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Second order logarithmic corrections to the Drell–Yan cross-section

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Abstract. We present a complete order α_s^2 calculation of the large logarithmic terms of the type $\ln^i(1-x)/(1-x)$ ($x = Q^2/s$), which appear in the Wilson coefficient of the total and differential DY cross-sections. These terms are computed using renormalization-group methods. It is shown that besides the well known constant part, they constitute the bulk of the radiative correction. This in particular holds for the higher τ-region which is still accessible to experiment. The large logarithmic corrections determine the shape of the K-factor and give a partial explanation of the phenomenon of anomalous scaling.

I Introduction

During the last ten years much effort was spent by theorists and experimentalists in confronting the data of many deep inelastic processes with predictions of perturbative QCD. The results are rather impressive, in particular as far as the size of the various total and differential cross-sections are concerned. Nevertheless, the theory is still not able to provide us with an accurate description of the precise details shown by the experimental distributions. As an example we want to mention the K-factor [1, 2] and the behaviour of the v parameter in the angular decay distribution [3] of the muon-pair in the Drell–Yan (DY) process $\pi + W \rightarrow \mu^+ \mu^- +$ "hadrons". Both of them are in disagreement with the predictions of perturbative QCD. These discrepancies are due to several problems which arise in experiment as well as in theory.

In experiment, both statistical and systematic errors lead to uncertainties in the determination of the nucleon structure functions. Further, if one measures these structure functions in deep inelastic scattering on nuclear targets one has to correct for the Fermi-motion and to deal with the so called EMC effect [4]. As has been shown in [5] the above uncertainties influence the parameters in the pion structure function and explain partially the effect of anomalous scaling of the K-factor.

As far as the theory is concerned, the measurement of the parameters in the parton-distribution functions and the running coupling constant depend heavily on our knowledge of the underlying deep inelastic processes. However, the nonperturbative properties of QCD are poorly understood. Here we want to mention the evolution of parton-jets into the observed hadrons and the question of higher twist effects. Our understanding of the perturbative aspects of the theory is much better, but one has to bear in mind that at energies used in the above experiment, the running coupling constant is rather big. This leads to a slow convergence of the perturbation series. Moreover, hitherto most processes have only been calculated up to the next-to-leading order. Some of these corrections turn out to be very large and are of the same size as the Born term. An example is the Wilson-coefficient of the DY total and differential cross-section [6–8].

A thorough analysis reveals that these large corrections can be attributed to parton subprocesses which contain the maximum number of gluons in the final state. They manifest themselves as π^2-terms coming from the virtual and soft gluon contributions and as large logarithms of the type $\ln^i(1-x)/(1-x)$ which appear in the hard gluon part. The latter gives a large contribution near the boundary of phase space where the Bjorken scaling variable x goes to one. Since second order corrections are not available yet, many authors [9–13] have tried to improve the predictions of perturbative QCD. They resum these large corrections using various techniques. This mostly results in the exponentiation of the lowest order term. A second way to improve the perturbation series is to use a suitable factorization scale. This has been done in the frame work of optimized perturbative QCD [14–16]. The drawback of these approaches is that they are based on extrapolations of the next-to-leading order terms, ignoring effects coming from higher order corrections. The sensitivity of the improved perturbation series to the higher order contributions will decrease, if the running coupling constant gets smaller like e.g. in W and Z production. Nevertheless, it is our opinion that a full knowledge of the second order correction is necessary in order to confront theory with the data. This in particular holds for the two types of corrections mentioned above.

In this paper we will present a complete calculation of all logarithms of the type $\ln^i(1-x)/(1-x)$ up to order α_s^2 which appear in the total DY cross-section $d\sigma/dQ^2(h1+h2\to\gamma^*+\text{"hadrons"})$. This will be an extension of a previous article where one of the authors [17] has calculated the contribution for $i=3,2$. These results have been confirmed by [18]. We also determine the same type of terms appearing in the differential cross-section $d\sigma/d\sqrt{\tau}\,dx_F$ and discuss its effect on the K-factor and the phenomenon of anomalous scaling. We will show that these corrections are very important to the shape of the K-factor.

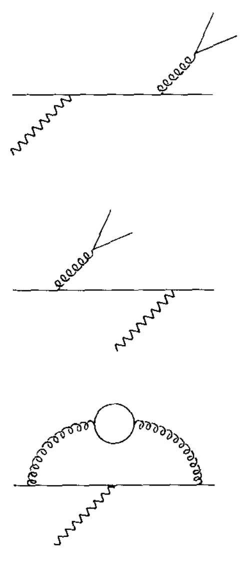

Fig. 1. Real and virtual gluon graphs contributing to the $O(\alpha_s^2)$ DI structure function of the process: $\gamma^*+q\to q+q+\bar{q}$

2 $O(\alpha_s^2)$ corrections to the total Drell-Yan cross-section

In this section we will present the calculation of the $O(\alpha_s^2)$ corrections to the K-factor of the Drell–Yan (DY) process

$$h1+h2\to\gamma^*+\text{"hadrons"}. \tag{2.1}$$

The colour average cross-section is defined by:

$$\frac{d\sigma}{dQ^2}=\frac{4\pi\alpha^2}{9Q^4}\tau W(\tau,Q^2) \quad \tau=\frac{Q^2}{s}. \tag{2.2}$$

Where s represents the centre of mass energy of the incoming hadrons $(h1,h2)$ and Q^2 stands for the virtual photon (γ^*) mass. From the DY mechanism it follows that the hadronic structure function $W(\tau,Q^2)$ can be written as

$$\begin{aligned}W(\tau,Q^2)=&\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx\,\delta(\tau-x_1x_2x)\\&\cdot[f^q(x_1,Q^2)f^{\bar{q}}(x_2,Q^2)\\&+f^{\bar{q}}(x_1,Q^2)f^q(x_2,Q^2)]\Delta(x,Q^2).\end{aligned} \tag{2.3}$$

Here $f^q(f^{\bar{q}})$ denotes the quark (anti-quark) distribution functions and $\Delta(x,Q^2)$ is the QCD correction term to the zeroth order DY process. In the subsequent sections we will limit ourselves to parton subprocesses with a quark and an anti-quark in the initial state. The correction term $\Delta(x,Q^2)$ can be obtained from the DY and deep inelastic (DI) parton structure functions $\hat{W}(\tau,Q^2)$ and $\hat{\mathscr{F}}_2(\tau,Q^2)$ using the following mass-factorization

$$\begin{aligned}\hat{W}(z,Q^2,\varepsilon)=&\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx\,\delta(z-x_1x_2x)\\&\cdot\hat{\mathscr{F}}_2(x_1,Q^2,\varepsilon)\hat{\mathscr{F}}_2(x_2,Q^2,\varepsilon)\Delta(x,Q^2).\end{aligned} \tag{2.4}$$

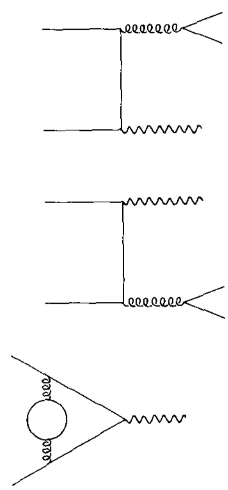

Fig. 2. Real and virtual gluon graphs contributing to the $O(\alpha_s^2)$ DY structure function of the process: $q+\bar{q}\to\gamma^*+q+\bar{q}$

In this case the ε represents the collinear divergences appearing in $\hat{W}$ and $\hat{\mathscr{F}}_2$, which are regularized by n-dimensional regularization $(\varepsilon=n-4)$. Equation (2.4) only holds for the non-singlet part of the parton structure functions. However, from the literature [6–8] it is known that the bulk of the correction can be traced back to the non-singlet part of $\Delta(x,Q^2)$. Hence we are justified to limit ourselves to the calculation of this quantity only.

Further, we want to point out that the mass-factorization chosen in (2.4) is the most natural one, since DI lepton-hadron scattering provides us with the best information about the phenomenological (anti-)quark distribution functions.

The $O(\alpha_s)$ DI parton subprocess, which contributes to $\hat{\mathscr{F}}_2$ is given by

$$\gamma^*(q)+q(\hat{p})\to q(\hat{p}')+g(\hat{k}_1) \tag{2.5}$$

and the $O(\alpha_s^2)$ contributions are coming from the processes

$$\gamma^*(q)+q(\hat{p})\to q(\hat{p}')+g(\hat{k}_1)+g(\hat{k}_2) \tag{2.6}$$

$$\gamma^*(q)+q(\hat{p})\to q(\hat{p}')+q(\hat{k}_1)+\bar{q}(\hat{k}_2). \tag{2.7}$$

Equation (2.7) stands for the process where a gluon in the final state decays into a quark anti-quark pair (Fig. 1). The corresponding DY processes which contribute to $\hat{W}$ are given by

$$q(\hat{p}_1)+\bar{q}(\hat{p}_2)\to\gamma^*(q)+g(\hat{k}_1) \tag{2.8}$$

and

$$q(\hat{p}_1)+\bar{q}(\hat{p}_2)\to\gamma^*(q)+g(\hat{k}_1)+g(\hat{k}_2) \tag{2.9}$$

$$q(\hat{p}_1)+\bar{q}(\hat{p}_2)\to\gamma^*(q)+q(\hat{k}_1)+\bar{q}(\hat{k}_2). \tag{2.10}$$

(See Fig. 2 for the process in (2.10).)

From mass-factorization and the renormalization group we can derive the following expressions for the non-singlet parton structure functions.

$$\begin{aligned}
&\hat{\mathscr{F}}_2(x,Q^2,\varepsilon)\\
&=\delta(1-x)+\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\left[P_0(x)\frac{1}{\varepsilon}+f_0(x)\right]\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\bigg[\{\tfrac{1}{2}(P_0\otimes P_0)(x)+\beta_0P_0(x)\}\frac{1}{\varepsilon^2}\\
&\quad+\{\tfrac{1}{2}P_1(x)+(P_0\otimes f_0)(x)\}\frac{1}{\varepsilon}\\
&\quad+\tfrac{1}{2}(P_0\otimes P_0)(x)\frac{1}{\varepsilon}\ln\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+\{\tfrac{1}{4}(P_0\otimes P_0)(x)-\tfrac{1}{4}\beta_0P_0(x)\}\ln^2\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+\{\tfrac{1}{2}P_1(x)+(P_0\otimes f_0)(x)-\beta_0f_0(x)\}\ln\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+f_1(x)\bigg].
\end{aligned} \tag{2.11}$$

Where x is the Bjorken scaling variable, defined by $x=Q^2/(2\hat{p}q)$.

$$\begin{aligned}
&\hat{W}(x,Q^2,\varepsilon)\\
&=\delta(1-x)+\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\left[2P_0(x)\frac{1}{\varepsilon}+w_0(x)\right]\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\bigg[\{2(P_0\otimes P_0)(x)+2\beta_0P_0(x)\}\frac{1}{\varepsilon^2}\\
&\quad+\{P_1(x)+(P_0\otimes w_0)(x)\}\frac{1}{\varepsilon}\\
&\quad+2(P_0\otimes P_0)(x)\frac{1}{\varepsilon}\ln\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+\{(P_0\otimes P_0)(x)-\tfrac{1}{2}\beta_0P_0(x)\}\ln^2\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+\{P_1(x)+2(P_0\otimes w_0)(x)-\beta_0w_0(x)\}\ln\left(\frac{Q^2}{\mu^2}\right)\\
&\quad+w_1(x)\bigg].
\end{aligned} \tag{2.12}$$

With $x=Q^2/\hat{s}$ and $\hat{s}=(\hat{p}_1+\hat{p}_2)^2$.

The convolution symbol $\otimes$ is defined by

$$(f\otimes g)(x)=\int_0^1 dx_1\int_0^1 dx_2\,\delta(x-x_1x_2)f(x_1)g(x_2). \tag{2.13}$$

The $P_i(x)$ are the well known splitting functions which have been calculated in the literature [19–21] and β_0 is the lowest order coefficient of the β-function. The splitting functions are related to the anomalous dimension $\gamma^{(n)}$ of the non-singlet operators in the following way.

$$\gamma^{(n)}(\alpha_s)=\gamma_0^{(n)}\left(\frac{\alpha_s}{4\pi}\right)+\gamma_1^{(n)}\left(\frac{\alpha_s}{4\pi}\right)^2+\cdots. \tag{2.14}$$

The coefficients $\gamma_i^{(n)}$ are obtained from $P_i(x)$ via the Mellin-transform

$$\gamma_i^{(n)}=-\int_0^1 dx\,x^{n-1}P_i(x) \tag{2.15}$$

and the β_0 is defined by

$$\beta(\alpha_s)=2\alpha_s\left\{-\beta_0\left(\frac{\alpha_s}{4\pi}\right)-\beta_1\left(\frac{\alpha_s}{4\pi}\right)^2+\cdots\right\} \tag{2.16}$$

(Notice that this definition differs from the usual one, see also (2.20).) From mass-factorization one finds that $\hat{\mathscr{F}}_2^{(n)}$ and $\hat{W}^{(n)}$ can be written as

$$\hat{\mathscr{F}}_2^{(n)}(Q^2,\varepsilon)=A_2^{(n)}(\varepsilon)C_{\mathrm{DI}}^{(n)}(Q^2) \tag{2.17a}$$

$$\hat{W}^{(n)}(Q^2,\varepsilon)=\{A_2^{(n)}(\varepsilon)\}^2C_{\mathrm{DY}}^{(n)}(Q^2). \tag{2.17b}$$

Where the operator matrix element $A_2^{(n)}(\varepsilon)$ is given by

$$\begin{aligned}
A_2^{(n)}(\varepsilon)&=1+\left(\frac{\alpha_s}{4\pi}\right)\left\{-\gamma_0^{(n)}\frac{1}{\varepsilon}\right\}\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{(\tfrac{1}{2}(\gamma_0^{(n)})^2-\beta_0\gamma_0^{(n)})\frac{1}{\varepsilon^2}-\tfrac{1}{2}\gamma_1^{(n)}\frac{1}{\varepsilon}\right\}.
\end{aligned} \tag{2.18}$$

Here ε denotes the collinear divergence and not the UV-singularity.

Using (2.17a) and (2.18) we obtain the Wilson coefficient $C_{\mathrm{DI}}^{(n)}(Q^2)$

$$\begin{aligned}
&C_{\mathrm{DI}}^{(n)}(Q^2)\\
&=1+\left(\frac{\alpha_s}{4\pi}\right)\left\{-\tfrac{1}{2}\gamma_0^{(n)}\ln\left(\frac{Q^2}{\mu^2}\right)-\tfrac{1}{8}\varepsilon\gamma_0^{(n)}\ln^2\left(\frac{Q^2}{\mu^2}\right)\right.\\
&\quad\left.+f_0^{(n)}+\frac{\varepsilon}{2}f_0^{(n)}\ln\left(\frac{Q^2}{\mu^2}\right)\right\}\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{[\tfrac{1}{8}(\gamma_0^{(n)})^2+\tfrac{1}{4}\beta_0\gamma_0^{(n)}]\ln^2\left(\frac{Q^2}{\mu^2}\right)\right.
\end{aligned}$$

$$+[-\tfrac{1}{2}\gamma_1^{(n)}-\tfrac{1}{2}\gamma_0^{(n)}f_0^{(n)}-\beta_0 f_0^{(n)}]\ln\left(\frac{Q^2}{\mu^2}\right)+f_1^{(n)}\Big\}. \tag{2.19}$$

An analogous formula exists for $C_{\mathrm{DY}}^{(n)}(Q^2)$. The definitions above have been chosen in such a way that the quantities $P_i(x)$ and β_i can be immediately obtained from the literature [19–21]. Using these definitions $C^{(n)}(Q^2)$ satisfies the Callan–Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}-\gamma^{(n)}(\alpha_s)\right]C^{(n)}(Q^2,\mu^2,\alpha_s)=0. \tag{2.20}$$

From (2.4), (2.11) and (2.12) one can derive that $\Delta(x,Q^2)$ is equal to

$$\Delta(x,Q^2)=\delta(1-x)+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)\Delta_0(x)+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2\Delta_1(x) \tag{2.21}$$

with

$$\Delta_0(x)=w_0(x)-2f_0(x) \tag{2.22a}$$

$$\Delta_1(x)=w_1(x)-2f_1(x)-(f_0\otimes f_0)(x)-2(f_0\otimes\Delta_0)(x) \tag{2.22b}$$

and $\alpha_s(Q^2)$ is the running coupling constant, defined by

$$\alpha_s(Q^2)=\frac{\alpha_s(\mu^2)}{1+\beta_0\dfrac{\alpha_s(\mu^2)}{4\pi}\ln\left(\dfrac{Q^2}{\mu^2}\right)}=\frac{4\pi}{\beta_0\ln\left(\dfrac{Q^2}{\Lambda^2}\right)}. \tag{2.23}$$

Where Λ is a parameter which has to be determined experimentally. For our subsequent calculation of the parton structure functions we introduce the following notations. The parton structure functions corresponding to the processes in (2.5)–(2.10) receive contributions from virtual, soft and hard gluons*. Therefore we will split them up into three parts [8].

$$\hat{\mathscr{F}}_2(x,Q^2,\varepsilon)=\hat{\mathscr{F}}_2^V(Q^2,\varepsilon)+\hat{\mathscr{F}}_2^S(Q^2,\varepsilon,\delta)+\hat{\mathscr{F}}_2^H(x,Q^2,\varepsilon) \tag{2.24a}$$

$$\hat{W}(x,Q^2,\varepsilon)=\hat{W}^V(Q^2,\varepsilon)+\hat{W}^S(Q^2,\varepsilon,\delta)+\hat{W}^H(x,Q^2,\varepsilon). \tag{2.24b}$$

Where $\hat{\mathscr{F}}_2^V$, $\hat{W}^V$; $\hat{\mathscr{F}}_2^S$, $\hat{W}^S$ and $\hat{\mathscr{F}}_2^H$, $\hat{W}^H$ stand for the virtual, soft and hard gluon contributions respectively. The δ in the soft gluon structure functions is a resolution parameter which distinguishes soft ($x>1-\delta$) and hard ($x<1-\delta$) gluons. Since $\hat{\mathscr{F}}_2^V$, $\hat{W}^V$ and $\hat{\mathscr{F}}_2^S$, $\hat{W}^S$ are nothing but multiplicative corrections to the Born–term they can be written as

$$\hat{\mathscr{F}}_2^V+\hat{\mathscr{F}}_2^S\equiv\delta(1-x)\hat{\mathscr{F}}_2^{S+V}(Q^2,\delta)=\delta(1-x)|F(Q^2)|^2B^{\mathrm{DI}}(Q^2,\delta)\quad Q^2<0 \tag{2.25a}$$

$$\hat{W}^V+\hat{W}^S\equiv\delta(1-x)\hat{W}^{S+V}(Q^2,\delta)=\delta(1-x)|F(Q^2)|^2B^{\mathrm{DY}}(Q^2,\delta)\quad Q^2>0. \tag{2.25b}$$

Here $F(Q^2)$ denotes the quark-formfactor and $B^{\mathrm{DI}}(Q^2,\delta)$ and $B^{\mathrm{DY}}(Q^2,\delta)$ represent the soft bremsstrahlungs corrections to the processes in (2.5)–(2.10). The latter quantities appear, when there is a soft gluon or a soft fermion pair in the final state, and they can be calculated using the following formulae.

$$B^{\mathrm{DI}}(Q^2,\delta)=1+\int_{1-\delta}^{1}dx\,\hat{\mathscr{F}}_2^H(x,Q^2,\varepsilon) \tag{2.26a}$$

$$B^{\mathrm{DY}}(Q^2,\delta)=1+\int_{1-\delta}^{1}dx\,\hat{W}^H(x,Q^2,\varepsilon). \tag{2.26b}$$

The $O(\alpha_s)$ contribution ((2.5) and (2.8)) to the above quantities can be found in the literature [7, 8]. It turns out that the correction to the zeroth order Drell–Yan cross-section is very large, about 80% at $Q^2=100\,\mathrm{GeV}^2$. This is mainly due to the large π^2 terms appearing in $\hat{\mathscr{F}}_2^{S+V}$ (2.25a), $\hat{W}^{S+V}$ (2.25b) which give rise to a change in the overall normalization of the cross-section. Another source of large corrections are the logarithms of the type $\ln^i(1-x)/(1-x)$ which appear in $\hat{\mathscr{F}}_2^H$ and $\hat{W}^H$. They affect the shape of the K-factor and are therefore relevant to the study of the phenomenon of anomalous scaling [1, 2]. Both corrections are due to soft gluon initial state radiation and in higher order also to soft fermion pair emission, see (2.7) and (2.10). The last process has been calculated and the result is given in Appendix A and [22]. From this calculation we infer that the virtual and soft parton structure functions in (2.25a), (2.25b) are a series expansion in the Riemann zeta function $\zeta(n)$*. The $\zeta(n)$ originate from the expansion of the gamma-functions which appear in the n-dimensionally regularized integrals. Their coefficients can only be obtained by an explicit calculation which we plan to do in the future for processes (2.6) and (2.9). In contrast to these terms the large logarithms mentioned above can be obtained by a trick, so that a long calculation is unnecessary. At this moment we can only determine the coefficients of $\ln^i(1-x)/(1-x)$ up to $O(\alpha_s^2)$, since the renormalization group coefficients mentioned in (2.14) and (2.16) are only known up to that order.

The coefficients of the large logarithms are determined in the following way. Let us start with the most general form of the correction term $\Delta(x,Q^2)$ [6–8].

$$\Delta(x,Q^2)=\delta(1-x)+\sum_{i=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^i\cdot\left[\sum_{j=0}^{2i-1}\left\{\theta(1-\delta-x)\Delta_{ij}\frac{\ln^j(1-x)}{1-x}+\delta(1-x)\right.\right.$$

* The calculation which involves the $\ln(\delta)$ approach proceeds along the way as is outlined in [6, 8]

* The π^2 term appearing in the $O(\alpha_s)$ part is a particular example ($\pi^2=6\zeta(2)$)

$$\cdot\left(\frac{1}{j+1}\Delta_{ij}\ln^{j+1}(\delta)+\Delta_i\right)\Bigg\}\Bigg]+\tilde{\Delta}(x,Q^2). \tag{2.27}$$

Where $\tilde{\Delta}(x,Q^2)$ represents all the remaining terms which satisfy the relation: $\int_0^1 dx\tilde{\Delta}(x,Q^2)$ is finite. The Δ_i stand for the sum of the virtual and soft gluon contributions as indicated in (2.25a), (2.25b). In the above equation we have used the fact that there exists an one to one correspondence between the coefficients of the large logarithms and those of the $\ln\delta$ terms. This follows from the constraint that $\int_0^1 dx\Delta(x,Q^2)$ is finite. Analogous expressions like (2.27) exist for the renormalization group coefficients $P_i(x)$ and the non-pole terms f_i (2.11) and ω_i (2.12). The $\ln\delta$ terms can be wholly attributed to the soft bremsstrahlungs contribution in (2.26a), (2.26b). The determination of their coefficients proceeds as follows.

Expanding the renormalized $F(Q^2)$ and $B(Q^2,\delta)$ in a power series of α_s we get

$$F(Q^2)=1+\left(\frac{\alpha_s}{4\pi}\right)F^{(1)}(Q^2)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(F^{(2)}(Q^2)+\frac{2}{\varepsilon}\beta_0F^{(1)}(Q^2)\right)+\cdots \tag{2.28}$$

$$B(Q^2,\delta)=1+\left(\frac{\alpha_s}{4\pi}\right)B^{(1)}(Q^2,\delta)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(B^{(2)}(Q^2,\delta)+\frac{2}{\varepsilon}\beta_0B^{(1)}(Q^2,\delta)\right)+\cdots. \tag{2.29}$$

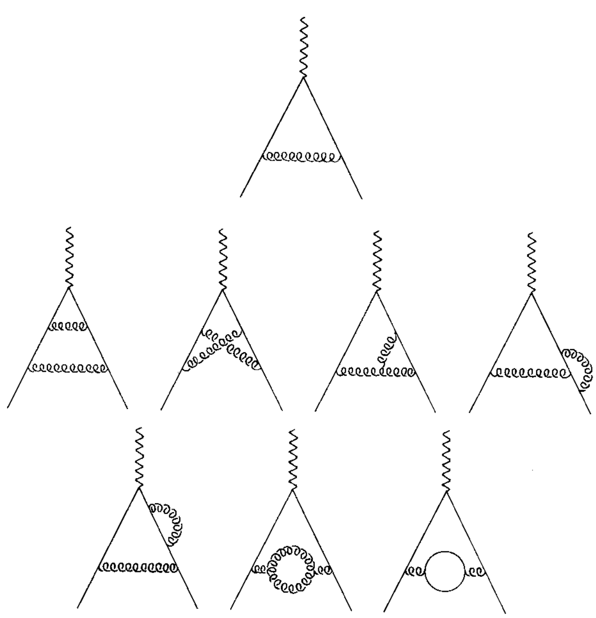

Fig. 3. Order α_s and α_s^2 contribution to the quark-formfactor

The $F^{(1)}$ and $F^{(2)}$ are obtained from the diagrams in Fig. 3. The $B^{(1)}$ and $B^{(2)}$ originate from the $O(\alpha_s)$ and $O(\alpha_s^2)$ contributions to $\hat{\mathscr{F}}_2$ and $\hat{W}$, see (2.26a), (2.26b), which can be computed from the following amplitudes.

The $O(\alpha_s)$ amplitudes corresponding to the processes (2.5) and (2.8) are denoted by $M_1^{\rm DI}(\hat{p}',\hat{k}_1)$ and $M_1^{\rm DY}(q,\hat{k}_1)$ respectively, so that $\hat{\mathscr{F}}_2^{(1)}$ and $\hat{W}^{(1)}$ are equal to

$$\hat{\mathscr{F}}_2^{(1)}(x,Q^2,\varepsilon)\sim\int dPS_2|M_1^{\rm DI}(\hat{p}',\hat{k}_1)|^2 \tag{2.30}$$

$$\hat{W}^{(1)}(x,Q^2,\varepsilon)\sim\int dPS_2|M_1^{\rm DY}(q,\hat{k}_1)|^2. \tag{2.31}$$

Where dPS_n stands for the n-body phase space (see Appendix B).

The $O(\alpha_s^2)$ contribution consists of two parts.

1. The $O(\alpha_s)$ virtual corrections to $M_1^{\rm DI}$ and $M_1^{\rm DY}$, which we will denote by $M_1^{{\rm DI},V}(\hat{p}',\hat{k}_1)$ and $M_1^{{\rm DY},V}(q,\hat{k}_1)$ respectively.

2. The $O(\alpha_s^2)$ amplitudes corresponding to the processes (2.6), (2.7) and (2.9), (2.10). They are represented by $M_2^{\rm DI}(\hat{p}',\hat{k}_1,\hat{k}_2)$ and $M_2^{\rm DY}(q,\hat{k}_1,\hat{k}_2)$.

The $O(\alpha_s^2)$ parton structure function can be written as

$$\hat{\mathscr{F}}_2^{(2)}(x,Q^2,\varepsilon)\sim 2\int dPS_2\{{\rm Re}(M_1^{{\rm DI},V}(\hat{p}',\hat{k}_1)\cdot M_1^{\rm DI}(\hat{p}',\hat{k}_1))\}+\int dPS_3|M_2^{\rm DI}(\hat{p}',\hat{k}_1,\hat{k}_2)|^2 \tag{2.32}$$

$$\hat{W}^{(2)}(x,Q^2,\varepsilon)\sim 2\int dPS_2\{{\rm Re}(M_1^{{\rm DY},V}(q,\hat{k}_1)\cdot M_1^{\rm DY}(q,\hat{k}_1))\}+\int dPS_3|M_2^{\rm DY}(q,\hat{k}_1,\hat{k}_2)|^2. \tag{2.33}$$

From the structure of the two and three body phase space integrals in Appendix B we can derive that the parton structure functions behave like

$$\hat{\mathscr{F}}_2^{(1)}(x,Q^2,\varepsilon)\sim(1-\varepsilon)^{\varepsilon/2}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}A_1^{\rm DI}(x,Q^2,\varepsilon) \tag{2.34}$$

$$\widehat{W(1)}(x,Q^2,\varepsilon)\sim(1-x)^{\varepsilon}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}A_1^{\rm DY}(x,Q^2,\varepsilon) \tag{2.35}$$

and

$$\hat{\mathscr{F}}_2^{(2)}(x,Q^2,\varepsilon)\sim(1-x)^{\varepsilon/2}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}A_1^{{\rm DI},V}(x,Q^2,\varepsilon)+(1-x)^{\varepsilon}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}A_2^{\rm DI}(x,Q^2,\varepsilon) \tag{2.36}$$

$$\hat{W}^{(2)}(x,Q^2,\varepsilon)\sim(1-x)^{\varepsilon}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}A_1^{{\rm DY},V}(x,Q^2,\varepsilon)+(1-x)^{2\varepsilon}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}A_2^{\rm DY}(x,Q^2,\varepsilon). \tag{2.37}$$

The processes mentioned in (2.5)–(2.10) lead to infrared (IR) singularities in the parton structure functions. These singularities manifest themselves as poles of the type $\ln^i(1-x)/(1-x)$ in the quantities $A^{\rm DI}$ and $A^{\rm DY}$ (2.34)–(2.37). From the above equations we can derive the $\ln\delta$ dependence of $B(Q^2)$ in (2.26a), (2.26b). Substituting (2.28) and (2.29) in (2.25a), (2.25b) we find

$$\hat{\mathscr{F}}_2^{S+V}(Q^2,\varepsilon,\delta) = 1+\left(\frac{\alpha_s}{4\pi}\right)\{2F^{(1)}(Q^2)+B^{(1),\mathrm{DI}}(Q^2,\delta)\} + \left(\frac{\alpha_s}{4\pi}\right)^2\Big\{2F^{(2)}(Q^2)+(F^{(1)}(Q^2))^2 + B^{(2),\mathrm{DI}}(Q^2,\delta)+2F^{(1)}(Q^2)B^{(1),\mathrm{DI}}(Q^2,\delta) + \frac{2}{\varepsilon}\beta_0(2F^{(1)}(Q^2)+B^{(1),\mathrm{DI}}(Q^2,\delta))\Big\} \tag{2.38}$$

with a similar expression for $\hat{W}^{S+V}$.

On the other hand the $\ln\delta$ terms can be obtained from the insertion of (2.34)–(2.37) in (2.26a), (2.26b). So that we can make the identification

$$B^{(1),\mathrm{DI}}(Q^2,\delta)\sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\int_{1-\delta}^{1}dx(1-x)^{\varepsilon/2}A_1^{\mathrm{DI}}(x,Q^2,\varepsilon) \tag{2.39}$$

$$B^{(1),\mathrm{DY}}(Q^2,\delta)\sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\int_{1-\delta}^{1}dx(1-x)^{\varepsilon}A_1^{\mathrm{DY}}(x,Q^2,\varepsilon) \tag{2.40}$$

$$B^{(2),\mathrm{DI}}(Q^2,\delta)\sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\int_{1-\delta}^{1}dx(1-x)^{\varepsilon}A_2^{\mathrm{DI}}(x,Q^2,\varepsilon) \tag{2.41}$$

$$B^{(2),\mathrm{DY}}(Q^2,\delta)\sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\int_{1-\delta}^{1}dx(1-x)^{2\varepsilon}A_2^{\mathrm{DY}}(x,Q^2,\varepsilon) \tag{2.42}$$

$$\left(2F^{(1)}(Q^2)+\frac{2}{\varepsilon}\beta_0\right)B^{(1),\mathrm{DI}}(Q^2,\delta) \sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\int_{1-\delta}^{1}dx(1-x)^{\varepsilon/2}A_1^{\mathrm{DI},V}(x,Q^2,\varepsilon) \tag{2.43}$$

$$\left(2F^{(1)}(Q^2)+\frac{2}{\varepsilon}\beta_0\right)B^{(1),\mathrm{DY}}(Q^2,\delta) \sim\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\int_{1-\delta}^{1}dx(1-x)^{\varepsilon}A_1^{\mathrm{DY},V}(x,Q^2,\varepsilon). \tag{2.44}$$

Hence we can derive the formulae for the soft bremsstrahlungs contributions.

$$B^{(n),\mathrm{DI}}=\delta^{n(\varepsilon/2)}\sum_{i=0}^{2n}\frac{d_{n,i}^{\mathrm{DI}}}{\varepsilon^i} \tag{2.45}$$

$$B^{(n),\mathrm{DY}}=\delta^{n\varepsilon}\sum_{i=0}^{2n}\frac{d_{n,i}^{\mathrm{DY}}}{\varepsilon^i}. \tag{2.46}$$

In the above equations we have generalized the expressions for $B^{(n),\mathrm{DI}}$ and $B^{(n),\mathrm{DY}}$ to all orders in perturbation theory, in particular as far as the behaviour of the $\ln\delta$ terms are concerned. This behaviour is essential for the determination of the coefficients of the large logarithms, as we will see below. First we observe that the pole terms of $\hat{\mathscr{F}}_2^{S+V}$ and $\hat{W}^{S+V}$ are determined up to order α_s^2. This can be inferred from (2.11) and (2.12) where P_0, P_1 and f_0, ω_0 are known from the literature [6–8, 19–21]. Next we know the quark-formfactor $F(Q^2)$ (2.28) from [23]. We have repeated the calculation with the help of dispersion relation techniques [24] and agree with their result as far as the pole terms are concerned. From the observations made above and in (2.38) we can determine all the pole terms of $B(Q^2,\delta)$. This enables us to compute all the coefficients of the $\ln\delta$ terms up to order α_s^2.

Let us start with the determination of the $\delta(1-x)$ piece of the parton structure functions $\hat{\mathscr{F}}_2(x,Q^2,\delta)$ (2.11) and $\hat{W}(x,Q^2,\delta)$ (2.12). In Appendix C we have listed all renormalization group coefficients in the limit $x\to1$. Substituting them in (2.11) and (2.12) we obtain for the soft and virtual part the following expressions

$$\begin{aligned}
\ln\hat{\mathscr{F}}_2^{S+V}(Q^2,\delta) &= \left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}C_F\Big[(8\ln\delta+6)\frac{1}{\varepsilon} \\
&+2\ln^2\delta-3\ln\delta-9-4\zeta(2)+\varepsilon\{\tfrac{1}{3}\ln^3\delta-\tfrac{3}{4}\ln^2\delta \\
&+(\tfrac{7}{2}-3\zeta(2))\ln\delta+\tfrac{3}{4}\zeta(2)+9\}\Big] \\
&+\left(\frac{\alpha_s}{4\pi}\right)^2\Big[C_F^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\Big\{-32\zeta(2)\frac{1}{\varepsilon^2} \\
&+(\tfrac{3}{2}+12\zeta(2)+56\zeta(3)-32\zeta(2)\ln\delta)\frac{1}{\varepsilon}\Big\} \\
&+C_AC_F\Big\{((\tfrac{268}{9}-8\zeta(2))\ln\delta+\tfrac{17}{6}+\tfrac{44}{3}\zeta(2)-12\zeta(3))\frac{1}{\varepsilon} \\
&+(\tfrac{88}{3}\ln\delta+22)\frac{1}{\varepsilon^2}+(-\tfrac{11}{2}-\tfrac{22}{3}\ln\delta)\ln^2\left(\frac{Q^2}{\mu^2}\right) \\
&+(-\tfrac{22}{3}\ln^2\delta+(\tfrac{367}{9}-8\zeta(2))\ln\delta \\
&+\tfrac{215}{6}+\tfrac{88}{3}\zeta(2)-12\zeta(3))\ln\left(\frac{Q^2}{\mu^2}\right)\Big\} \\
&+n_fC_F\Big\{(-\tfrac{40}{9}\ln\delta-\tfrac{1}{3}-\tfrac{8}{3}\zeta(2))\frac{1}{\varepsilon} \\
&+(-\tfrac{16}{3}\ln\delta-4)\frac{1}{\varepsilon^2}+(1+\tfrac{4}{3}\ln\delta)\ln^2\left(\frac{Q^2}{\mu^2}\right) \\
&+(\tfrac{4}{3}\ln^2\delta-\tfrac{58}{9}\ln\delta-\tfrac{19}{3}-\tfrac{16}{3}\zeta(2))\ln\left(\frac{Q^2}{\mu^2}\right)\Big\}\Big]
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
\ln\hat{W}^{S+V}(Q^2,\delta) &= C_F\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2} \\
&\cdot\Big[(16\ln\delta+12)\frac{1}{\varepsilon}+8\ln^2\delta-16+8\zeta(2) \\
&+\varepsilon(\tfrac{8}{3}\ln^3\delta-6\zeta(2)\ln\delta-\tfrac{21}{2}\zeta(2)+16)\Big]
\end{aligned}$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left[C_F^2\left(\frac{Q^2}{\mu^2}\right)^\varepsilon\left\{-128\zeta(2)\frac{1}{\varepsilon^2}\right.\right.$$
$$\left.+(3-24\zeta(2)+304\zeta(3)-256\zeta(2)\ln\delta)\frac{1}{\varepsilon}\right\}$$
$$+C_AC_F\left\{((\tfrac{536}{9}-16\zeta(2))\ln\delta+\tfrac{17}{3}+\tfrac{88}{3}\zeta(2)-24\zeta(3))\frac{1}{\varepsilon}\right.$$
$$+(\tfrac{176}{3}\ln\delta+44)\frac{1}{\varepsilon^2}+(-\tfrac{44}{3}\ln\delta-11)\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$+(-\tfrac{88}{3}\ln^2\delta+(\tfrac{536}{9}-16\zeta(2))\ln\delta+\tfrac{193}{3}-24\zeta(3))$$
$$\left.\cdot\ln\left(\frac{Q^2}{\mu^2}\right)\right\}+n_fC_F\left\{(-\tfrac{80}{9}\ln\delta-\tfrac{2}{3}-\tfrac{16}{3}\zeta(2))\frac{1}{\varepsilon}\right.$$
$$+(-\tfrac{32}{3}\ln\delta-8)\frac{1}{\varepsilon^2}+(\tfrac{8}{3}\ln\delta+2)\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$\left.\left.+(\tfrac{16}{3}\ln^2\delta-\tfrac{80}{9}\ln\delta-\tfrac{34}{3})\ln\left(\frac{Q^2}{\mu^2}\right)\right\}\right]. \qquad (2.48)$$

We have suppressed terms of the type $\gamma_E-\ln(4\pi)$ ($\gamma_E=$ Euler constant) in the above and subsequent equations. They originate from n-dimensional regularization but disappear in the finite correction term $\Delta(x,Q^2)$*.

The renormalized quark-formfactor for space-like and time-like Q^2 is given by

$$\ln|F(Q^2)|$$
$$=\left(\frac{\alpha_s}{4\pi}\right)|F^{(1)}(Q^2)|+\left(\frac{\alpha_s}{4\pi}\right)^2$$
$$\cdot\left\{|F^{(2)}(Q^2)|-\tfrac{1}{2}|F^{(1)}(Q^2)|^2+\frac{2}{\varepsilon}\beta_0|F^{(1)}(Q^2)|\right\}$$
$$=C_F\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\left[-\frac{8}{\varepsilon^2}+\frac{6}{\varepsilon}+\zeta(2)-8\right.$$
$$\left.+\varepsilon(8-\tfrac{3}{4}\zeta(2)-\tfrac{7}{3}\zeta(3))\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left[C_F^2\left(\frac{Q^2}{\mu^2}\right)^\varepsilon\left\{(\tfrac{3}{2}-12\zeta(2)+24\zeta(3))\frac{1}{\varepsilon}\right.\right.$$
$$\left.-\tfrac{1}{8}+29\zeta(2)-30\zeta(3)-\tfrac{44}{5}\zeta(2)^2\right\}$$
$$+C_AC_F\left\{-\frac{44}{\varepsilon^3}+(\tfrac{64}{9}+4\zeta(2))\frac{1}{\varepsilon^2}\right.$$
$$+(\tfrac{961}{54}+11\zeta(2)-26\zeta(3))\frac{1}{\varepsilon}$$
$$+\tfrac{11}{9}\ln^3\left(\frac{Q^2}{\mu^2}\right)+(-\tfrac{233}{18}+2\zeta(2))\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$+\left(-\frac{44}{3}\frac{1}{\varepsilon^2}+(-\tfrac{134}{9}+4\zeta(2))\frac{1}{\varepsilon}\right.$$
$$\left.+\tfrac{2545}{54}+\tfrac{22}{3}\zeta(2)-26\zeta(3)\right)\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$\left.-\tfrac{51157}{648}-\tfrac{337}{18}\zeta(2)+\tfrac{313}{9}\zeta(3)+\tfrac{44}{5}\zeta(2)^2\right\}$$
$$+n_fC_F\left\{\frac{8}{\varepsilon^3}-\frac{16}{9}\frac{1}{\varepsilon^2}+(-\tfrac{65}{27}-2\zeta(2))\frac{1}{\varepsilon}\right.$$
$$-\tfrac{2}{9}\ln^3\left(\frac{Q^2}{\mu^2}\right)+\tfrac{19}{9}\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$+\left(\frac{8}{3}\frac{1}{\varepsilon^2}+\frac{20}{9}\frac{1}{\varepsilon}-\frac{209}{27}-\frac{4}{3}\zeta(2)\right)\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$\left.\left.+\tfrac{4085}{324}+\tfrac{23}{9}\zeta(2)+\tfrac{2}{9}\zeta(3)\right\}\right]$$
$$+\theta(Q^2)\left[C_F\left(\frac{\alpha_s}{4\pi}\right)\{6\zeta(2)-\tfrac{9}{2}\varepsilon\}\right.$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{C_AC_F\left(-22\zeta(2)\ln\left(\frac{Q^2}{\mu^2}\right)\right.\right.$$
$$\left.+\tfrac{233}{3}\zeta(2)-12\zeta(2)^2\right)$$
$$\left.\left.+n_fC_F\left(4\zeta(2)\ln\left(\frac{Q^2}{\mu^2}\right)-\tfrac{38}{3}\zeta(2)\right)\right\}\right]. \qquad (2.49)$$

Note that the first three terms of the C_F part of the formfactor exponentiate. Since the leading pole terms in the quantities B^{DI} and B^{DY} have to cancel the corresponding ones appearing in the quark-formfactor we make the following ansatz, see (2.39)–(2.44).

$$\ln B^{\mathrm{DI}}(Q^2,\delta)$$
$$=\left(\frac{\alpha_s}{4\pi}\right)B^{(1),\mathrm{DI}}(Q^2,\delta)$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{B^{(2),\mathrm{DI}}(Q^2,\delta)-\tfrac{1}{2}(B^{(1),\mathrm{DI}}(Q^2,\delta))^2\right.$$
$$\left.+\frac{2}{\varepsilon}\beta_0B^{(1),\mathrm{DI}}(Q^2,\delta)\right\}$$
$$=\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2}\delta^{\varepsilon/2}C_F\left[1+\frac{2}{\varepsilon}\beta_0\left(\frac{\alpha_s}{4\pi}\right)\right]$$
$$\cdot\left[\frac{16}{\varepsilon^2}-\frac{6}{\varepsilon}+7-6\zeta(2)+\varepsilon(-7+\tfrac{9}{4}\zeta(2)+\tfrac{14}{3}\zeta(3))\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{Q^2}{\mu^2}\right)^\varepsilon\delta^\varepsilon\left[C_F^2\left\{\frac{a_{22}^{\mathrm{DI}}}{\varepsilon^2}+\frac{a_{21}^{\mathrm{DI}}}{\varepsilon}+a_{20}^{\mathrm{DI}}\right\}\right.$$
$$+C_AC_F\left\{\frac{b_{23}^{\mathrm{DI}}}{\varepsilon^3}+\frac{b_{22}^{\mathrm{DI}}}{\varepsilon^2}+\frac{b_{21}^{\mathrm{DI}}}{\varepsilon}+b_{20}^{\mathrm{DI}}\right\}$$
$$\left.+n_fC_F\left\{\frac{c_{23}^{\mathrm{DI}}}{\varepsilon^3}+\frac{c_{22}^{\mathrm{DI}}}{\varepsilon^2}+\frac{c_{21}^{\mathrm{DI}}}{\varepsilon}+c_{20}^{\mathrm{DI}}\right\}\right] \qquad (2.50)$$

* We use the $\overline{MS}$ scheme to remove UV-singularities

$$\ln B^{DY}(Q^2,\delta)$$

$$= \left(\frac{\alpha_s}{4\pi}\right) B^{(1),DY}(Q^2,\delta) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ B^{(2),DY}(Q^2,\delta) - \tfrac{1}{2}(B^{(1),DY}(Q^2,\delta))^2 + \frac{2}{\varepsilon}\beta_0 B^{(1),DY}(Q^2,\delta) \right\}$$

$$= \left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2} \delta^\varepsilon C_F \left[1 + \frac{2}{\varepsilon}\beta_0\left(\frac{\alpha_s}{4\pi}\right)\right] \cdot \left[\frac{16}{\varepsilon^2} - 6\zeta(2) + \varepsilon \tfrac{14}{3}\zeta(3)\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \delta^{2\varepsilon} \left[C_F^2 \left\{\frac{a_{22}^{DY}}{\varepsilon^2} + \frac{a_{21}^{DY}}{\varepsilon} + a_{20}^{DY}\right\} + C_A C_F \left\{\frac{b_{23}^{DY}}{\varepsilon^3} + \frac{b_{22}^{DY}}{\varepsilon^2} + \frac{b_{21}^{DY}}{\varepsilon} + b_{20}^{DY}\right\} + n_f C_F \left\{\frac{c_{23}^{DY}}{\varepsilon^3} + \frac{c_{22}^{DY}}{\varepsilon^2} + \frac{c_{21}^{DY}}{\varepsilon} + c_{20}^{DY}\right\}\right]. \quad (2.51)$$

The residues of the pole-terms in the above equations can be derived from the equalities (2.25a), (2.25b) i.e.

$$\ln \hat{\mathscr{F}}_2^{S+V}(Q^2,\delta) = 2\ln|F(Q^2)| + \ln B^{DI}(Q^2,\delta) \quad (2.52a)$$

$$\ln \hat{W}^{S+V}(Q^2,\delta) = 2\ln|F(Q^2)| + \ln B^{DY}(Q^2,\delta) \quad (2.52b)$$

and (2.49)–(2.51). The results are given in Table 1. Notice that in contrast to $F(Q^2)$ the C_F^2 part of $B(Q^2,\delta)$ only exponentiates in the first *two* terms. The coefficients a_{20} and b_{20} have not been calculated yet. However, the c_{20} term has been calculated in [22] and Appendix A and is given in Table 1. These coefficients are needed for the determination of the constant part of the K-factor i.e. the Δ_i in (2.27). Using Table 1 and the arguments given below (2.27) we are able to compute the coefficients f_1 and w_1 in (2.11) and (2.12). The results are

$$f_1(x) \overset{x\to 1}{=} \frac{\theta(1-\delta-x)}{(1-x)}[C_F^2\{\tfrac{56}{3}\ln^3(1-x) - 30\ln^2(1-x) + (20-96\zeta(2))\ln(1-x) + \tfrac{237}{2} + 36\zeta(2) + 8\zeta(3)\} + C_A C_F\{-\tfrac{22}{3}\ln^2(1-x) + (\tfrac{367}{9} - 8\zeta(2))\ln(1-x) - \tfrac{3155}{54} + \tfrac{44}{3}\zeta(2) + 40\zeta(3)\} + n_f C_F\{\tfrac{4}{3}\ln^2(1-x) - \tfrac{58}{9}\ln(1-x) + \tfrac{247}{27} - \tfrac{8}{3}\zeta(2)\}] + \delta(1-x)[C_F^2\{\tfrac{14}{3}\ln^4\delta - 10\ln^3\delta + (10-48\zeta(2))\ln^2\delta + (\tfrac{237}{2} + 36\zeta(2) + 8\zeta(3))\ln\delta\} + C_A C_F\{-\tfrac{22}{9}\ln^3\delta + (\tfrac{367}{18} - 4\zeta(2))\ln^2\delta + (-\tfrac{3155}{44} + \tfrac{44}{3}\zeta(2) + 40\zeta(3))\ln\delta\} + n_f C_F\{\tfrac{4}{9}\ln^3\delta - \tfrac{29}{9}\ln^2\delta + (\tfrac{247}{27} - \tfrac{8}{3}\zeta(2))\ln\delta\}] \quad (2.53)$$

Table 1. The residues of the pole-terms in the soft bremsstrahlungs corrections B^{DI} and B^{DY}. (See (2.50) and (2.51).)

	DI	DY
a_{22}	$-32\zeta(2)$	$-128\zeta(2)$
a_{21}	$-\frac{3}{2} + 36\zeta(2) + 8\zeta(3)$	$256\zeta(3)$
a_{20}	?	?
b_{23}	$-\frac{88}{3}$	$-\frac{88}{3}$
b_{22}	$\frac{466}{9} - 8\zeta(2)$	$\frac{268}{9} - 8\zeta(2)$
b_{21}	$-\frac{4541}{54} + \frac{110}{3}\zeta(2) + 40\zeta(3)$	$-\frac{808}{27} + \frac{154}{3}\zeta(2) + 28\zeta(3)$
b_{20}	?	?
c_{23}	$\frac{16}{3}$	$\frac{16}{3}$
c_{22}	$-\frac{76}{9}$	$-\frac{40}{9}$
c_{21}	$\frac{373}{27} - \frac{20}{3}\zeta(2)$	$\frac{112}{27} - \frac{28}{3}\zeta(2)$
c_{20}	$-\frac{7081}{324} + \frac{95}{9}\zeta(2) + \frac{64}{9}\zeta(3)$	$-\frac{328}{81} + \frac{70}{9}\zeta(2) + \frac{124}{9}\zeta(3)$

$$w_1(x) \overset{x\to 1}{=} \frac{\theta(1-\delta-x)}{(1-x)}[C_F^2\{\tfrac{896}{3}\ln^3(1-x) + 96\ln^2(1-x) - (256 + 576\zeta(2))\ln(1-x) + 256 - 240\zeta(2) + 512\zeta(3)\} + C_A C_F\{-\tfrac{176}{3}\ln^2(1-x) + (\tfrac{1072}{9} - 32\zeta(2)) \cdot\ln(1-x) - \tfrac{1616}{27} + \tfrac{176}{3}\zeta(2) + 56\zeta(3)\} + n_f C_f\{\tfrac{32}{3}\ln^2(1-x) - \tfrac{160}{9}\ln(1-x) + \tfrac{224}{27} - \tfrac{32}{3}\zeta(2)\}] + \delta(1-x)[C_F^2\{\tfrac{224}{3}\ln^4\delta + 32\ln^3\delta - (128 + 228\zeta(2))\ln^2\delta + (256 - 240\zeta(2) + 512\zeta(3))\ln\delta\} + C_A C_F\{-\tfrac{176}{9}\ln^3\delta + (\tfrac{536}{9} - 16\zeta(2))\ln^2\delta + (-\tfrac{1616}{27} + \tfrac{176}{3}\zeta(2) + 56\zeta(3))\ln\delta\} + n_f C_F\{\tfrac{32}{9}\ln^3\delta - \tfrac{80}{9}\ln^2\delta + (\tfrac{224}{27} - \tfrac{32}{3}\zeta(2))\ln\delta\}]. \quad (2.54)$$

In order to obtain expression (2.27) by means of mass-factorization it is convenient to calculate the Mellin-transform of the parton structure functions $\hat{\mathscr{F}}_2$ (2.11) and $\hat{W}$ (2.12). In Appendix B of ref. [10] one can find the Mellin-transforms of the distributions in the limit $n\to\infty$ which is equivalent to $x\to 1$. Using these formulae, we find

$$\ln \hat{\mathscr{F}}_2^{(n)}(Q^2,\varepsilon)$$

$$= \left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2} C_F\left\{(-8\ln n + 6) \cdot\frac{1}{\varepsilon} + 2\ln^2 n + 3mn - 2\zeta(2) - g\right\} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[C_F^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \left\{(\tfrac{3}{2} - 12\zeta(2) + 24\zeta(3))\frac{1}{\varepsilon} + (\tfrac{3}{2} - 12\zeta(2) + 24\zeta(3))\ln n\right\} + C_A C_F\left\{((-\tfrac{268}{9} + 8\zeta(2))\ln n + \tfrac{17}{6} + \tfrac{44}{3}\zeta(2) - 12\zeta(3))\frac{1}{\varepsilon} + (-\tfrac{88}{3}\ln n + 22)\frac{1}{\varepsilon^2} + (\tfrac{22}{3}\ln n - \tfrac{11}{2})\ln^2\left(\frac{Q^2}{\mu^2}\right) + (-\tfrac{22}{3}\ln^2 n + (-\tfrac{367}{9} + 8\zeta(2))\ln n\right.$$

$$+\tfrac{215}{6}+22\zeta(2)-12\zeta(3))\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$+\tfrac{22}{9}\ln^3 n+(\tfrac{367}{18}-4\zeta(2))\ln^2 n$$
$$+(\tfrac{3155}{54}-\tfrac{22}{3}\zeta(2)-40\zeta(3))\ln n\Big\}$$
$$+n_f C_F\Big\{(\tfrac{40}{9}\ln n-\tfrac{1}{3}-\tfrac{8}{3}\zeta(2))\frac{1}{\varepsilon}$$
$$+(\tfrac{16}{3}\ln n-4)\frac{1}{\varepsilon^2}+(-\tfrac{4}{3}\ln n+1)\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$+(\tfrac{4}{3}\ln^2 n+\tfrac{58}{9}\ln n-\tfrac{19}{3}-4\zeta(2))\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$-\tfrac{4}{9}\ln^3 n-\tfrac{29}{9}\ln^2 n+(-\tfrac{247}{27}+\tfrac{4}{3}\zeta(2))\ln n\Big\}\Big]. \tag{2.55}$$

$\ln \hat{W}^{(n)}(Q^2,\varepsilon)$

$$=\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon/2} C_F\Big\{(-16\ln n+12)\frac{1}{\varepsilon}$$
$$+8\ln^2 n-16+16\zeta(2)\Big\}$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left[C_F^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\Big\{(3-24\zeta(2)+48\zeta(3))\frac{1}{\varepsilon}\Big\}\right.$$
$$+C_A C_F\Big\{((-\tfrac{536}{9}+16\zeta(2))\ln n$$
$$+\tfrac{17}{3}+\tfrac{88}{3}\zeta(2)-24\zeta(3))\frac{1}{\varepsilon}$$
$$+(-\tfrac{176}{3}\ln n+44)\frac{1}{\varepsilon^2}+(\tfrac{44}{3}\ln n-11)\ln^2\left(\frac{Q^2}{\mu^2}\right)$$
$$+(-\tfrac{88}{3}\ln^2 n+(-\tfrac{536}{9}+16\zeta(2))\ln n$$
$$+\tfrac{193}{3}-\tfrac{88}{3}\zeta(2)-24\zeta(3))\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$+\tfrac{176}{9}\ln^3 n+(\tfrac{536}{9}-16\zeta(2))\ln^2 n$$
$$+(\tfrac{1616}{27}-56\zeta(3))\ln n\Big\}$$
$$+n_f C_F\Big\{(\tfrac{80}{9}\ln n-\tfrac{2}{3}-\tfrac{16}{3}\zeta(2))\frac{1}{\varepsilon}+(\tfrac{32}{3}\ln n-8)\frac{1}{\varepsilon^2}$$
$$+(-\tfrac{8}{3}\ln n+2)\ln^2\left(\frac{Q^2}{\mu^2}\right)+(\tfrac{16}{3}\ln^2 n+\tfrac{80}{9}\ln n$$
$$-\tfrac{34}{3}+\tfrac{16}{3}\zeta(2))\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$\left.-\tfrac{32}{9}\ln^3 n-\tfrac{80}{9}\ln^2 n-\tfrac{224}{27}\ln n\Big\}\right]. \tag{2.56}$$

Mass factorization (see 2.4) yields

$\ln \Delta^{(n)}(Q^2)$

$$=\ln \hat{W}^{(n)}(Q^2,\varepsilon)-2\ln\hat{\mathscr{F}}_2^{(n)}(Q^2,\varepsilon)$$
$$\overset{n\to\infty}{=}\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F\{4\ln^2 n-6\ln n+2+20\zeta(2)\}$$
$$+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2[C_F^2\{(-3+24\zeta(2)-48\zeta(3))\ln n\}$$
$$+C_A C_F\{\tfrac{44}{3}\ln^3 n+(\tfrac{169}{9}-8\zeta(2))\ln^2 n$$
$$+(-57+\tfrac{44}{3}\zeta(2)+24\zeta(3))\ln n\}$$
$$+n_f C_F\{-\tfrac{8}{3}\ln^3 n-\tfrac{22}{9}\ln^2 n+(10-\tfrac{8}{3}\zeta(2))\ln n\}]. \tag{2.57}$$

Where the $\beta_0 \ln(Q^2/\mu^2)$ term was removed using the definition of the running coupling constant (2.23).

Taking the inverse Mellin-transform of $\Delta^{(n)}(Q^2)$ we get

$\Delta(x, Q^2)$

$$\overset{x\to 1}{=}\frac{\theta(1-\delta-x)}{(1-x)}\left[\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F(8\ln(1-x)+6)\right.$$
$$+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2\{C_F^2(32\ln^3(1-x)+72\ln^2(1-x)$$
$$+(52+64\zeta(2))\ln(1-x)+15+24\zeta(2)$$
$$+112\zeta(3))+C_A C_F(-44\ln^2(1-x)$$
$$+(\tfrac{338}{9}-16\zeta(2))\ln(1-x)$$
$$+57+\tfrac{88}{3}\zeta(2)-24\zeta(3))+n_f C_f$$
$$\left.\cdot(8\ln^2(1-x)-\tfrac{44}{9}\ln(1-x)-10-\tfrac{16}{3}\zeta(2))\}\right]$$
$$+\delta(1-x)\left[1+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)\right.$$
$$\cdot C_F(4\ln^2\delta+6\ln\delta+2+16\zeta(2))+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2$$
$$\cdot\{C_F^2(8\ln^4\delta+24\ln^3\delta+(26+32\zeta(2))\ln^2\delta$$
$$+(15+24\zeta(2)+112\zeta(3))\ln\delta$$
$$+2+14\zeta(2)+48\zeta(3)+\tfrac{624}{5}\zeta(2)^2)$$
$$+C_A C_F(-\tfrac{44}{3}\ln^3\delta+(\tfrac{169}{9}-8\zeta(2))\ln^2\delta$$
$$+(57+\tfrac{88}{3}\zeta(2)-24\zeta(3))\ln\delta$$
$$-\tfrac{169}{9}\zeta(2)-\tfrac{88}{3}\zeta(3)+8\zeta(2)^2)$$
$$+n_f C_F(\tfrac{8}{3}\ln^3\delta-\tfrac{22}{9}\ln^2\delta-(10+\tfrac{16}{3}\zeta(2))\ln\delta$$
$$\left.+\tfrac{22}{9}\zeta(2)+\tfrac{16}{3}\zeta(3))\}\right]. \tag{2.58}$$

The coefficients of the leading and next-to-leading terms in (2.57) and (2.58) were already found by an explicit Feynman diagram calculation in [17]. This calculation was confirmed in [18], where it was also shown that the well known exponentiation [9, 10] of the $4\ln^2 n+2\pi^2$ could be generalized to the exponentiation of the complete lowest order assymptotic term. In IR behaved theories like QCD the typical form of (2.57) and (2.58) is characteristic of all quantities which are calculated in the limit where only soft gluons and fermion pairs are contributing. As an example we mention the $O(\alpha_s^2)$ calculations of the Sterman–Weinberg formula in [23] and the low p_T DY differential cross-section in [25]. In all these expressions large logarithms appear, of which the coefficients can be derived from the non-singlet anomalous dimension in (2.14). In the limit $n \to \infty$ the latter can be written as follows (see [20] and Appendix C).

$$\gamma^{(\infty)} = \gamma^{(K)} \ln n + \tilde{\gamma}. \tag{2.59}$$

Where $\gamma^{(K)}$ is the residue of the IR singularities appearing in the formfactor and in the soft bremsstrahlungs factor as given in (2.49)–(2.51). Its definition is given in [26]. Notice that it is a renormalization group invariant in contrast to $\tilde{\gamma}$. The latter merely represents the residues of the collinear divergences. From (2.14) it follows that $\gamma^{(K)}$ and $\tilde{\gamma}$ can be expanded in α_s

$$\gamma^{(K)} = \left(\frac{\alpha_s}{4\pi}\right)\gamma_0^{(K)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \gamma_1^{(K)} + \cdots \tag{2.60a}$$

$$\tilde{\gamma} = \left(\frac{\alpha_s}{4\pi}\right)\tilde{\gamma}_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \tilde{\gamma}_1 + \cdots \tag{2.60b}$$

The coefficients $\gamma_i^{(K)}$ and $\tilde{\gamma}_i$ can be found in Appendix C. Expression (2.57) can be rewritten into

$$\begin{aligned} \ln \Delta^{(n)}(Q^2) &= \left(\frac{\alpha_s(Q^2)}{4\pi}\right)[\tfrac{1}{2}\gamma_0^{(K)} \ln^2 n + \tilde{\gamma}_0 \ln n + C_F(2 + 20\zeta(2))] \\ &+ \left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2 [\tfrac{1}{2}\beta_0 \gamma_0^{(K)} \ln^3 n + \tfrac{1}{2}(\gamma_1^{(K)} + \beta_0 \tilde{\gamma}_0) \\ &\cdot \ln^2 n + (\tilde{\gamma}_1 + \beta_0 C_F(12\zeta(2) - 14)) \ln n]. \end{aligned} \tag{2.61}$$

We checked that the same type of expression can be found for (72) in [23] and for (15) in [25] and we believe that this can be generalized for all quantities which are dominated by soft gluons and soft fermion pairs.

3 $O(\alpha_s^2)$ corrections to the differential Drell–Yan cross-section

The DY differential cross-section will be denoted by

$$\frac{d^2\sigma}{dx_1^0 dx_2^0} = \frac{4\pi\alpha^2}{9Q^2} W(x_1^0, x_2^0) \quad x_F = \frac{2q_L}{\sqrt{s}}. \tag{3.1}$$

Where q_L is the longitudinal momentum of the virtual photon. Further, x_1^0 and x_2^0 are defined by $x_1^0 x_2^0 = \tau$ and $x_F = x_1^0 - x_2^0$. Since we deal with the same QCD processes as in the previous sections, the hadronic structure function becomes

$$\begin{aligned} &W(x_1^0, x_2^0, Q^2) \\ &= \int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1^0 - t_1 x_1)\delta(x_2^0 - t_2 x_2) \\ &\cdot [f^q(t_1, Q^2) f^{\bar{q}}(t_2, Q^2) + f^{\bar{q}}(t_1, Q^2) f^q(t_2, Q^2)] \\ &\cdot \Delta(x_1, x_2). \end{aligned} \tag{3.2}$$

Where $\Delta(x_1, x_2)$ is the non-singlet QCD correction term, which is determined in the following way

$$\begin{aligned} &\hat{W}(z_1, z_2, Q^2, \varepsilon) \\ &= \int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dx_1 \int_0^1 dx_2 \delta(z_1 - t_1 x_1)\delta(z_2 - t_2 x_2) \\ &\cdot \hat{\mathscr{F}}_2(t_1, Q^2, \varepsilon) \hat{\mathscr{F}}_2(t_2, Q^2, \varepsilon) \Delta(x_1, x_2, Q^2). \end{aligned} \tag{3.3}$$

The $\hat{W}$ stands for the parton DY structure function, which is defined in the same way as the hadronic one in (3.1), whereas $\hat{\mathscr{F}}_2$ is given by (2.11). From the lowest order calculation [7, 27] of $\Delta(x_1, x_2)$ we infer that it can be written as (Appendix D).

$$\begin{aligned} &\Delta(x_1, x_2, Q^2) \\ &= \delta(1 - x_1)\delta(1 - x_2)\Delta^{S+V}(Q^2, \delta) \\ &\quad + \delta(1 - x_1)\theta(1 - \delta - x_2)\Delta^{S+V+H}(x_2, Q^2, \delta) \\ &\quad + \theta(1 - \delta - x_1)\delta(1 - x_2)\Delta^{S+V+H}(x_1, Q^2, \delta) \\ &\quad + \theta(1 - \delta - x_1)\theta(1 - \delta - x_2)\Delta^H(x_1, x_2). \end{aligned} \tag{3.4}$$

From (3.3) and the expression for $\hat{\mathscr{F}}_2$ in (2.11) we can derive the same form for $\hat{W}(x_1, x_2, Q^2, \varepsilon)$ as for $\Delta(x_1, x_2)$. From the lowest order result for $\Delta(x_1, x_2)$ and $\hat{\mathscr{F}}_2$ we are now able to determine all pole terms in $\hat{W}$ which leads to an analogous expression as given in (2.12). We aim to determine $\Delta(x_1, x_2)$ in the limit $x_1, x_2 \to 1$, which will be the analogon of formula (2.58). We can proceed along the same line as has been done in the previous section. However, here we encounter some complications which are due to the C_F^2 part. The problem of the calculation of this part is revealed, if one looks at the perturbative expansion of $\hat{W}(z_1, z_2)$, which cannot be presented here since it is too long. The origin of the difficulty can be traced back to the second and third part of (3.4). In contrast to the total DY cross-section the $\ln \delta$ terms do not only appear in the double δ-function term, but also in the single ones. Since the quark form factor only contributes to the double δ-function, it is very difficult to compute the soft bremsstrahlungs factor $B^{DY}(\varepsilon, \delta)$ (2.26b) for the single δ-function. The situation is different for the $C_A C_F$ and $n_f C_F$ terms, because in these cases the calculation is very similar to the lowest order one, as we will show in Appendix D. Therefore we decided to choose a different approach which will be inspired by our findings in the last part of Sect. 2 and by the form of the lowest order correction term.

Let us start from the double Mellin-transform, which is defined by

$$f^{(n_1, n_2)} = \int_0^1 dz_1 \int_0^1 dz_2 z_1^{n_1 - 1} z_2^{n_2 - 1} f(z_1, z_2). \tag{3.5}$$

The lowest order contribution [13] to $\Delta^{(n_1,n_2)}$ is given by

$$\Delta^{(n_1,n_2)} \overset{n_1,n_2\to\infty}{=} 1+\left(\frac{\alpha_s}{4\pi}\right)C_F\{4\ln n_1\ln n_2-3\ln n_1 -3\ln n_2+2+20\zeta(2)\}. \quad (3.6)$$

A comparison of (3.6) with the lowest order expression for $\Delta^{(n)}$ in (2.57) shows that they are equal for $n_1=n_2=n$. Our first assumption will be that $\Delta^{(n_1,n_2)}$ is equal to $\Delta^{(n)}$ for $n_1=n_2=n$ and $n\to\infty$ to all orders in α_s. The second assumption is that $\Delta^{(n_1,n_2)}$ has the same structure as given in (2.61). This implies that the lowest order correction exponentiates completely and that the other coefficients are given by the β-function, the non-singlet anomalous dimension $\gamma^{(n)}$ and some unknown constants, which follow from the first assumption.

We make the following ansatz

$$\begin{aligned}\ln\Delta^{(n_1,n_2)}(Q^2) &=\left(\frac{\alpha_s(Q^2)}{4\pi}\right)[\tfrac{1}{2}\gamma_0^{(K)}\ln n_1\ln n_2+\tfrac{1}{2}\tilde{\gamma}_0(\ln n_1+\ln n_2)\\&+C_F(2+20\zeta(2))]\\&+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2[\tfrac{1}{4}\beta_0\gamma_0^{(K)}(\ln^2 n_1\ln n_2+\ln n_1\ln^2 n_2)\\&+\tfrac{1}{2}\gamma_1^{(K)}\ln n_1\ln n_2+\tfrac{1}{4}\beta_0\tilde{\gamma}_0(\ln^2 n_1+\ln^2 n_2)\\&+\tfrac{1}{2}(\tilde{\gamma}_1+\beta_0 C_F(12\zeta(2)-14))(\ln n_1+\ln n_2)]. \quad (3.7)\end{aligned}$$

The splitting of the $\ln n_1 \ln n_2$ and the $\ln^2 n_i$ terms for the $C_A C_F$ and $n_f C_F$ part seems artificial. However, in Appendix D we will show that the structure of the above formula can be derived in the same way as has been done in the previous section. An interesting feature of the above equation is that $\gamma^{(K)}$ only contributes to the mixed logarithms $\ln^i n_1 \ln^j n_2$, whereas $\tilde{\gamma}$ only determines the coefficients of unmixed ones $\ln^i n_k$.

Taking now the inverse Mellin-transform of (3.7) we get

$$\begin{aligned}\Delta(x_1,x_2,Q^2) &=\delta(1-x_1)\delta(1-x_2)\\&\cdot\Big[1+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F(4\ln^2\delta+6\ln\delta+2+20\zeta(2))\\&+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2\{C_F^2(8\ln^4\delta+24\ln^3\delta\\&+(26+64\zeta(2))\ln^2\delta+(15+72\zeta(2)+48\zeta(3))\ln\delta\\&+2+31\zeta(2)+208\zeta(2)^2)+C_AC_F(-\tfrac{44}{3}\ln^3\delta\\&+(\tfrac{169}{9}-8\zeta(2))\ln^2\delta+(57-24\zeta(3))\ln\delta+11\zeta(2))\\&+n_fC_F(\tfrac{8}{3}\ln^3\delta-\tfrac{22}{9}\ln^2\delta-10\ln\delta-2\zeta(2))\}\Big]\\&+\frac{\delta(1-x_1)\theta(1-\delta-x_2)}{(1-x_2)}\\&\cdot\Big[\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F(3+4\ln\delta)\\&+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2\{C_F^2((9-16\zeta(2))\ln(1-x_2)\\&+\tfrac{15}{2}+36\zeta(2)+24\zeta(3)+(16\ln^2\delta+24\ln\delta)\\&\cdot\ln(1-x_2)+12\ln^2\delta+(17+80\zeta(2))\ln\delta)\\&+C_AC_F(-11\ln(1-x_2)+\tfrac{57}{2}-12\zeta(3)\\&-\tfrac{44}{3}\ln\delta\ln(1-x_2)-\tfrac{22}{3}\ln^2\delta+(\tfrac{268}{9}-8\zeta(2))\ln\delta)\\&+n_fC_F(2\ln(1-x_2)-5+\tfrac{8}{3}\ln\delta\ln(1-x_2)\\&+\tfrac{4}{3}\ln^2\delta-\tfrac{40}{9}\ln\delta)\}\Big]+[\mathbf{x}_1\leftrightarrow\mathbf{x}_2]\\&+\frac{\theta(1-\delta-x_1)\theta(1-\delta-x_2)}{(1-x_1)(1-x_2)}\Big[\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F(4)\\&+\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2\{C_F^2(32\ln(1-x_1)\ln(1-x_2)\\&+24(\ln(1-x_1)+\ln(1-x_2))+17+80\zeta(2))\\&+C_AC_F(-\tfrac{44}{3}(\ln(1-x_1)+\ln(1-x_2))\\&+\tfrac{268}{9}-8\zeta(2))\\&+n_fC_F(\tfrac{8}{3}(\ln(1-x_1)+\ln(1-x_2))-\tfrac{40}{9})\}\Big]. \quad (3.8)\end{aligned}$$

The $\ln\delta$ terms follow from the condition that $\int_0^1 dx_1 \cdot \int_0^1 dx_2 \Delta(x_1,x_2)$ is finite.

4 Discussion of the results

Here we will discuss the usefulness of the approximate expressions for the DY correction terms, which have been derived in the previous sections. We will look into the range of applicability of the approximation and its relevance to the anomalous scaling of the K-factor. This phenomenon has been discovered by the NA10 group [1, 2] and it shows up at medium values of τ.

The approximate formula for the DY correction terms are given in (2.58) and (3.8). They have been derived in the limit $x\to 1$ and $x_1, x_2\to 1$ respectively. We will show that in practice these expressions can be applied in a much larger range than only near the boundary of phase-space, where $\tau\approx 1$.

Starting with the total cross-section (2.2), we define the theoretical K-factor by

$$K_{th}=\sum_{n=0}^{\infty}K^{(n)}. \quad (4.1)$$

Where $K^{(n)}$ represents the order α_s^n contribution to the K-factor. It is defined as

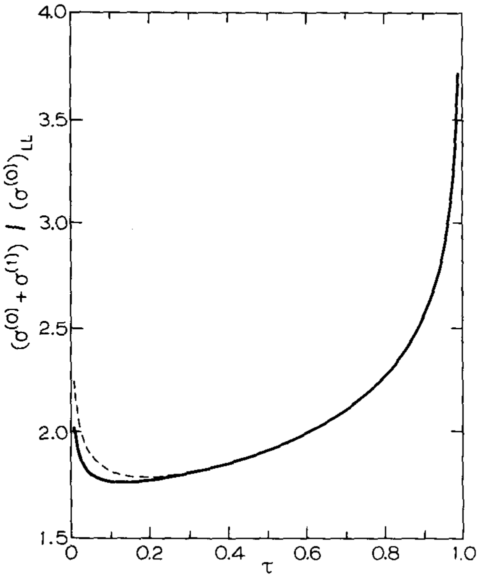

Fig. 4. $K^{(0)}+K^{(1)}_{\text{exact}}$: solid line, $K^{(0)}+K^{(1)}_{\text{app}}$: dashed line at $\sqrt{s}=19.1$ GeV

$$K^{(n)}=\frac{\left(\dfrac{d\sigma^{(n)}}{dQ^2}\right)}{\left(\dfrac{d\sigma^{(0)}}{dQ^2}\right)_{LL}}=\frac{W^{(n)}(\tau,Q^2)}{W^{(0)}_{LL}(\tau,Q^2)}. \tag{4.2}$$

The relation between $\frac{d\sigma^{(n)}}{dQ^2}$ and $W^{(n)}$ is given in (2.2). The latter is determined by the order α_s^n term in $\Delta(x,Q^2)$ using (2.3). The quantity $\left(\frac{d\sigma^{(0)}}{dQ^2}\right)_{LL}$ is the naive DY cross-section with the parton distribution functions in the leading log approximation. The nucleon structure functions used in the numerator and denominator of (4.2) can be found in the first column of Table 1 in [2]. The pion structure functions in the numerator and denominator are obtained from the NLLA and LLA parametrizations respectively. These can be found in Table 2 of the same reference. Our figures are presented for a CM-energy of $\sqrt{s}=19.1$ GeV, where the data of the NA10 group have been taken.

We want to compare the exact DY correction term (see 2.27), which we will call Δ_{exact}, with the approximated one. The latter, denoted by Δ_{app}, is defined as

$$\Delta_{\text{app}}(x,Q^2)=\Delta_{\text{exact}}(x,Q^2)-\tilde{\Delta}(x,Q^2). \tag{4.3}$$

The definition of $\tilde{\Delta}(x,Q^2)$ is given in (2.27). It contains all terms which are integrable in $x=1$. Up to order α_s the expression for $\tilde{\Delta}(x,Q^2)$ can be obtained from (2.22a), (C.6) and (C.7). It is equal to

$$\tilde{\Delta}(x,Q^2)=\left(\frac{\alpha_s(Q^2)}{4\pi}\right)C_F[-4(1+x)\ln(1-x)-8x-12]. \tag{4.4}$$

The coefficients Δ_{ij} and Δ_i in (2.27) determine $\Delta_{\text{app}}(x,Q^2)$. Up to order α_s^2 they can be found in (2.58).

In Fig. 4 we have plotted $K^{(0)}+K^{(1)}_{\text{exact}}$ (solid line) and $K^{(0)}+K^{(1)}_{\text{app}}$ (dashed line). In this figure we see that the difference between these quantities is negligible for $\tau>0.3$. This is remarkable, for one would expect that the relation: $K_{\text{exact}}=K_{\text{app}}$ is only valid for $\tau\approx 1$. The $O(\alpha_s^2)$ expression for $\tilde{\Delta}(x,Q^2)$ is not yet known. However, one can argue that it is still negligible with respect to the other contributions to $\Delta(x,Q^2)$. This can be inferred by studying e.g. the corrections to processes which do not contain soft and virtual contributions and therefore only consist of $\tilde{\Delta}(x,Q^2)$. An example is $q+q\to q+q+\gamma^*$ which has been calculated in [28]. One finds that the contribution of this process to the DY cross-section is negligible and that the $O(\alpha_s^2)$ expression for $\tilde{\Delta}(x,Q^2)$ is a sum of polylogarithms which go to zero, if $x\to 1$. The same observation was made in [29]. In this paper one has calculated the order α^2 QED initial state photonic correction to the process $e^+e^-\to\mu^+\mu^-$. This corresponds to the Abelian part of the DY process in (2.9) and (2.10). Here we observe that the terms in $\tilde{\Delta}(x,Q^2)$ are small with respect to those originating from soft photon radiation, which appear in Δ_{app}. Summarizing our discussion we conclude that for $\tau>0.3$ we can replace Δ_{exact} by Δ_{app}. For the study of anomalous scaling only the high τ region is relevant, so we will use Δ_{app} in the subsequent part.

Using (2.3) and (2.27) $W^{(n)}(\tau,Q^2)$ can be written as

$$W^{(n)}_{\text{app}}(\tau,Q^2)=W^{(0)}(\tau,Q^2)\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^n\Delta_n+W^{(0)}(\tau,Q^2)\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^n\cdot\sum_{j=0}^{2n-1}\frac{1}{j+1}\Delta_{nj}\ln^{j+1}(1-\tau)+R^{(n)}(\tau,Q^2). \tag{4.5}$$

Where the coefficients Δ_n, Δ_{nj} for $n=1,2$ can be read off from (2.58). The expression for $W^{(1)}_{\text{app}}$ is completely known. However, since the C_F^2 and C_AC_F contributions to Δ_2 are not calculated yet, we are not able to determine $W^{(2)}_{\text{app}}$. Therefore one assumes that Δ_2 is obtained from Δ_1 by exponentiation, so that $\Delta_2=1/2(\Delta_1)^2$. In this case $W^{(2)}_{\text{app}}$ is given by the $O(\alpha_s^2)$ part of $\Delta(x,Q^2)$ in (2.58). In the literature one assumes that the contribution to the K-factor can be wholly attributed to the Δ_n and the leading $\ln^{2n}(1-\tau)$ term in (4.5). However, as we can see in Fig. 5, this assumption is only correct, if τ is almost equal to one. (In Fig. 5 the numbers 1 (4), 2 (5) and 3 (6) correspond to the contributions to $K^{(1)}$ ($K^{(2)}$) coming from the first, second and last part of (4.5).) In the medium τ region, which is experimentally accessible, the leading $\ln(1-\tau)$ terms are nearly completely cancelled by the next-to-leading ones and the correction is dominated by the first and last terms in (4.5). This holds for $K^{(1)}$ as well as $K^{(2)}$. From this we infer that almost in the whole τ range the shape of the K-factor is determined

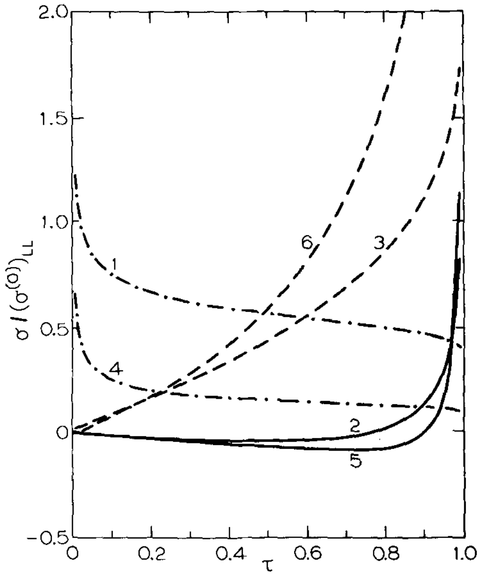

Fig. 5. Contributions to $K^{(1)}_{\mathrm{app}}$ (order α_s) at $\sqrt{s} = 19.1\,\mathrm{GeV}$. Equation (4.5) is split into three parts for each order of α_s: 1: Δ_1, 2: $\ln^i(1-\tau)$, 3: $R^{(1)}(\tau, Q^2)$. Contributions to $K^{(2)}_{\mathrm{app}}$ (order α_s^2) at $\sqrt{s} = 19.1\,\mathrm{GeV}$. 4: Δ_2, 5: $\ln^i(1-\tau)$, 6: $R^{(2)}(\tau, Q^2)$

by $R^{(n)}(\tau, Q^2)$. It also gives an important contribution to the overall normalization of the K-factor, in particular for $\tau > 0.5$. The same conclusions hold, when the leading log parton distribution functions in $W^{(0)}_{LL}$ of (4.2) are replaced by the scale independent ones. The importance of the term $R^{(n)}(\tau, Q^2)$ to the phenomenon of anomalous scaling will become clear when we discuss its analogon in the case of the differential DY cross-section.

In the literature one assumes that the radiative correction is dominated by the C_F part of the Δ_n and the (leading) $\ln^{2n}(1-\tau)$ term in (4.5). In order to improve the perturbation series one performs a resummation of these terms, which is usually done by exponentiation. Generalizing the exponentiation to the total $O(\alpha_s)$ term we obtain

$$\begin{aligned} W_{\mathrm{app}}(\tau, Q^2) &= \sum_{n=0}^{\infty} W^{(n)}_{\mathrm{app}}(\tau, Q^2) \\ &= \sum_{n=1}^{\infty} R^{(n)}(\tau, Q^2) + W^{(0)}(\tau, Q^2) \exp\left[\left(\frac{\bar{\alpha}_s}{4\pi}\right) C_F \right. \\ &\quad \left. \cdot \{4\ln^2(1-\tau) + 6\ln(1-\tau) + 16\zeta(2) + 2\}\right] \\ &\quad \cdot \left[1 + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^2 \{C_F^2(-32\zeta(2)\ln^2(1-\tau) \right. \\ &\quad + (112\zeta(3) - 72\zeta(2) + 3)\ln(1-\tau) \\ &\quad - \tfrac{16}{5}\zeta(2)^2 + 48\zeta(3) - 18\zeta(2)) \\ &\quad + C_A C_F((\tfrac{367}{9} - 8\zeta(2))\ln^2(1-\tau) \\ &\quad + (-24\zeta(3) + 88\zeta(2) + \tfrac{193}{3})\ln(1-\tau) \\ &\quad + 8\zeta(2)^2 - \tfrac{88}{3}\zeta(3) - \tfrac{169}{9}\zeta(2)) \\ &\quad + n_f C_F(-\tfrac{58}{9}\ln^2(1-\tau) - (16\zeta(2) + \tfrac{34}{3})\ln(1-\tau) \\ &\quad \left. + \tfrac{16}{3}\zeta(3) + \tfrac{22}{9}\zeta(2))\} + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^3 \{\cdots\} + \cdots\right]. \end{aligned} \tag{4.6}$$

Following [11] we have absorbed the $\ln(1-\tau)$ in the running coupling constant $\alpha_s(Q^2)$, so $\bar{\alpha}_s$ is given by

$$\bar{\alpha}_s = \alpha_s(Q^2(1-\tau)). \tag{4.7}$$

The argument of the exponent was already derived in [17, 18]. This resummation is only useful if the following conditions are satisfied.

- The perturbation series is dominated by the $\ln(1-\tau)$ terms.
- Δ_n does not deviate too much $(\Delta_1)^n/n!$.
- $R^{(n)}(\tau, Q^2)$ is small.
- $Q^2(1-\tau) \gg \Lambda^2$ (See (2.23)).

However, for medium τ values ($\tau < 0.7$) these conditions are not satisfied, since the sum of the $\ln^i(1-\tau)$ terms is small compared to $R^{(n)}(\tau, Q^2)$ (see Fig. 5). Notice that the medium τ-region is accessible to experiment [1, 2]. Further, it is not clear whether Δ_n really exponentiates. This question we hope to answer in the future when the exact value of Δ_2 is known.

We now call our attention to the differential DY cross-section (3.1), which is of more relevance to experiment than the total one. In the subsequent part of this section we will use the differential cross-section in the following form (3.1)

$$\frac{d^2\sigma}{d\sqrt{\tau}\,dx_F} = \frac{2\sqrt{\tau}}{x_1^0 + x_2^0} \frac{d^2\sigma}{dx_1^0 dx_2^0} \tag{4.8}$$

Proceeding in the same way as at the beginning of this section, we will first discuss the validity of the approximate DY correction term $\Delta_{\mathrm{app}}(x_1, x_2)$, given by (3.8), in the experimental τ-region. For this purpose we compare the $O(\alpha_s)$ contribution to $\Delta_{\mathrm{app}}(x_1, x_2)$ with the exact $O(\alpha_s)$ correction $\Delta_{\mathrm{exact}}(x_1, x_2)$, which can be found in [7, 16, 27]. Let us first define the differential K-factor.

$$dK^{(n)} = \frac{\left(\dfrac{d^2\sigma^{(n)}}{d\sqrt{\tau}\,dx_F}\right)}{\left(\dfrac{d^2\sigma^{(0)}}{d\sqrt{\tau}\,dx_F}\right)_{LL}} = \frac{W^{(n)}(x_1^0, x_2^0, Q^2)}{W^{(0)}_{LL}(x_1^0, x_2^0, Q^2)}. \tag{4.9}$$

The quantities which appear in the numerator and denominator of the above equation are the analogons of those defined below (4.2). In Fig. 6 we have plotted $dK^{(0)} + dK^{(1)}_{\mathrm{exact}}$ and $dK^{(0)} + dK^{(1)}_{\mathrm{app}}$ versus τ for $x_F = 0$. Form this figure we infer that the difference between the exact and approximated K-factor disappears for $\tau > 0.4$. This situation was also observed in the case of the total cross-section. Following the same arguments as have been given before, we assume that the difference between $dK^{(n)}_{\mathrm{exact}}$ and $dK^{(n)}_{\mathrm{app}}$ will

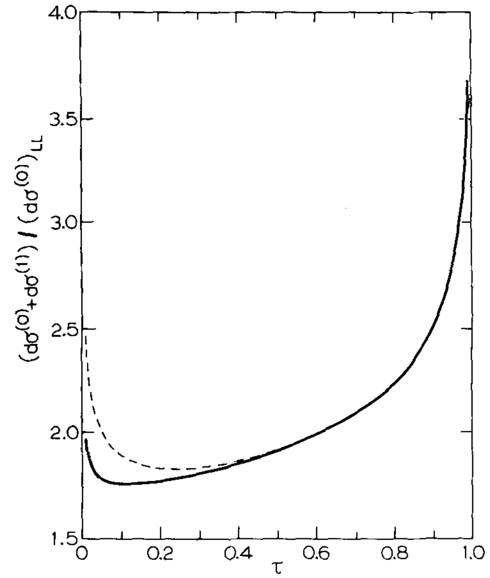

Fig. 6. $dK^{(0)} + dK^{(1)}_{\text{exact}}$: solid line, $dK^{(0)} + dK^{(1)}_{\text{app}}$: dashed line at $\sqrt{s} = 19.1$ GeV

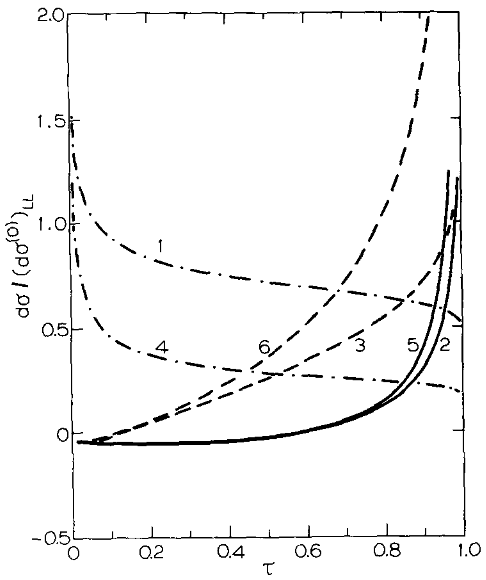

Fig. 7. Contributions to $dK^{(1)}_{\text{app}}$ (order α_s) at $\sqrt{s} = 19.1$ GeV. Equation (4.10) is split into three parts for each order of α_s: 1: $\bar{\Delta}_1$, 2: $\ln^i(1-x_1^0)\ln^j(1-x_2^0)$, 3: $\bar{R}^{(1)}(x_1^0, x_2^0, Q^2)$. Contributions to $dK^{(2)}_{\text{app}}$ (order α_s^2) at $\sqrt{s} = 19.1$ GeV. 4: $\bar{\Delta}_2$, 5: $\ln^i(1-x_1^0)\ln^j(1-x_2^0)$, 6: $\bar{R}^{(2)}(x_1^0, x_2^0, Q^2)$

vanish in the same τ-region for $n > 1$. After the convolution of $\Delta(x_1, x_2)$ with the parton structure functions, we can write

$$W^{(n)}(x_1^0, x_2^0, Q^2) = W^{(0)}(x_1^0, x_2^0, Q^2)\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^n \bar{\Delta}_n + W^{(0)}(x_1^0, x_2^0, Q^2)\left(\frac{\alpha_s(Q^2)}{4\pi}\right)^n \cdot \sum_{i=0}^{n}\sum_{j=0}^{n} \bar{\Delta}_{nij} \ln^i(1-x_1^0)\ln^j(1-x_2^0) + \bar{R}^{(n)}(x_1^0, x_2^0, Q^2). \tag{4.10}$$

The coefficients $\bar{\Delta}_n$, $\bar{\Delta}_{nij}$ can be derived from (3.8). Notice that the exact $\bar{\Delta}_2$ is not known yet since we derived it from Δ_2 in (4.5), which was obtained by exponentiation. In Fig. 7 we have plotted the order α_s (α_s^2) contributions from the first 1 (4), the second 2 (5) and the third term 3 (6) of (4.10) to $dK^{(1)}$ ($dK^{(2)}$) for $x_F = 0$. As has been observed in the case of the total cross-section, the first and the last part of (4.10) constitute the bulk of the radiative correction. The logarithms of the type $\ln^i(1-x_1^0)\ln^j(1-x_2^0)$ cancel each other in the experimentally accessible τ-region and they only contribute at very high τ values. Further, we see that $\bar{R}^{(n)}$ determines the shape of the K-factor and dominates the correction for $\tau > 0.7$. These features can also be observed for x_F values other than zero. In order to show this we adopt the presentation of the experimental data in [1, 2] and compute the cross-section

$$d\sigma^{(n)} = \int dx_F \int d\sqrt{\tau}\, \frac{d^2\sigma^{(n)}}{d\sqrt{\tau}\, dx_F}. \tag{4.11}$$

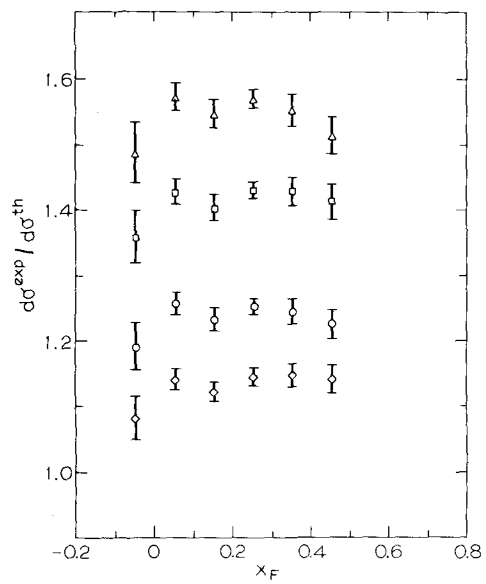

Fig. 8. $0.24 < \sqrt{\tau} < 0.30$, $\sqrt{s} = 19.1$ GeV, △: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}})$, □: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{app}})$, ○: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}} + d\sigma^{(2)}_{\text{app}})$, ◇: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}} + d\sigma^{(2)}_{\text{app}})$, where the denominator is exponentiated according to (4.12)

The integrals in (4.11) are taken over the $\sqrt{\tau} - x_F$ cells given in Table 2 of [1]. The parton structure functions in (4.11) are the same as those appearing in the numerator of (4.2) and (4.9). In Figs. 8 and 9 we have presented the quantities $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}})$ and $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{app}})$ for $0.24 < \sqrt{\tau} < 0.30$ and

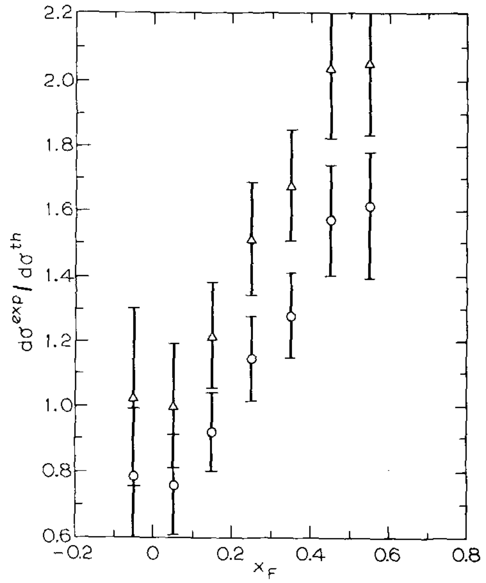

Fig. 9. $0.54 < \sqrt{\tau} < 0.72$ $\sqrt{s} = 19.1$ GeV, $\triangle$: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}})$, $\bigcirc$: $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}} + d\sigma^{(2)}_{\text{app}})$

$0.54 < \sqrt{\tau} < 0.72$ respectively (see also Fig. 6 in [2]). In the lowest $\sqrt{\tau}$-bin the two expressions do not overlap, whereas in the highest bin we cannot distinguish anymore the approximated from the exact order α_s contribution. Hence we can conclude that for the highest $\sqrt{\tau}$-bin the expression $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{app}} + d\sigma^{(2)}_{\text{app}})$ will give a good description of its exact counterpart. This is important because in this bin the anomalous scaling of the K-factor has been observed by the NA10 experiment [1, 2]. As is shown in [2] the $O(\alpha_s)$ contribution to $\Delta(x_1, x_2)$ is unable to explain the experimental data for the whole τ-range. In the lowest $\sqrt{\tau}$-bin the ratio $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}})$ is constant (~ 1.5) in the range $0.0 < x_F < 0.6$. For the highest bin this ratio changes from 1 to 2 if x_F increases from 0.0 to 0.6. This phenomenon is known as anomalous scaling. The agreement between theory and experiment improves if we include the approximated $O(\alpha_s^2)$ correction (3.8). This can been seen in Figs. 8 and 9, where we have plotted the quantity $d\sigma_{\text{exp}}/(d\sigma^{(0)} + d\sigma^{(1)}_{\text{exact}} + d\sigma^{(2)}_{\text{app}})$. For the lower τ values experiment and theory agree fairly well and for the highest $\sqrt{\tau}$-bin the agreement gets better. A further improvement is achieved, in particular for the lowest $\sqrt{\tau}$-bin, if we resum the first and second parts of (4.10).

$$\begin{aligned} W_{\text{app}}(x_1^0, x_2^0, Q^2) &= \sum_{n=0}^{\infty} W^{(n)}_{\text{app}}(x_1^0, x_2^0, Q^2) \\ &= \sum_{n=1}^{\infty} \bar{R}^{(n)}(x_1^0, x_2^0, Q^2) + W^{(0)}(x_1^0, x_2^0, Q^2) \\ &\cdot \exp\left[\left(\frac{\tilde{\alpha}_s}{4\pi}\right) C_F\{4L_1L_2 + 3L_1 + 3L_2 + 2 + 20\zeta(2)\}\right] \\ &\cdot \left[1 + \left(\frac{\tilde{\alpha}_s}{4\pi}\right)^2 \{C_F^2(-8\zeta(2)(L_1^2 + L_2^2) + (24\zeta(3) \right. \\ &\quad - 24\zeta(2) + \tfrac{3}{2})(L_1 + L_2) + 8\zeta(2)^2 - 9\zeta(2)) \\ &\quad + C_A C_F((\tfrac{367}{9} - 8\zeta(2))L_1L_2 + (\tfrac{193}{6} + \tfrac{110}{3}\zeta(2) \\ &\quad - 12\zeta(3))(L_1 + L_2) + 11\zeta(2)) \\ &\quad + n_f C_F(-\tfrac{58}{9}L_1L_2 - (\tfrac{17}{3} + \tfrac{20}{3}\zeta(2)) \\ &\quad \left. \cdot(L_1 + L_2) - 2\zeta(2))\} + \left(\frac{\tilde{\alpha}_s}{4\pi}\right)^3 \{\cdots\} + \cdots\right] \end{aligned} \tag{4.12}$$

where L_1 and L_2 are given by

$$L_1 = \ln(1 - x_1^0) \quad L_2 = \ln(1 - x_2^0) \tag{4.13}$$

and $\tilde{\alpha}_s$ is the modified coupling constant defined by

$$\tilde{\alpha}_s = \alpha_s\left(Q^2\sqrt{(1 - x_1^0)(1 - x_2^0)}\right). \tag{4.14}$$

We have checked that the leading term in the exponent: $\tilde{\alpha}_s L_1 L_2$ agrees with the corresponding one in (25) of [13], provided one makes an expansion in the renormalized coupling constant α_s. As we have pointed out below (4.7), the above resummation of the L_i is only useful if they dominate the radiative correction, which occurs if $x_1^0, x_2^0 \to 1$ (see Fig. 7). Outside this region the correction is dominated by either $\bar{\Delta}_n$ of $\bar{R}^{(n)}$ and the leading and subleading terms in L_i cancel each other. Therefore in the experimentally accessible τ-region only the exponentiation of $\bar{\Delta}_1 (= 2 + 20\zeta(2))$ is useful. This in particular holds for the lowest $\sqrt{\tau}$-bin (Fig. 8) where the agreement between experiment and theory is rather good. In the highest $\sqrt{\tau}$-bin the distinction between the cross-section perturbatively corrected up to $O(\alpha_s^2)$ and the exponentiated* one disappears. Here the term $\bar{R}^{(n)}$ in (4.12) plays an important role. This can be seen in Fig. 10, where we show the relative contributions from $\bar{\Delta}_2$ and $\bar{R}^{(2)}$ to the quantity $W^{(2)}_{\text{app}}/(W^{(0)} + W^{(1)}_{\text{app}})$ for two values of τ. They correspond to the endpoints of the highest $\sqrt{\tau}$-bin, i.e. $\tau = 0.25$ and $\tau = 0.55$. For $\tau = 0.25$ the $\bar{\Delta}_2$ term dominates $\bar{R}^{(2)}$ but at $\tau = 0.55$ their roles are reversed. For both values of τ one observes that the $\bar{\Delta}_2$ contribution remains almost constant, whereas at $\tau = 0.55$ the $\bar{R}^{(2)}$ term rises very steeply as a function of x_F. This implies that the slope of the data in Fig. 9 is determined by $\bar{R}^{(2)}$ rather than by $\bar{\Delta}_2$. The latter only accounts for the normalization. This situation will not change if the exact value of $\bar{\Delta}_2$ is known.

In the above we have seen that for high τ values $\bar{R}^{(n)}$ ($n \leqq 2$) give large contributions to the K-factor. We expect that for $n > 2$ the $\bar{R}^{(n)}$ will further decrease the slope, since the perturbation series is very slowly convergent. In contrast to $\bar{\Delta}_n$ and the logarithms $\ln(1 - x_i^0)$ in (4.10), we do not know how to resume $\bar{R}^{(n)}$. We believe that the $\bar{R}^{(n)}$ in (4.10) partially explain

* In this $\sqrt{\tau}$-bin exponentiation according to (4.12) breaks down for the highest two x_F values, because the requirement $Q^2\sqrt{(1 - x_1^0)(1 - x_2^0)} \gg \Lambda^2$ is no longer fulfilled

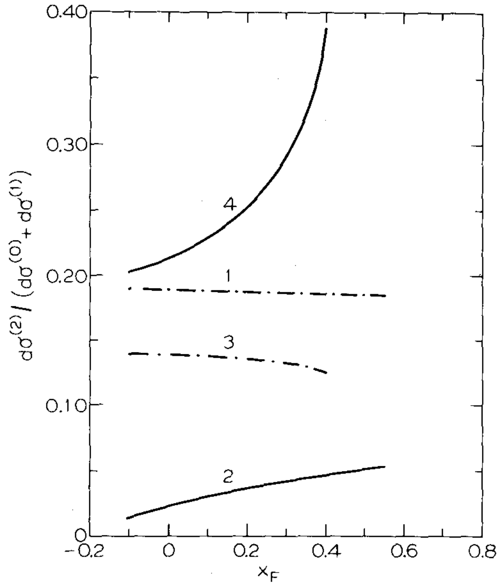

Fig. 10. Contributions to $W^{(2)}_{\text{app}}/(W^{(0)}+W^{(1)}_{\text{app}})$. $\sqrt{s}=19.1$ GeV, See (4.10). $\tau=0.25$: 1: $\bar{\Delta}_2$, 2: $\bar{R}^{(2)}(x_1^0,x_2^0,Q^2)$. $\tau=0.55$: 3: $\bar{\Delta}_2$, 4: $\bar{R}^{(2)}(x_1^0,x_2^0,Q^2)$

the phenomenon of anomalous scaling of the K-factor. A second important result of our findings is that the $\ln^i(1-x)/(1-x)$ terms appearing in (2.58) and (3.8) constitute a large part of the radiative correction in a τ-region far beyond the boundary of phase space.

Finally we want to comment on the relation between the Δ_2 in (4.5) and the $\bar{\Delta}_2$ in (4.10). The $\bar{\Delta}_2$ was derived from the requirement that the Mellin-transforms of the corrections terms in (2.58) and (3.8) should be equal in the limit $n=n_1=n_2\to\infty$. We have checked that the differential cross-section integrated over x_F and the total cross-section give the same numerical results.

5 Summary and conclusions

In this paper we have calculated up to order α_s^2 all large logarithms of the type $\ln^i(1-x)/(1-x)$ which contribute to the correction term of the total DY cross-section. The coefficients of these logarithms are derived from the behaviour of the phase space integrals and the renormalization group parameters, which are known up to order α_s^2. Furthermore, we give a plausible derivation of the large logarithms in the correction term of the differential DY cross-section. The above calculation is an extension of the work done in [17, 18]. In the higher τ-region which is still accessible to experiment we were able to show that the above terms together with the constant part of the correction term dominate the radiative correction to the naive DY process. After convoluting the correction term with the parton distribution functions, the contribution of the large logarithms can be split into two parts, viz. the $\ln^i(1-\tau)$ terms and a piece called $R^{(n)}(\tau,Q^2)$. In the experimentally accessible region the first mentioned terms cancel each other. Therefore, except for the very high τ-region, the resummation of the $\ln^i(1-\tau)$ does not improve the perturbation series and one is certainly not justified to exponentiate the leading logarithm only.

We have seen that the $\bar{R}^{(n)}(x_1^0,x_2^0,Q^2)$ account for the shape of the K-factor and give a partial explanation of anomalous scaling. We believe that the agreement between theory and experiment will improve, if higher order contributions of $\bar{R}^{(n)}(x_1^0,x_2^0,Q^2)$ are included.

Notice that in our analysis we have used the pion and nucleon structure functions of [2]. The pion parameters were determined by fitting the DY differential cross-section corrected up to order α_s to the data in the lowest $\sqrt{\tau}$-bin. This fitting procedure should have been repeated to include the order α_s^2 contribution. However, we have postponed this until the coefficient $\bar{\Delta}_2$ in (4.10) is exactly known, since this is the most important part of the correction term for the lower τ values. Finally, as has been pointed out in [5], it is important to measure the nuclear structure functions with a higher degree of accuracy.

Acknowledgements. We are indebted to J.G.M. Kuijf for his assistance in making the computer programs. One of us, W.L. van Neerven would like to thank the Eidgenössische Technische Hochschule in Zürich where a part of this work was done.

Appendix A

In this appendix we present the results of the parton structure functions corresponding to the fermion pair production processes in (2.7) and (2.10). Their calculation is very similar to the lowest order one. The difference is that the outgoing gluon, which decays into the fermions, becomes virtual, so that we have to insert the gluon self energy and its absorptive part in the appropriate graphs. The unrenormalized structure functions are given by

$$\hat{\mathcal{F}}_2^{(2),V}(Q^2,\varepsilon) = \delta(1-x)n_f C_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon} \cdot\left[-\frac{16}{3}\frac{1}{\varepsilon^3}+\frac{112}{9}\frac{1}{\varepsilon^2}+\left(-\frac{706}{27}-\frac{4}{3}\zeta(2)\right)\frac{1}{\varepsilon} +\frac{7541}{162}+\frac{28}{9}\zeta(2)-\frac{52}{9}\zeta(3)\right] \tag{A.1}$$

$$\hat{\mathcal{F}}_2^{(2),S}(Q^2,\varepsilon,\delta) = \delta(1-x)n_f C_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\delta^{\varepsilon} \cdot\left[\frac{16}{3}\frac{1}{\varepsilon^3}-\frac{76}{9}\frac{1}{\varepsilon^2}+\left(\frac{373}{27}-\frac{20}{3}\zeta(2)\right)\frac{1}{\varepsilon} -\frac{7081}{324}+\frac{95}{9}\zeta(2)+\frac{64}{9}\zeta(3)\right] \tag{A.2}$$

$$\hat{\mathscr{F}}_2^{(2),H}(x,Q^2,\varepsilon)$$
$$= \theta(1-\delta-x) n_f C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon$$
$$\cdot \left[\frac{8}{3}\frac{1+x^2}{1-x}\frac{1}{\varepsilon^2} + \left\{\frac{1+x^2}{1-x}(\tfrac{8}{3}\ln(1-x) - 4\ln x - \tfrac{38}{9}) + \tfrac{10}{3} + 6x\right\}\frac{1}{\varepsilon} + \frac{1+x^2}{1-x}(\tfrac{4}{3}\ln^2(1-x) - \tfrac{38}{9}\ln(1-x) - 4\ln x \ln(1-x) - \tfrac{4}{3}Li_2(1-x) + \tfrac{7}{3}\ln^2 x + \tfrac{22}{3}\ln x + \tfrac{373}{54} - \tfrac{10}{3}\zeta(2)) + (\tfrac{10}{3} + 6x)\ln(1-x) - (\tfrac{16}{3} + 8x)\ln x - \tfrac{125}{18} - \tfrac{33}{2}x\right] \tag{A.3}$$

$$\hat{W}^{(2),V}(Q^2,\varepsilon)$$
$$= \delta(1-x) n_f C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \cdot \left[-\frac{16}{3}\frac{1}{\varepsilon^3} + \frac{112}{9}\frac{1}{\varepsilon^2} + (-\tfrac{706}{27} + \tfrac{44}{3}\zeta(2))\frac{1}{\varepsilon} + \tfrac{7541}{162} - \tfrac{308}{9}\zeta(2) - \tfrac{52}{9}\zeta(3)\right] \tag{A.4}$$

$$\hat{W}^{(2),S}(Q^2,\varepsilon,\delta)$$
$$= \delta(1-x) n_f C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \delta^{2\varepsilon} \cdot \left[\frac{16}{3}\frac{1}{\varepsilon^3} - \frac{40}{9}\frac{1}{\varepsilon^2} + (\tfrac{112}{27} - \tfrac{28}{3}\zeta(2))\frac{1}{\varepsilon} - \tfrac{328}{81} + \tfrac{70}{9}\zeta(2) + \tfrac{124}{9}\zeta(3)\right] \tag{A.5}$$

$$\hat{W}^{(2),H}(x,Q^2,\varepsilon)$$
$$= \theta(1-\delta-x) n_f C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \left[\frac{16}{3}\frac{1+x^2}{1-x}\frac{1}{\varepsilon^2} + \left\{\frac{1+x^2}{1-x}(\tfrac{32}{3}\ln(1-x) - 8\ln x - \tfrac{40}{9}) - \tfrac{16}{3}(1-x)\right\}\frac{1}{\varepsilon} + \frac{1+x^2}{1-x}(\tfrac{32}{3}\ln^2(1-x) - \tfrac{80}{9}\ln(1-x) - 16\ln x\ln(1-x) - \tfrac{4}{3}Li_2(1-x) + \tfrac{16}{3}\ln^2 x + \tfrac{20}{3}\ln x + \tfrac{112}{27} - \tfrac{28}{3}\zeta(2)) - \tfrac{32}{3}(1-x)\ln(1-x) + \tfrac{4}{3}(1+x)Li_2(1-x) + \tfrac{2}{3}(1+x)\ln^2 x + \tfrac{8}{3}(2-3x)\ln x + \tfrac{148}{9}\ln(1-x)\right]. \tag{A.6}$$

After coupling constant renormalization ($\overline{MS}$ scheme) we obtain the renormalized $n_f C_F$ part of the correction term $\Delta(x)$.

$$\Delta^{(2)}(x,Q^2) = \hat{W}^{(2)}(x,Q^2,\varepsilon) - 2\hat{\mathscr{F}}_2^{(2)}(x,Q^2,\varepsilon) - \tfrac{4}{3}n_f\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{\varepsilon}\Delta_0(x) \tag{A.7}$$

Where $\Delta_0(x)$, see (2.22a), can be derived from (C.6) and (C.7).

Finally we obtain

$$\Delta^{(2)}(x,Q^2)$$
$$= n_f C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\theta(1-\delta-x)\left\{\frac{1+x^2}{1-x}(4\ln^2(1-x) - \tfrac{22}{9}\ln(1-x) - \tfrac{16}{3}\ln x\ln(1-x) + \tfrac{4}{3}Li_2(1-x) + \tfrac{2}{3}\ln^2 x - 6\ln x - 5 - \tfrac{8}{3}\zeta(2)) + (1+x)(\tfrac{4}{3}Li_2(1-x) + \tfrac{2}{3}\ln^2 x) + (-\tfrac{34}{3} + 2x)\ln(1-x) + (10 + \tfrac{14}{3}x)\ln x + \tfrac{123}{9} + \tfrac{143}{9}x\right\} + \delta(1-x)\{\tfrac{8}{3}\ln^3\delta - \tfrac{22}{9}\ln^2\delta - (10 + \tfrac{16}{3}\zeta(2))\ln\delta - \tfrac{38}{9} - \tfrac{340}{9}\zeta(2) + \tfrac{16}{3}\zeta(3)\}\right] + \tfrac{2}{3}n_f\left(\frac{\alpha_s}{4\pi}\right)^2 \ln\left(\frac{Q^2}{\mu^2}\right)\Delta_0(x). \tag{A.8}$$

The last term in the above equation can be removed by using the definition of the running coupling constant, see (2.23).

Appendix B

The most general phase space integral for the DI process

$$\gamma^*(q) + q(\hat{p}) \to q(\hat{p}') + g(\hat{k}_1) + \cdots + g(\hat{k}_{i-1}) \tag{B.1}$$

will be denoted by

$$\int dPS_i^{\mathrm{DI}} = \int d^n\hat{p}' d^n\hat{k}_1 \cdots d^n\hat{k}_{i-1}\delta^+(\hat{p}'^2)\delta^+(\hat{k}_1^2)\cdots\delta^+(\hat{k}_{i-1}^2) \cdot \delta^{(n)}\left(q + \hat{p} - \hat{p}' - \sum_{j=1}^{i-1}\hat{k}_j\right). \tag{B.2}$$

For $i=2$ and $i=3$ we obtain respectively

$$\int dPS_2^{\mathrm{DI}} = 2^{2-n}\frac{\pi^{(n/2)-1}}{\Gamma\left(\frac{n}{2}-1\right)}\hat{s}^{(n/2)-1}\int_0^\pi d\theta(\sin\theta)^{n-3} \tag{B.3}$$

$$\int dPS_3^{\mathrm{DI}} = \frac{\pi^{n-3}}{8}\frac{\hat{s}^{1-(n/2)}}{\Gamma(n-3)}\int_0^\pi d\theta_2(\sin\theta_2)^{n-4}\int_0^\pi d\theta_1(\sin\theta_1)^{n-3} \cdot \int_0^{\hat{s}} ds_1 \int_0^{\hat{s}-s_1} ds_2 (s_1 s_2(\hat{s} - s_1 - s_2))^{(n/2)-2}. \tag{B.4}$$

Where we have integrated over all angles which do not appear in the matrix element. Further, we have defined $\hat{s} \equiv (\hat{p}+q)^2$, $s_1 \equiv (\hat{p}' + \hat{k}_1)^2$ and $s_2 \equiv (\hat{p}' + \hat{k}_2)^2$.

If we now define y and z by

$$s_1 = \hat{s}(1-y) \tag{B.5a}$$

$$s_2 = \hat{s}\frac{y(1-z)}{1-zy} \tag{B.5b}$$

(B.4) can be rewritten into [24]

$$\int dPS_3^{\mathrm{DI}} = \frac{\pi^{n-3}}{8}\frac{1}{\Gamma(n-3)}\hat{s}^{n-3}\int_0^\pi d\theta_2(\sin\theta_2)^{n-4} \cdot\int_0^\pi d\theta_1(\sin\theta_1)^{n-3}\int_0^1 dz\int_0^1 dy(z(1-z))^{(n/2)-2} \cdot(y(1-y))^{n-3}(1-zy)^{2-n}. \tag{B.6}$$

It is convenient to introduce the variable x defined by: $s \equiv -q^2(1-x)/x$.

The general phase space integral for the DY process

$$q(\hat{p}_1) + \bar{q}(\hat{p}_2) \to \gamma^*(q) + g(\hat{k}_1) + \cdots + g(\hat{k}_{i-1}) \tag{B.7}$$

is given by

$$\int dPS_i^{\mathrm{DY}} = \int d^n q d^n\hat{k}_1 \cdots d^n\hat{k}_{i-1}\delta^+(q^2-Q^2)\delta^+(\hat{k}_1^2)\cdots\delta^+(\hat{k}_{i-1}^2) \cdot\delta^{(n)}\left(\hat{p}_1+\hat{p}_2-q-\sum_{j=1}^{i-1}\hat{k}_j\right). \tag{B.8}$$

For $i=2$ and $i=3$ we have respectively

$$\int dPS_2^{\mathrm{DY}} = 2^{2-n}\frac{\pi^{(n/2)-1}}{\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{\hat{s}-Q^2}{\hat{s}}\right)^{n-3}\hat{s}^{(n/2)-2} \cdot\int_0^\pi d\theta(\sin\theta)^{n-3} \tag{B.9}$$

$$\int dPS_3^{\mathrm{DY}} = \frac{\pi^{n-3}}{8}\frac{\hat{s}^{1-(n/2)}}{\Gamma(n-3)}\int_0^\pi d\theta_2(\sin\theta_2)^{n-4} \cdot\int_0^\pi d\theta_1(\sin\theta_1)^{n-3}\int_{Q^2}^{\hat{s}} ds_1\int_{\hat{s}Q^2/s_1}^{\hat{s}-s_1+Q^2} ds_2 \cdot(s_1s_2-\hat{s}Q^2)^{(n/2)-2}(\hat{s}+Q^2-s_1-s_2)^{(n/2)-2} \tag{B.10}$$

with $\hat{s} \equiv (\hat{p}_1+\hat{p}_2)^2$, $s_1 \equiv (q+\hat{k}_1)^2$ and $s_2 \equiv (q+\hat{k}_2)^2$.

Making the following substitutions

$$x = \frac{Q^2}{\hat{s}} \tag{B.11a}$$

$$s_1 = (1-y(1-x))\hat{s} \tag{B.11b}$$

$$s_2 = \left\{x + y(1-x) - \frac{(1-x)^2y(1-y)z}{1-y(1-x)}\right\}\hat{s} \tag{B.11c}$$

(B.10) becomes

$$\int dPS_3^{\mathrm{DY}} = \frac{\pi^{n-3}}{8}\frac{1}{\Gamma(n-3)}\left(\frac{\hat{s}-Q^2}{\hat{s}}\right)^{2n-5}\hat{s}^{n-3} \cdot\int_0^\pi d\theta_2(\sin\theta_2)^{n-4}\int_0^\pi d\theta_1(\sin\theta_1)^{n-3}\int_0^1 dz \cdot\int_0^1 dy(y(1-y))^{n-3}(z(1-z))^{(n/2)-2} \cdot(1-y(1-x))^{1-(n/2)}. \tag{B.12}$$

Appendix C

The non-singlet splitting functions can be obtained from [19–21]. In the limit $x \to 1$ they become

$$P_0(x) = C_F\left\{8\frac{\theta(1-\delta-x)}{1-x} + \delta(1-x)(6+8\ln\delta)\right\} \tag{C.1}$$

$$P_1(x) = \frac{\theta(1-\delta-x)}{1-x}\{C_AC_F(\tfrac{536}{9}-16\zeta(2)) + n_fC_F(-\tfrac{80}{9})\} + \delta(1-x)\{C_F^2(3-24\zeta(2)+48\zeta(3)) + C_AC_F(\tfrac{17}{3}+\tfrac{88}{3}\zeta(2)-24\zeta(3) + (\tfrac{536}{9}-16\zeta(2))\ln\delta) + n_fC_F(-\tfrac{2}{3}-\tfrac{16}{3}\zeta(2)-\tfrac{80}{9}\ln\delta)\}. \tag{C.2}$$

In the limit $n \to \infty$ the anomalous dimension $\gamma^{(n)}$ (2.14) becomes equal to (2.59). The coefficients in (2.60a), (2.60b) are then given by

$$\gamma_0^{(K)} = 8C_F$$
$$\gamma_1^{(K)} = C_AC_F(\tfrac{536}{9}-16\zeta(2)) - n_fC_F\tfrac{80}{9} \tag{C.3}$$

$$\tilde{\gamma}_0 = -6C_F$$
$$\tilde{\gamma}_1 = C_F^2(-3+24\zeta(2)-48\zeta(3)) + C_AC_F(-\tfrac{17}{3}-\tfrac{88}{3}\zeta(2)+24\zeta(3)) + n_fC_F(\tfrac{2}{3}+\tfrac{16}{3}\zeta(2)). \tag{C.4}$$

The lowest order coefficient of the β-function is equal to

$$\beta_0 = \tfrac{11}{3}C_A - \tfrac{2}{3}n_f. \tag{C.5}$$

The non-pole contributions f_0 and ω_0 can be found in [7, 8]. Extending them to include terms of order ε we find

$$f_0(x) = C_F\Bigg[\theta(1-\delta-x)\Bigg\{2\frac{1+x^2}{1-x}\ln\frac{1-x}{x} + 4x + 6 - \frac{3}{1-x} + \varepsilon\Bigg(\frac{1}{2}\frac{1+x^2}{1-x}\ln^2\frac{1-x}{x} + \left(2x+3-\frac{3}{2}\frac{1}{1-x}\right)\ln\frac{1-x}{x} - 6 - 4x + \frac{7}{2}\frac{1}{1-x} - \frac{3}{2}\frac{1+x^2}{1-x}\zeta(2)\Bigg)\Bigg\} + \delta(1-x)$$

$$\cdot\{-9-4\zeta(2)+2\ln^2\delta-3\ln\delta+\varepsilon(9+\tfrac{3}{4}\zeta(2)$$
$$+\tfrac{1}{3}\ln^3\delta-\tfrac{3}{4}\ln^2\delta+(\tfrac{7}{2}-3\zeta(2))\ln\delta)\}\Big] \quad (C.6)$$

$$\omega_0(x)=C_F\Big[\theta(1-\delta-x)\Big\{4\frac{1+x^2}{1-x}\ln\frac{(1-x)^2}{x}$$
$$+\varepsilon\Big(\frac{1+x^2}{1-x}\ln^2\frac{(1-x)^2}{x}+4(1-x)$$
$$-3\frac{1+x^2}{1-x}\zeta(2)\Big)\Big\}$$
$$+\delta(1-x)\{-16+8\zeta(2)+8\ln^2\delta$$
$$+\varepsilon(16-\tfrac{21}{2}\zeta(2)+\tfrac{8}{3}\ln^3\delta-6\zeta(2)\ln\delta)\}\Big]. \quad (C.7)$$

Appendix D

Here we will present the calculation of the $C_A C_F$ and $n_f C_F$ parts of $\Delta(x_1,x_2)$ in (3.8).

Starting with the unrenormalized DY structure function $\hat{W}(x_1,x_2)$ in (3.3) we can make the following ansatz for the sum of its soft and hard parts in the limit $x_1,x_2\to 1$.

$$\hat{W}^{(2),S+H}(x_1,x_2,Q^2)$$
$$=XC_F\Big(\frac{\alpha_s}{4\pi}\Big)^2\Big(\frac{Q^2}{\mu^2}\Big)^\varepsilon\Big[\delta(1-x_1)\delta(1-x_2)\delta^{2\varepsilon}$$
$$\cdot\Big\{\frac{a_{23}}{\varepsilon^3}+\frac{a_{22}}{\varepsilon^2}+\frac{a_{21}}{\varepsilon}+a_{20}\Big\}+\delta(1-x_1)\theta(1-\delta-x_2)$$
$$\cdot\delta^\varepsilon\frac{(1-x_2)^\varepsilon}{(1-x_2)}\Big\{\frac{b_{22}}{\varepsilon^2}+\frac{b_{21}}{\varepsilon}+b_{20}\Big\}+[x_1\leftrightarrow x_2]$$
$$+\theta(1-\delta-x_1)\theta(1-\delta-x_2)\frac{(1-x_1)^\varepsilon}{(1-x_1)}\frac{(1-x_2)^\varepsilon}{(1-x_2)}$$
$$\cdot\Big\{\frac{c_{21}}{\varepsilon}+c_{20}\Big\}\Big]. \quad (D.1)$$

Here X stands for the colourfactor C_A or n_f.

This ansatz follows from mass-factorization (3.3) and renormalization. We have verified that the above equation is correct for $X=n_f$. In Sect. 2 we made the observation that the expressions for the $n_f C_F$ and $C_A C_F$ parts of $\hat{W}(x)$ have the same form. Therefore we assume that the same holds for $\hat{W}(x_1,x_2)$.

The coefficients of the pole terms in the above equation can be determined in the same way as in Sect. 2. First we add to (D.1) the virtual part $\hat{W}^{(2),V}$, which is given by the unrenormalized time-like formfactor.

$$\hat{W}^{(2),V}(x_1,x_2,Q^2)$$
$$=\delta(1-x_1)\delta(1-x_2)\{2\,\mathrm{Re}\,F_X^{(2)}(Q^2)\}. \quad (D.2)$$

For $X=n_f$ we have

$$\mathrm{Re}\,F_X^{(2)}(Q^2)$$
$$=n_f C_f\Big(\frac{\alpha_s}{4\pi}\Big)^2\Big(\frac{Q^2}{\mu^2}\Big)^\varepsilon\Big[-\frac{8}{3}\frac{1}{\varepsilon^3}+\frac{59}{9}\frac{1}{\varepsilon^2}$$
$$+(-\tfrac{353}{27}+\tfrac{22}{3}\zeta(2))\frac{1}{\varepsilon}+\tfrac{7541}{324}-\tfrac{154}{9}\zeta(2)-\tfrac{26}{9}\zeta(3)\Big] \quad (D.3)$$

and for $X=C_A$

$$\mathrm{Re}\,F_X^{(2)}(Q^2)$$
$$=C_A C_F\Big(\frac{\alpha_s}{4\pi}\Big)^2\Big(\frac{Q^2}{\mu^2}\Big)^\varepsilon\Big[\frac{44}{3}\frac{1}{\varepsilon^3}+(-\tfrac{332}{9}+4\zeta(2))\frac{1}{\varepsilon^2}$$
$$+(\tfrac{4129}{54}-\tfrac{121}{3}\zeta(2)-26\zeta(3))\frac{1}{\varepsilon}-\tfrac{89173}{648}+\tfrac{1754}{18}\zeta(2)$$
$$+\tfrac{467}{9}\zeta(3)-\tfrac{16}{5}\zeta(2)^2\Big]. \quad (D.4)$$

The renormalized DY structure function is equal to

$$\hat{W}^{(2),R}=\hat{W}^{(2),S+H}+\hat{W}^{(2),V}+X\frac{2}{\varepsilon}\beta_0^{(X)}\Big(\frac{\alpha_s}{4\pi}\Big)\hat{W}^{(1)}. \quad (D.5)$$

with

$$\beta_0^{(n_f)}=-\tfrac{2}{3}\quad \beta_0^{(C_A)}=\tfrac{11}{3} \quad (D.6)$$

and $\hat{W}^{(1)}$ is the lowest order contribution which has to be expanded up to order ε.

$$\hat{W}^{(1)}(x_1,x_2,Q^2)$$
$$=C_F\Big(\frac{\alpha_s}{4\pi}\Big)\Big[\delta(1-x_1)\delta(1-x_2)\Big\{(12+16\ln\delta)\frac{1}{\varepsilon}$$
$$+8\ln^2\delta-16+12\zeta(2)+(6+8\ln\delta)\ln\Big(\frac{Q^2}{\mu^2}\Big)$$
$$+\varepsilon\Big((\tfrac{3}{2}+2\ln\delta)\ln^2\Big(\frac{Q^2}{\mu^2}\Big)+(4\ln^2\delta-8+6\zeta(2))$$
$$\cdot\ln\Big(\frac{Q^2}{\mu^2}\Big)+\tfrac{8}{3}\ln^3\delta-2\zeta(2)\ln\delta+16$$
$$-\tfrac{21}{2}\zeta(2)-4\zeta(3)\Big)\Big\}+\delta(1-x_1)\frac{\theta(1-\delta-x_2)}{1-x_2}$$
$$\cdot\Big\{\frac{8}{\varepsilon}+4\ln\delta+4\ln(1-x_2)+4\ln\Big(\frac{Q^2}{\mu^2}\Big)$$
$$+\varepsilon\Big(\ln^2\Big(\frac{Q^2}{\mu^2}\Big)+2\ln\delta\ln\Big(\frac{Q^2}{\mu^2}\Big)+2\ln(1-x_2)$$
$$\cdot\ln\Big(\frac{Q^2}{\mu^2}\Big)+\ln^2\delta+2\ln\delta\ln(1-x_2)$$
$$+\ln^2(1-x_2)-\zeta(2)\Big)\Big\}+[x_1\leftrightarrow x_2]$$
$$+\frac{\theta(1-\delta-x_1)\theta(1-\delta-x_2)}{(1-x_1)(1-x_2)}$$

$$\cdot\left\{4+\varepsilon\left(2\ln(1-x_1)+2\ln(1-x_2)+2\ln\left(\frac{Q^2}{\mu^2}\right)\right)\right\}\Bigg]. \qquad \text{(D.7)}$$

The triple poles, which are of infra-red origin, are cancelled in the combination $\hat{W}^{(2),S+H}+\hat{W}^{(2),V}$ and the single (UV) pole in the last part of (D.1) is removed by $X(2/\varepsilon)\beta_0^{(X)}\hat{W}^{(1)}$ of (D.5). This determines the coefficients a_{23} and c_{21} respectively. The remaining pole terms are cancelled by mass-factorization, so that the quantity

$$\begin{aligned}&\Delta^{(2)}(x_1,x_2,Q^2)\\&=\hat{W}^{(2),R}(x_1,x_2,Q^2)-\delta(1-x_1)\delta(1-x_2)\\&\quad\cdot\{2\hat{\mathscr{F}}^{S+V}_{2,X}(Q^2,\delta)\}-\delta(1-x_1)\hat{\mathscr{F}}^{H}_{2,X}(x_2,Q^2)\\&\quad-\delta(1-x_2)\hat{\mathscr{F}}^{H}_{2,X}(x_1,Q^2)\end{aligned} \qquad \text{(D.8)}$$

is finite. Where $\hat{\mathscr{F}}^{S+V}_{2,X}$ (2.47) and $\hat{\mathscr{F}}^{H}_{2,X}$ ((2.11) and (2.53)) are the X parts of the renormalized DI parton structure function. This provides us with the remaining coefficients, except for b_{20} and c_{20}. These are obtained from the requirement

$$\int_0^1 dx_1 \int_0^1 dx_2 \Delta^{(2)}(x_1,x_2,Q^2)<\infty. \qquad \text{(D.9)}$$

The coefficients a_{ij}, b_{ij} and c_{ij} are given in Table 2.

We can now determine all $\ln^i(1-x_1)/(1-x_1)\times\ln^j(1-x_2)/(1-x_2)$ terms of $\Delta^{(2)}(x_1,x_2)$. The result agrees with (3.8).

Remark. Notice that (D.1) can be written as

$$\begin{aligned}&\hat{W}^{(2),S+H}(x_1,x_2,Q^2)\\&=XC_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}(1-x_1)^{-1+\varepsilon}(1-x_2)^{-1+\varepsilon}\\&\quad\cdot\left\{\frac{a_{23}}{\varepsilon}+a_{22}+a_{21}\varepsilon+a_{20}\varepsilon^2\right\}.\end{aligned} \qquad \text{(D.10)}$$

Table 2. The residues of the pole-terms in the C_A and n_f parts of the unrenormalized DY structure function $\hat{W}(x_1,x_2)$. (See (D.1).)

	C_A	n_f
a_{23}	$-\frac{88}{3}$	$\frac{16}{3}$
a_{22}	$\frac{268}{9}-8\zeta(2)$	$-\frac{40}{9}$
a_{21}	$-\frac{808}{27}+22\zeta(2)+28\zeta(3)$	$\frac{112}{27}-4\zeta(2)$
a_{20}	?	?
b_{22}	$-\frac{88}{3}$	$\frac{16}{3}$
b_{21}	$\frac{268}{9}-8\zeta(2)$	$-\frac{40}{9}$
b_{20}	$-\frac{808}{27}+22\zeta(2)+28\zeta(3)$	$\frac{112}{27}-4\zeta(2)$
c_{21}	$-\frac{88}{3}$	$\frac{16}{3}$
c_{20}	$\frac{268}{9}-8\zeta(2)$	$-\frac{40}{9}$

Where $(1-x_1)^{-1+\varepsilon}(1-x_2)^{-1+\varepsilon}$ has to be interpreted as

$$\begin{aligned}&(1-x_1)^{-1+\varepsilon}(1-x_2)^{-1+\varepsilon}\\&=\delta(1-x_1)\delta(1-x_2)\frac{1}{\varepsilon^2}\delta^{2\varepsilon}\\&\quad+\theta(1-\delta-x_1)\delta(1-x_2)\frac{1}{\varepsilon}\delta^{\varepsilon}\frac{(1-x_1)^{\varepsilon}}{(1-x_1)}\\&\quad+\delta(1-x_1)\theta(1-\delta-x_2)\frac{1}{\varepsilon}\delta^{\varepsilon}\frac{(1-x_2)^{\varepsilon}}{(1-x_2)}\\&\quad+\theta(1-\delta-x_1)\theta(1-\delta-x_2)\frac{(1-x_1)^{\varepsilon}}{(1-x_1)}\frac{(1-x_2)^{\varepsilon}}{(1-x_2)}.\end{aligned} \qquad \text{(D.11)}$$

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