construction described above (they do not depend on the special representation of A as a linear combination of words). We have $\|F_i(A)\| \leqq \|A\|$.*

For the proof of this proposition we first need a lemma. Let n be finite and let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ with $\varepsilon_i \in \{1, \dots, n\}$ be a sequence which is aperiodic in the sense that there is no $i_0 > 0$ such that $\{\varepsilon_i\}_{i \geqq i_0}$ becomes periodic. Given $r \in \mathbb{N}$, write $U_r = S_{\varepsilon_1} \dots S_{\varepsilon_r}$ and $P_r = U_r U_r^*$.

1.8. Lemma. *Let $M_1, \dots, M_m$ be words in $S_1, \dots, S_n$ and $S_1^*, \dots, S_n^*$ and let k be a natural number. Suppose that each M_i has the form $M_i = S_\mu S_\nu^*$ where $\ell(\mu) \neq \ell(\nu)$. Then there is $r \in \mathbb{N}$ such that*

$$P_r S_\alpha^* M_i S_\beta P_r = 0$$

for $i = 1, \dots, m$ and for all $\alpha, \beta \in W_k^n$.

Proof. If $M_i = S_\mu S_\nu^*$ where $\ell(\mu) \neq \ell(\nu)$, then $S_\alpha^* M_i S_\beta = 0$ or we have after cancellation $S_\alpha^* M_i S_\beta = S_\gamma S_\delta^*$ in lowest terms where $\ell(\gamma) - \ell(\delta) = \ell(\mu) - \ell(\nu)$ (cf. 1.3). This shows that $S_\alpha^* M_i S_\beta$ also satisfies the hypothesis on M_i of the Lemma for any $\alpha, \beta \in W_k^n$. Thus it suffices to show that for any finite collection $M_1, \dots, M_{m'}$ of words of the form $M_i = S_{\mu_i} S_{\nu_i}^*$ with $\ell(\mu_i) \neq \ell(\nu_i)$, there is $r \in \mathbb{N}$ such that $P_r M_i P_r = 0$ $(i = 1, \dots, m')$. It suffices to prove this for the case $m' = 1$.

Let $\ell(\mu_1) = p$ and $\ell(\nu_1) = q$. Then, for $r > p, q$, the expression $L_r = U^{*r} M_1 U^r$ can be non-zero only if $S_{\mu_1} = U_p$ and $S_{\nu_1} = U_q$ (1.2b)). Thus $L_r = S_{\varepsilon_r}^* \dots S_{\varepsilon_{p+1}}^* S_{\varepsilon_{q+1}} \dots S_{\varepsilon_r}$. But then L_r must be zero for sufficiently large r since by assumption $p \neq q$ and since $\{\varepsilon_i\}$ is aperiodic.

Proof of Proposition 1.7. Since for $i \geqq 0$, by construction $F_{i+1}(A) = F_i(AV^*)$ and for $i \leqq 0$, $F_{i-1}(A) = F_i(VA)$, it suffices to prove the assertions for $F_0(A)$.

We consider first the case that n is finite. Choose an aperiodic sequence $\{\varepsilon_i\}$ as in the preceding lemma. Let k be so large that $F_0(A)$ is in $\mathscr{F}_k^n$. Using Lemma 1.8 we find $r \in \mathbb{N}$, $r > k$ such that $P_r S_\alpha^* V^j A_j S_\beta P_r = 0$ for $j = -N, \dots, -1$ and $P_r S_\alpha^* A_j V^j S_\beta P_r = 0$ for $j = 1, \dots, N$ and for all $\alpha, \beta \in W_k^n$. We set

$$Q = \sum_{\alpha \in W_k^n} S_\alpha P_r S_\alpha^* .$$

Then $QV^j A_j Q = 0$ for $j = -N, \dots, -1$ and $QA_j V^j Q = 0$ for $j = 1, \dots, N$. On the other hand Q commutes with every $X \in \mathscr{F}_k^n$ and $X \mapsto QXQ$ is an isomorphism of $\mathscr{F}_k^n$ onto

$Q\mathscr{F}_k^n Q$. In fact, $QS_\alpha S_\beta^* = S_\alpha S_\beta^* Q = S_\alpha P_r S_\beta^*$ and the set $\{S_\alpha P_r S_\beta^* | \alpha, \beta \in W_k^n\}$ is a self-adjoint system of matrix units generating $Q\mathscr{F}_k^n Q$. Thus

$$\|F_0(A)\| = \|QF_0(A)Q\| = \|QAQ\| \leqq \|A\| .$$

Consider now the case $n = \infty$. There is a finite subset $\mathbb{I}$ of $\mathbb{N}$ such that A is a linear combination of words in S_i, S_i^* ($i \in \mathbb{I}$). We assume that $C^*(S_1, S_2, \ldots)$ is represented on Hilbert space and choose an isometry $\hat{S}$ such that $\hat{S}^*\hat{S} = \mathbf{1}$ and

$$\hat{S}\hat{S}^* = P = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^* .$$

We may assume that $1 \in \mathbb{I}$ and define $\hat{F}_i(X)$ for X in the star algebra $\tilde{\mathscr{P}}$ generated algebraically by S_i, $i \in \mathbb{I}$ and $\hat{S}$, as above with respect to $V = S_1$. Then $\hat{F}_0(A) = F_0(A)$ since A is an expression in S_i, S_i^* only. We know already from above that there is a projection Q in $\tilde{\mathscr{P}}$ such that $QAQ = Q\hat{F}_0(A)Q$ and $\|Q\hat{F}_0(A)Q\| = \|\hat{F}_0(A)\|$. Hence

$$\|F_0(A)\| = \|\hat{F}_0(A)\| = \|Q\hat{F}_0(A)Q\| = \|QAQ\| \leqq \|A\| .$$

Since in the finite *and* in the infinite case the mapping $F_0(A) \mapsto QF_0(A)Q$ is an isomorphism, we finally see that $F_0(A)$ is uniquely determined by $QF_0(A)Q$, hence by A.

1.9. Suppose that $\{\hat{S}_i\}_{i=1}^n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{S}_i\hat{S}_i^* = \mathbf{1}$ and let $\hat{\mathscr{P}}$ be the star algebra generated algebraically by this family. It follows from 1.4 that $\mathscr{F}^n \cap \mathscr{P}$ and $\hat{\mathscr{F}}^n \cap \hat{\mathscr{P}}$ are algebraically isomorphic. Since these algebras are inductive limits of finite-dimensional C^*-algebras, they carry a unique C^*-norm. We may therefore identify $\mathscr{F}^n$ and $\hat{\mathscr{F}}^n$. With this identification, if $A \in \mathscr{P}$ and $\hat{A}$ is the corresponding linear combination of words in $\hat{\mathscr{P}}$, then $F_i(A) = F_i(\hat{A})$ for all $i \in \mathbb{Z}$. In particular, $A = 0$ if and only if $\hat{A} = 0$. This shows that $\mathscr{P}$ and $\hat{\mathscr{P}}$ are algebraically star isomorphic. We equip $\mathscr{P}$ with the largest C^*-norm

$$\|X\|_0 = \sup\{\|\varrho(X)\| \,|\varrho \text{ is a star representation of } \mathscr{P} \text{ on a separable Hilbert space}\}.$$

Let $\mathscr{L}$ be the $\|\cdot\|_0$-completion of $\mathscr{P}$. Since $\|\cdot\|_0$ is a C^*-norm which majorizes the initial norm on $\mathscr{P}$, the C^*-algebra $C^*(S_1, \ldots, S_n)$ is a quotient of $\mathscr{L}$. We shall show that $\mathscr{L} \cong C^*(S_1, \ldots, S_n)$. This will imply

$$C^*(S_1, \ldots, S_n) \cong \mathscr{L} \cong \hat{\mathscr{L}} \cong C^*(\hat{S}_1, \ldots, \hat{S}_n)$$

1.10. The mappings $F_i : \mathscr{P} \to \mathscr{F}^n (i \in \mathbb{Z})$ extend according to Proposition 1.7 to normdecreasing linear mappings $F_i : C^*(S_1, \ldots, S_n) \to \mathscr{F}^n$ and $F_i : \mathscr{L} \to \mathscr{F}^n$ (the use of the same notation for both mappings will not cause confusion). F_0 is a conditional expectation [17, p. 101].

Proposition. *Let $X \in \mathscr{L}$. If $F_i(X) = 0$ for all $i \in \mathbb{Z}$, then $X = 0$.*

Proof. We use an argument which appears in [14, 1.2.5]. Let $\mathscr{L}$ be faithfully represented on $\mathscr{H}$. By definition of the norm on $\mathscr{L}$ the mapping $\varrho_\lambda : S_i \mapsto \lambda S_i (i = 1, \ldots, n)$ extends, for every $\lambda \in \mathbb{C}$ with modulus 1 to a continuous star representation ϱ_λ of $\mathscr{L}$ on $\mathscr{H}$. Note that $\varrho_\lambda(X) = X$ for every $X \in \mathscr{F}^n$.

Given $\xi, \eta \in \mathscr{H}$ with $\|\xi\| = \|\eta\| = 1$, let f be the function on the unit circle $\mathbb{T}$ in $\mathbb{C}$ which is defined by

$$f(\lambda) = (\varrho_\lambda(X)\xi|\eta) \quad (\lambda \in \mathbb{T}).$$

Let $\{A_k\}$ be a sequence in $\mathscr{P}$ which converges in $\mathscr{L}$ to X. Consider the functions

$$h_k(\lambda) = (\varrho_\lambda(A_k)\xi|\eta) \quad (\lambda \in \mathbb{T}).$$

Since $\|\varrho_\lambda(X) - \varrho_\lambda(A_k)\|_0 \leqq \|X - A_k\|_0$, the functions h_k tend to f uniformly on $\mathbb{T}$. We have

$$h_k(\lambda) = \sum_{i=-N}^{-1} (\lambda^i V^i F_i(A_k)\xi|\eta) + (F_0(A_k)\xi|\eta) + \sum_{i=1}^{N} (F_i(A_k)\lambda^i V^i \xi|\eta) = \sum_{i=-N}^{N} a_{ik}\lambda^i.$$

The i-th Fourier-coefficient a_{ik} of h_k converges to the i-th Fourier-coefficient f_i of f as $k \to \infty$.

But $\lim_{k\to\infty} |a_{ik}| \leqq \lim_{k\to\infty} \|F_i(A_k)\|_0 = 0$ by assumption for all $i \in \mathbb{Z}$ so that $f = 0$ and $X = 0$, since ξ, η were arbitrary.

Remark 1. The idea of the proof of 1.10 really consists in interpreting $F_i(X)$ as i-th Fourier coefficient of the function $\lambda \mapsto \varrho_\lambda(X)$ $(\lambda \in \mathbb{T})$. In fact, the equation $F_i(X) = \int_{\mathbb{T}} \varrho_\lambda(X)\lambda^{-i} d\lambda$ holds for every $X \in \mathscr{L}$.

Remark 2. Let $A_k \in \mathscr{P}$ converge to $X \in \mathscr{L}$. Since

$$F_0(X^*X) = \lim_{k\to\infty} \left[\sum_{i<0} F_i(A_k)^* F_i(A_k) + F_0(A_k)^* F_0(A_k) + \sum_{i>0} V^{-i} F_i(A_k)^* F_i(A_k) V^i\right]$$

we see from the proposition that F_0 is faithful in $\mathscr{L}$.

This fact and Proposition 1.10 itself could have been derived in a slightly different approach from the general theory of crossed products [18]. We preferred the proof given above because it is very elementary and fits exactly into the framework of this paper.

1.11. Proposition. *$\mathscr{L}$ is canonically isomorphic to $C^*(S_1, \ldots, S_n)$.*

Proof. The identity mapping $\pi: \mathscr{P} \to \mathscr{P}$ extends to a continuous star homomorphism π of $\mathscr{L}$ onto $C^*(S_1, \ldots, S_n)$. We show that π is injective. We obviously have $F_i \circ \pi = \pi \circ F_i$ [after identification of $\mathscr{F}^n$ and $\pi^{-1}(\mathscr{F}^n)$]. If $\pi(X) = 0$ then $F_i(\pi(X)) = 0$ whence $\pi(F_i(X)) = F_i(X) = 0$ for all $i \in \mathbb{Z}$.

1.12. Theorem. *If $\{\hat{S}_i\}_{i=1}^n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* = \mathbf{1}$ $\left(\text{or } \sum_{i=1}^r \hat{S}_i \hat{S}_i^* \leqq \mathbf{1} \text{ for every } r \in \mathbb{N}, \text{ if } n = \infty\right)$, then $C^*(\hat{S}_1, \ldots, \hat{S}_n)$ is canonically isomorphic to $C^*(S_1, \ldots, S_n)$ (i.e. the map $\hat{S}_i \to S_i$ extends to an isomorphism from $C^*(\hat{S}_1, \ldots, \hat{S}_n)$ onto $C^*(S_1, \ldots, S_n)$).*

Proof. This follows from 1.9 and 1.11. Note that in 1.9 all isomorphisms are canonical.

In view of this it makes sense to write $\mathcal{O}_n$ for $C^*(S_1, \ldots, S_n)$ since the isomorphism class of $\mathcal{O}_n$ does not depend on the choice of $\{S_i\}_{i=1}^n$. We remark that Theorem 1.12 also shows that $\mathcal{O}_n$ is simple. In fact, let $\mathcal{J}$ be a maximal ideal in $\mathcal{O}_n = C^*(S_1, \ldots, S_n)$ and $\pi: \mathcal{O}_n \to \mathcal{O}_n/\mathcal{J}$ the canonical projection mapping. Then, by Theorem 1.12, the simple C^*-algebra $\mathcal{O}_n/\mathcal{J} = C^*(\pi(S_1), \ldots, \pi(S_n))$ is isomorphic to $\mathcal{O}_n$. But we are now going to show that $\mathcal{O}_n$ has a property which is much stronger than simplicity (in [8] we raised the question if every infinite simple C^*-algebra with unit has this property).

1.13. Theorem. *Let n be finite and let X be a non-zero element of $\mathcal{O}_n$. Then there are A, $B \in \mathcal{O}_n$ such that $AXB = \mathbf{1}$.*

Proof. By 1.10 we have $F_0(X^*X) \neq 0$. Without loss of generality assume that $\|F_0(X^*X)\| = 1$. Let $Y \in \mathcal{P}$ be a positive element such that $\|X^*X - Y\| < \varepsilon \leqq 1/4$. Then $\|F_0(Y)\| \geqq 1 - \varepsilon$ (1.7). In the proof of Proposition 1.7 we constructed a projection $Q \in \mathcal{F}^n \cap \mathcal{P}$ such that $\|QF_0(Y)Q\| = \|F_0(Y)\|$ and $QYQ = QF_0(Y)Q$. Let k be so large that $QF_0(Y)Q$ is in $\mathcal{F}_k^n$. Since $\mathcal{F}_k^n$ is a finite-dimensional C^*-algebra, QYQ has the form $QYQ = \sum_{i=1}^{s} \lambda_i R_i$ where R_i are minimal projections in $\mathcal{F}_k^n$ and λ_i are positive real numbers. There is i_0, $1 \leqq i_0 \leqq s$ such that $\lambda_{i_0} \geqq 1 - \varepsilon$ and there is a partial isometry U in $\mathcal{F}_k^n$ such that $U^*U = R_{i_0}$ and $UU^* = S_1^k S_1^{*k}$ (note that $S_1^k S_1^{*k}$ is a minimal projection in $\mathcal{F}_k^n$). Then with $A = S_1^{*k} UQ$ we have $AYA^* = \lambda_{i_0} \mathbf{1}$ and

$$\|AX^*XA^* - \mathbf{1}\| \leqq \|AX^*XA^* - AYA^*\| + \|AYA^* - \mathbf{1}\| \leqq 2\varepsilon$$

(since $\|A\| = 1$ and $1 - \varepsilon \leqq \lambda_{i_0} \leqq 1 + \varepsilon$). This shows that AX^*XA^* is invertible and we are done.

Remark. If in the situation of the preceding theorem $X \geqq 0$ and $\|F_0(X)\| = 1$, then it is obvious from the proof given above that A and B can be chosen such that $\|A\|$, $\|B\| \leqq 1 + \varepsilon$, for any given $\varepsilon > 0$. (Moreover A, B can be chosen such that $B = A^*$.) We will use this in Section 3 where we will prove a version of Theorem 1.13 for $\mathcal{O}_\infty$. A different proof of 1.13 for the case $n = \infty$ could also be given using methods similar (but more complicated) to those employed in the proof above.

2. Representation of $\mathcal{O}_n$ as a Crossed Product

2.1. Let $n \geqq 2$ be finite and let $j \in \mathbb{Z}$. Then $\mathcal{F}^n$ can be represented as an infinite tensor product [17, 1.23.11]

$$\mathcal{F}^n = \bigotimes_{i=j}^{\infty} \mathcal{N}_i = \mathcal{A}_j \quad \text{where} \quad \mathcal{N}_i \cong \mathcal{M}_n \quad \text{for all } i.$$

Define embeddings

$$\mathcal{A}_0 \hookrightarrow \mathcal{A}_{-1} \hookrightarrow \mathcal{A}_{-2} \hookrightarrow \ldots$$

by $\mathcal{A}_j \ni X \mapsto e_{11} \otimes X \in \mathcal{A}_{j-1} = \mathcal{M}_n \otimes \mathcal{A}_j$, where $\{e_{ij} | i, j = 1, \ldots n\}$ denotes a self-adjoint system of matrix units in $\mathcal{M}_n$. If we take the C^*-inductive limit [17, 1.23] of this sequence we get a C^*-algebra $\mathcal{C}_n$ isomorphic to $\mathcal{K} \otimes \mathcal{F}^n$. We may, of course, continue the above sequence of embeddings to positive integers

$$\dots \hookrightarrow \mathscr{A}_2 \hookrightarrow \mathscr{A}_1 \hookrightarrow \mathscr{A}_0 \hookrightarrow \mathscr{A}_{-1} \hookrightarrow \dots$$

in the same way by $\mathscr{A}_j \ni X \mapsto e_{11} \otimes X \in \mathscr{A}_{j-1}$ $(j \in \mathbb{Z})$. Since all $\mathscr{A}_j$ are isomorphic we may consider the automorphism Φ of $\mathscr{C}_n$ which is induced by the shift to the left, mapping an element in $\mathscr{A}_j$ to the corresponding element in $\mathscr{A}_{j+1}$. One may express the action of Φ somewhat informally by $\Phi(X) = e_{11} \otimes X \in e_{11} \otimes \mathscr{A}_j \cong \mathscr{A}_j$ for $X \in \mathscr{A}_{j-1}$.

Let the crossed product $C^*(\mathscr{C}_n, \Phi)$ be faithfully represented on the Hilbert space $\mathscr{H}$. Then there is a unitary U on $\mathscr{H}$ such that $\Phi(X) = UXU^*$ $(X \in \mathscr{C}_n)$ and $C^*(\mathscr{C}_n, \Phi)$ is the closure of the set of finite sums of the form $A = \sum_{i=-N}^{N} X_i U^i$ $(X_i \in \mathscr{C}_n)$. With $\tilde{X}_i = U^{-i} X_i U^i$ this expression becomes

$$A = \sum_{i<0} U^i \tilde{X}_i + X_0 + \sum_{i>0} X_i U^i \quad (\tilde{X}_i, X_i \in \mathscr{C}_n).$$

Let P be the unit of $\mathscr{A}_0 \subset C^*(\mathscr{C}_n, \Phi)$. Since $UPU^* = e_{11} \otimes P \in \mathscr{A}_0 = \mathscr{M}_n \otimes \mathscr{A}_1$ we have $UP = PUP$ and $PX_i U^i P = (PX_i P)(UP)^i$ for $i > 0$ and $PU^i \tilde{X}_i P = (UP)^{*-i} P \tilde{X}_i P$ for $i < 0$. With $V = UP$ we get

$$PAP = \sum_{i<0} V^i P \tilde{X}_i P + PX_0 P + \sum_{i>0} PX_i P V^i.$$

Thus $\mathscr{E}_n = PC^*(\mathscr{C}_n, \Phi)P$ is generated by $\mathscr{A}_0 = P\mathscr{C}_n P$ together with V.

Let $S_i = (e_{i1} \otimes P)V$ $(i = 1, \dots, n)$. Then $S_i^* S_i = P$ and $\sum_{i=1}^{n} S_i S_i^* = P$. Further $\mathscr{A}_0$ is generated by all elements of the form $S_\mu S_\nu^*$ where $\mu, \nu \in W_\infty^n$ and $\ell(\mu) = \ell(\nu)$. In fact, if $\mu = (j_1, \dots, j_k)$ and $\nu = (i_1, \dots, i_k)$, then $S_\mu S_\nu^* = e_{j_1 i_1} \otimes e_{j_2 i_2} \otimes \dots \otimes e_{j_k i_k} \otimes P \in \mathscr{A}_0 = \mathscr{M}_n \otimes \dots \otimes \mathscr{M}_n \otimes \mathscr{A}_k$. Hence $\mathscr{E}_n = C^*(S_1, \dots, S_n) \cong \mathscr{O}_n$.

Let P_k be the unit of $\mathscr{A}_k$ $(k \leqq 0)$. Then $C^*(\mathscr{C}_n, \Phi)$ is the inductive limit of $P_k C^*(\mathscr{C}_n, \Phi) P_k$ $(k \to -\infty)$. It is not hard to see that $P_{k-1} C^*(\mathscr{C}_n, \Phi) P_{k-1}$ is generated by $P_k C^*(\mathscr{C}_n, \Phi) P_k$ together with $\{e_{ij} \otimes P_k \mid 1 \leqq i, j \leqq n\} \subset \mathscr{A}_{k-1}$ and that, consequently, $C^*(\mathscr{C}_n, \Phi)$ is isomorphic to $\mathscr{K} \otimes \mathscr{O}_n$.

2.2. Let now $n = \infty$. For $j \in \mathbb{N}$ let $\mathscr{A}_j$ be the C^*-subalgebra of $\mathscr{O}_\infty$ defined by $\mathscr{A}_j = S_1^j \mathscr{F}^\infty S_1^{*j}$. Then $\mathscr{A}_{j-1} \cong \mathbb{C}1 \oplus (\mathscr{K} \otimes \mathscr{A}_j)$. On the other hand we also have $\mathscr{A}_i \cong \mathscr{A}_0 = \mathscr{F}^\infty$ for all $i \in \mathbb{N}$. Define $\mathscr{A}_j$ for negative j inductively by $\mathscr{A}_{j-1} = \mathbb{C}1 \oplus (\mathscr{K} \otimes \mathscr{A}_j)$. We fix a minimal projection R in $\mathscr{K}$ and consider the sequence of embeddings

$$\mathscr{A}_0 \hookrightarrow \mathscr{A}_{-1} \hookrightarrow \mathscr{A}_{-2} \hookrightarrow \dots$$

defined by $\mathscr{A}_j \ni X \mapsto R \otimes X \in \mathscr{K} \otimes \mathscr{A}_j \subset \mathscr{A}_{j-1}$. Let $\mathscr{C}_\infty$ be the inductive limit of this sequence. Clearly $\mathscr{C}_\infty$ is an AF-algebra. If as above we let Φ be the automorphism of $\mathscr{C}_\infty$ which is induced by the shift to the left on the above sequence (continued to positive integers) then $\mathscr{O}_\infty \cong PC^*(\mathscr{C}_\infty, \Phi)P$ where P is the unit of $\mathscr{A}_0 \subset C^*(\mathscr{C}_\infty, \Phi)$.

2.3. We have seen that $\mathscr{O}_n$ $(n = 2, \dots, \infty)$ is isomorphic to the crossed product of an AF-algebra by a single automorphism, cut down by a projection. By recent results of Connes [7, 6.8, 6.5, Theorem 6] and Choi and Effros [4, Corollary 3.2] this proves that $\mathcal{O}_n$ is nuclear. I am indebted to A. Connes and S. Sakai who called my attention to this fact. We show now that $\mathcal{O}_n$ can not be obtained as an inductive limit of type I C^*-algebras.

Proposition. *Let n be finite and let $S_1, \ldots, S_n$ be isometries on a Hilbert space $\mathcal{H}$ satisfying $\sum_{i=1}^{n} S_i S_i^* = P \leqq \mathbf{1}$. Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a C^*-algebra containing elements $A_1, \ldots, A_n$ such that $\|A_i - S_i\| < \varepsilon$. If ε is sufficiently (depending on n) small then there are $\tilde{A}_1, \ldots, \tilde{A}_n \in \mathcal{A}$ such that $\tilde{A}_i^* \tilde{A}_i = \mathbf{1}$ and $\sum_{i=1}^{n} \tilde{A}_i \tilde{A}_i^* \leqq \mathbf{1}$. If $P = \mathbf{1}$ then $\tilde{A}_1, \ldots, \tilde{A}_n$ can be chosen such that the sum of the range projections of $\tilde{A}_i$ equals $\mathbf{1}$.*

Proof. Let $\varepsilon < 1/10$. We have

$$\|A_i^* A_i - \mathbf{1}\| \leqq \|A_i^* A_i - A_i^* S_i\| + \|A_i^* S_i - S_i^* S_i\| \leqq (1+\varepsilon)\varepsilon + \varepsilon < 3\varepsilon .$$

Hence $A_i^* A_i$ is invertible and

$$\|A_i - A_i (A_i^* A_i)^{-\frac{1}{2}}\| \leqq \|A_i\| \, \|1 - A_i^* A_i^{-\frac{1}{2}}\| < (1+\varepsilon)\, 3\varepsilon < 4\varepsilon .$$

Now $V_i = A_i (A_i^* A_i)^{-\frac{1}{2}}$ is an isometry and

$$\|V_i V_i^* - S_i S_i^*\| \leqq \|V_i V_i^* - S_i V_i^*\| + \|S_i V_i^* - S_i S_i^*\| < 5\varepsilon + 5\varepsilon = 10\varepsilon .$$

Further

$$\|(V_i V_i^*)(V_j V_j^*)\| \leqq \|(S_i S_i^*)(S_j S_j^*)\| + \|(S_i S_i^* - V_i V_i^*) S_j S_j^*\| + \|V_i V_i^* (V_j V_j^* - S_j S_j^*)\| < 20\varepsilon \quad \text{for} \quad i \neq j .$$

Given $\delta > 0$, by [12, 1.7], if ε is sufficiently small there is a family of pairwise orthogonal projections $E_1, \ldots, E_n$ in $\mathcal{A}$ such that $\|E_i - V_i V_i^*\| < \delta$. Then $\|E_i V_i - V_i\| < \delta$. Thus $V_i^* E_i V_i$ is invertible for small δ and the elements $\tilde{A}_i = (E_i V_i)(V_i^* E_i V_i)^{-\frac{1}{2}}$ are isometries. Moreover the elements $\tilde{A}_i \tilde{A}_i^* = = E_i$ are pairwise orthogonal projections and $Q = \sum_{i=1}^{n} \tilde{A}_i \tilde{A}_i^*$ is a projection such that

$$\|Q - P\| = \left\| \sum_{i=1}^{n} (E_i - S_i S_i^*) \right\| \leqq n(\delta + 10\varepsilon) .$$

In particular $Q = \mathbf{1}$ if $P = \mathbf{1}$ and ε and δ are sufficiently small.

Corollary 1. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{O}_n$ (n finite) containing elements $A_1, \ldots, A_n$ such that $\|A_i - S_i\| < \varepsilon$. If ε is sufficiently (depending on n) small then any such $\mathcal{A}$ must contain a C^*-subalgebra which is isomorphic to $\mathcal{O}_n$.*

Corollary 2. *An infinite simple C^*-algebra $\mathcal{B}$ with unit can not be an inductive limit of type I C^*-algebras.*

Proof. By [8, 2.2] $\mathcal{B}$ contains isometries V_1, V_2 such that $V_1 V_1^* + V_2 V_2^* \leqq \mathbf{1}$. Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}$ containing elements A_1, A_2 such that $\|A_i - V_i\| < \varepsilon$. If ε is sufficiently small, then $\mathcal{A}$ contains isometries $\tilde{A}_1, \tilde{A}_2$ such that $\tilde{A}_1 \tilde{A}_1^* + \tilde{A}_2 \tilde{A}_2^* \leqq \mathbf{1}$. Since a quotient of $C^*(\tilde{A}_1, \tilde{A}_2)$ is isomorphic to $\mathcal{O}_2$ (3.1) and $\mathcal{O}_2$ is clearly not of Type I, $\mathcal{A}$ can not be of type I.

2.4. As $\mathcal{O}_n$ is simple, so is $\mathcal{K}\otimes\mathcal{O}_n$. But $\mathcal{K}\otimes\mathcal{O}_n$ is even algebraically simple (i.e. has no non-trivial not necessarily closed two-sided ideals). This follows from the following general theorem.

Theorem. *Let $\mathcal{A}$ be a simple C^*-algebra with unit. Then $\mathcal{K}\otimes\mathcal{A}$ is algebraically simple if and only if there is $k\in\mathbb{N}$ such that $\mathcal{M}_k\otimes\mathcal{A}$ is infinite.*

Proof. "Only if part". We use the notation of [8]. Assume that $\mathcal{M}_k\otimes\mathcal{A}$ is finite and let P be a projection of dimension r and Q a projection of dimension 1 in $\mathcal{M}_k$. Then $(P\otimes\mathbf{1}/Q\otimes\mathbf{1})=r$ in $\mathcal{M}_k\otimes\mathcal{A}$. In fact, we have $a=(P\otimes\mathbf{1}/Q\otimes\mathbf{1})\leqq r$. On the other hand $a<r$ would imply $(P\otimes\mathbf{1}/R\otimes\mathbf{1})=1$ for any projection $R\leqq P$ of dimension a in $\mathcal{M}_k$. Since $P\otimes\mathbf{1}$ is a finite projection in $\mathcal{M}_k\otimes\mathcal{A}$ [8, 2.4], this is impossible [8, 2.1]. Assume now that $\mathcal{M}_k\otimes\mathcal{A}$ is finite for any $k\in\mathbb{N}$. If P is a projection of dimension r and Q a projection of dimension 1 in $\mathcal{K}$ then $(P\otimes\mathbf{1}/Q\otimes\mathbf{1})$ in $\mathcal{K}\otimes\mathcal{A}$ equals $(P\otimes\mathbf{1}/Q\otimes\mathbf{1})$ in $(P\otimes\mathbf{1})(\mathcal{K}\otimes\mathcal{A})(P\otimes\mathbf{1})\cong\mathcal{M}_r\otimes\mathcal{A}$ hence equals r (we may assume $Q\leqq P$). Let $P_1, P_2, \ldots$ be a sequence of one-dimensional orthogonal projections in $\mathcal{K}$ and let $H=\sum_{i=1}^{\infty}\lambda_i P_i$ where $\lambda_i>0$ and $\lambda_i\to 0$.

Then for any $r\in\mathbb{N}$ and for any one-dimensional projection Q in $\mathcal{K}$ we have

$$H\gtrapprox\sum_{i=1}^{r}P_i=A_r \quad\text{and}\quad (H\otimes\mathbf{1}/Q\otimes\mathbf{1})\geqq(A_r\otimes\mathbf{1}/Q\otimes\mathbf{1})=r.$$

This shows that the ideal generated algebraically by $Q\otimes\mathbf{1}$ in $\mathcal{K}\otimes\mathcal{A}$ does not contain $H\otimes\mathbf{1}$.

"If part". The proof is essentially contained already in [10, 3.1.4]. We have only to combine Dixmier's argument with [8, 2.2]. We may assume that $\mathcal{A}$ itself is infinite. Let $E_1, E_2, \ldots$ be a sequence of pairwise orthogonal one-dimensional projections in $\mathcal{K}$ such that the sequence $\{H_k\}_{k=1}^{\infty}$, defined by $H_k=\sum_{i=1}^{k}E_i$, is an approximate identity for $\mathcal{K}$. It is easy to see that $H_k\otimes\mathbf{1}$ is an approximate identity for $\mathcal{K}\otimes\mathcal{A}$ (it is enough to check this for the algebraic tensor product of $\mathcal{K}$ and $\mathcal{A}$).

Let $\mathcal{J}$ be a non-zero ideal of $\mathcal{K}\otimes\mathcal{A}$. If $X\neq 0$ is in $\mathcal{J}$ then there is k such that $(H_k\otimes\mathbf{1})X(H_k\otimes\mathbf{1})\neq 0$ hence there are i,j, $1\leqq i,j\leqq k$ such that $(E_i\otimes\mathbf{1})X(E_j\otimes\mathbf{1})\neq 0$. If $E_{ij}\in\mathcal{K}$ is a partial isometry with support projection E_j and range projection E_i then $(E_i\otimes 1)X(E_{ij}\otimes 1)^*$ is in $\mathcal{J}$ and is non-zero. Thus $\mathcal{J}\cap E_i\otimes\mathcal{A}$ is non-zero, hence equals $E_i\otimes\mathcal{A}$ since $\mathcal{A}\cong E_i\otimes\mathcal{A}$ is algebraically simple.

From [8, 2.2] using induction we get the existence of infinitely many pairwise orthogonal projections F_i and elements V_i in $\mathcal{A}$ such that $V_i^*V_i=\mathbf{1}$ and $V_iV_i^*=F_i$ $(i=1,2,\ldots)$. We have $E_1\otimes F_i\sim E_1\otimes\mathbf{1}\sim E_i\otimes\mathbf{1}$ in $\mathcal{K}\otimes\mathcal{A}$. Let U_i be a partial isometry in $\mathcal{K}\otimes\mathcal{A}$ with range projection $E_1\otimes F_i$ and support projection $E_i\otimes\mathbf{1}$. With $G_k=\sum_{i=1}^{k}F_i$ and $Y_k=\sum_{i=1}^{k}U_i$ we have $Y_kY_k^*=E_1\otimes G_k$ and $Y_k^*Y_k=H_k\otimes\mathbf{1}$.

To complete the proof it is enough to show that any positive element X of $\mathcal{K}\otimes\mathcal{A}$ is in $\mathcal{J}$. Since $(H_k\otimes\mathbf{1})X^{\frac{1}{2}}$ is a Cauchy sequence also $Y_kX^{\frac{1}{2}}$ is a Cauchy sequence converging to an element Y of $\mathcal{K}\otimes\mathcal{A}$. Since $(E_1\otimes\mathbf{1})\,Y=Y$ and $E_1\otimes\mathbf{1}\in\mathcal{J}$ we have Y, $Y^*\in\mathcal{J}$. Therefore $Y^*Y=X$ is in $\mathcal{J}$.

Remark. Let A, $B\in\mathcal{K}\otimes\mathcal{O}_n$ and $B\neq 0$. There are i, $j\in\mathbb{N}$ such that $(E_i\otimes\mathbf{1})B(E_j\otimes\mathbf{1})\neq 0$. Let $C=(E_{1i}\otimes\mathbf{1})(E_i\otimes\mathbf{1})B(E_j\otimes\mathbf{1})(E_{j1}\otimes\mathbf{1})$ (E_{ij}= partial isometry in $\mathcal{K}$ with support projection E_j and range projection E_i). Then $C\neq 0$ and $C\in E_1\otimes\mathcal{O}_n$. There are F, G in $\mathcal{O}_n$ such that $(E_1\otimes F)C(E_1\otimes G)=E_1\otimes\mathbf{1}$ (1.13, 3.4).

Further there are $X_1,\ldots,X_r$ and $Y_1,\ldots,Y_r$ in $\mathcal{K}\otimes\mathcal{O}_n$ such that $A=\sum_{i=1}^{r} X_i(E_1\otimes\mathbf{1})Y_i$ (the ideal generated by $E_1\otimes\mathbf{1}$ in $\mathcal{K}\otimes\mathcal{O}_n$ consists exactly of all finite sums of this form). Let $V_1,\ldots,V_r$ be isometries in $\mathcal{O}_n$ such that $V_1V_1^*,\ldots,V_rV_r^*$ are pairwise orthogonal projections in $\mathcal{O}_n$. Then

$$A=\left(\sum_{i=1}^{r} X_i(E_1\otimes V_i^*)\right)(E_1\otimes\mathbf{1})\left(\sum_{i=1}^{r}(E_1\otimes V_i)\,Y_i\right).$$

Together this shows that there are X, $Y\in\mathcal{K}\otimes\mathcal{O}_n$ such that $A=XBY$.

3. Extensions of $\mathcal{O}_n$

3.1. Proposition. *Let $V_1,\ldots,V_n$ be isometries on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^{n} V_iV_i^*\leqq\mathbf{1}$ (n finite). Then the projection $P=\mathbf{1}-\sum_{i=1}^{n} V_iV_i^*$ generates a closed two-sided ideal $\mathcal{J}$ in $C^*(V_1,\ldots,V_n)$ which is isomorphic to $\mathcal{K}$ and contains P as a minimal projection. The quotient $C^*(V_1,\ldots,V_n)/\mathcal{J}$ is isomorphic to $\mathcal{O}_n$.*

Proof. Define, given $\mu\in W_\infty^n$, an isometry V_μ in the same way S_μ was defined in Section 1. The closure of the set $\mathcal{J}$ of all linear combinations of elements of the form $V_\mu P V_\nu^*$ ($\mu,\nu\in W_\infty^n$) is clearly a two-sided ideal in $C^*(V_1,\ldots,V_n)$. On the other hand $\mathcal{J}$ is contained in every two-sided ideal containing P.

Consider the product $X=(V_\mu PV_\nu^*)\,(V_\alpha PV_\beta^*)$ ($\mu,\nu,\alpha,\beta\in W_\infty^n$). After cancellation we have $V_\nu^*V_\alpha=V_\gamma V_\delta^*$ ($\gamma,\delta\in W_\infty^n$) in lowest terms (1.3). But $PV_\gamma V_\delta^*P\neq 0$ if and only if $V_\gamma V_\delta^*=\mathbf{1}$, since $PV_i=0$ ($i=1,\ldots,n$). Thus $X\neq 0$ if and only if $PV_\nu^*V_\alpha P\neq 0$ if and only if $\nu=\alpha$ (1.2). Hence

$$(V_\mu PV_\nu^*)(V_\alpha PV_\beta^*)=\delta_{\nu\alpha}V_\mu PV_\beta^*$$

and

$$(V_\mu PV_\nu^*)^*=V_\nu PV_\mu^*.$$

In other words the set $\{V_\mu PV_\nu^*|\mu,\nu\in W_\infty^n\}$ is a self-adjoint system of matrix units generating $\mathcal{J}$. Therefore $\mathcal{J}$ can be mapped isomorphically onto a dense star subalgebra of $\mathcal{K}$ which is an inductive limit of finite-dimensional C^*-algebras, hence carries a unique C^*-norm. This mapping must be isometric and extends to an isomorphism of $\mathcal{J}=\bar{\mathcal{J}}$ onto $\mathcal{K}$.

Remark 1. It seems to be interesting to study more general extensions of $\mathcal{O}_n$ by the compacts.

Remark 2. In the situation of the proposition, given i ($1\leqq i\leqq n$) and $\mu,\nu\in W_\infty^n$, there is $k\in\mathbb{N}$ such that $V_i^{*k}V_\mu P\,V_\nu^*V_i^k=0$. This shows that $V_i^{*k}AV_i^k$ tends to zero as $k\to\infty$ for each $A\in\mathcal{J}$.

3.2. Let $\mathscr{A}$ be a simple C^*-algebra with unit. It follows by induction from [8, 2.2] that $\mathscr{A}$ contains a sequence $V_1, V_2, \ldots$ of isometries satisfying $\sum_{i=1}^{k} V_i V_i^* \leqq \mathbf{1}$ for every $k \in \mathbb{N}$. We know already from Section 1 that $C^*(V_1, V_2, \ldots) \cong \mathcal{O}_\infty$. From 3.1 we see that $C^*(V_1, \ldots, V_n)$ ($n \geqq 2$ finite) contains a closed two-sided ideal $\mathscr{J}$ such that $C^*(V_1, \ldots, V_n)/\mathscr{J} \cong \mathcal{O}_n$. Therefore $\mathcal{O}_\infty$ is contained (with the same unit) in $\mathscr{A}$ and $\mathcal{O}_n$ is for any finite $n \geqq 2$ contained up to quotients in $\mathscr{A}$.

3.3. Consider $\mathcal{O}_2 = C^*(S_1, S_2)$. We put $\hat{S}_1 = S_1^2$, $\hat{S}_2 = S_1 S_2$, and $\hat{S}_3 = S_2$. Then $\hat{S}_i^* \hat{S}_i = \mathbf{1}$ and $\sum_{i=1}^{3} \hat{S}_i \hat{S}_i^* = \mathbf{1}$ so that $\mathcal{O}_3 \cong C^*(\hat{S}_1, \hat{S}_2, \hat{S}_3) \subset \mathcal{O}_2$. By induction we get the following chain of inclusions

$$\mathcal{O}_2 \supset \mathcal{O}_3 \supset \mathcal{O}_4 \supset \ldots \supset \mathcal{O}_\infty .$$

3.4. We use 3.1 to prove a version of 1.13 for $\mathcal{O}_\infty$.

Theorem. *Let X be a non-zero element of $\mathcal{O}_\infty$. Then there are $A, B \in \mathcal{O}_\infty$ such that $AXB = \mathbf{1}$.*

Proof. We may assume that $X \geqq 0$ and $\|F_0(X)\| = 1$. Let Y be a positive element of the star algebra generated algebraically by $S_1, S_2, \ldots$ such that $\|X - Y\| < \varepsilon < 1/4$. Without loss of generality we may assume that $\|F_0(Y)\| = 1$.

There is a finite subset $\mathbb{I}$ of $\mathbb{N}$ such that Y is a linear combination of words in S_i, S_i^* ($i \in \mathbb{I}$). We assume that $\mathcal{O}_\infty$ is represented on the Hilbert space $\mathscr{H}$ and choose an isometry $\hat{S}$ on $\mathscr{H}$ such that $\hat{S}\hat{S}^* = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^*$. Further we fix $i_0 \in \mathbb{N}$ such that $i_0 \notin \mathbb{I}$. We consider the C^*-algebras $\mathscr{A}_1$, generated by S_i ($i \in \mathbb{I}$) together with $\hat{S}$, and $\mathscr{A}_2$, generated by S_i ($i \in \mathbb{I}$) together with S_{i_0}. The projection $P = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^* - S_{i_0} S_{i_0}^*$ generates a non-trivial closed two-sided ideal $\mathscr{J}$ in $\mathscr{A}_2$ (3.1) and $\mathscr{A}_2/\mathscr{J}$ is canonically isomorphic to $\mathscr{A}_1$ (1.12).

We may assume that $1 \in \mathbb{I}$ and define $\hat{F}_i$ in $\mathscr{A}_1$ with respect to S_1 and $\tilde{F}_i$ in $\mathscr{A}_2/\mathscr{J}$ with respect to $\varrho(S_1)$ (where $\varrho : \mathscr{A}_2 \to \mathscr{A}_2/\mathscr{J}$ is the canonical mapping) in the same way in which F_i was defined in Section 1. Then $\hat{F}_0(Y) = F_0(Y)$ since Y is an expression in S_i, S_i^* ($i \in \mathbb{I}$) only. Therefore

$$\|\tilde{F}_0(\varrho(Y))\| = \|\hat{F}_0(Y)\| = \|F_0(Y)\| = 1 .$$

By the remark in 1.13 there are $A, B \in \mathscr{A}_2/\mathscr{J}$ such that $A\varrho(Y)B = \mathbf{1}$ and $\|A\|$, $\|B\| < 1 + \varepsilon$. Then A, B can be lifted to elements $\tilde{A}$, $\tilde{B}$ in $\mathscr{A}_2$ such that $\|\tilde{A}\|$, $\|\tilde{B}\| < 1 + 2\varepsilon$. We have $\tilde{A} Y \tilde{B} = \mathbf{1} + K$ with $K \in \mathscr{J}$. By Remark 2 in 3.1 we get $S_i^{*k}(\tilde{A} Y \tilde{B}) S_i^k \to \mathbf{1}$ as $k \to \infty$ for each $i \in \mathbb{I}$. Since

$$\|S_i^{*k}(\tilde{A} X \tilde{B}) S_i^k - S_i^{*k}(\tilde{A} Y \tilde{B}) S_i^k\| < (1 + 2\varepsilon)^2 \varepsilon < 1$$

this shows that $S_i^{*k}(\tilde{A} X \tilde{B}) S_i^k$ is invertible for sufficiently large k.

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