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On the Lattice of Extensions of the Modal Logics $\mathbf{KAlt_n}$

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Introduction

Some results concerning the lattice of normal modal logics $\Lambda(\mathbf{K})$ (see for instance [B1, B2]) show the extreme complexity of this structure and the consequent impossibility of a complete description of it. On the other hand, many other results which appear in the literature give descriptions of significative parts of $\Lambda(\mathbf{K})$ (see [B3, Ma, Ra]). In our paper we concentrate the attention on the lattice of the extensions of the logics $\mathbf{KAlt_n}$, originally introduced in [Se]. We recall that $\mathbf{KAlt_n}$ is the normal modal propositional logic characterized by the axiom

$$\mathrm{Alt_n} = Lp_0 \vee L(p_0 \to p_1) \vee \ldots \vee L(p_0 \wedge \ldots \wedge p_{n-1} \to p_n),$$

and a Kripke frame $\mathbf{F} = \langle W, R \rangle$ is a frame of $\mathbf{KAlt_n}$ iff for each $w \in W\, |\{v \in W : wRv\}| \leqq n$ (the formula $\mathrm{Alt_n}$ has not to be confused with the formula $\mathrm{I_n}$, see [Fi], which holds at those points which have at most n mutually incomparable successors). Because of this semantical content, which represents a sort of local finiteness, the set of extensions of $\mathbf{KAlt_n}$ can be intuitively regarded as the set which follows, in order of complexity, that of finite logics. This intuitive idea is confirmed at the end of Sect. 1, where we prove that, for each $n \in \omega$, each extension of $\mathbf{KAlt_n}$ is Kripke-complete and canonical; this result shows that, even if the incompleteness phenomena make the Kripke semantic unadequate for the study of the whole lattice $\Lambda(\mathbf{K})$, this semantic is adequate to the study of a class of logics bigger than that of finite logics (relatively to which the Kripke and the algebraic semantics are equivalent, thanks to the perfect correspondence between finite Kripke frames and dual spaces of finite modal algebras). In Sect. 2 we give a complete description of $\Lambda(\mathbf{KAlt_1})$, from which it results that this structure has the following properties: it is denumerable; contains only one pretabular logic; its non-finite logics, all having f.m.p., form a linear order of type $\langle \omega+1, \geqq \rangle$. From this last property it follows that, for each n, there exists a logic having exactly n non-finite extensions. All these properties disappear passing from $\mathbf{KAlt_1}$ to $\mathbf{KAlt_n}$ when $n \geqq 2$ (Sect. 3); in fact in $\Lambda(\mathbf{KAlt_2})$ we find a continuum of logics without f.m.p. and infinitely many pretabular logics.

1. Preliminaries and Basic Results

For the basic logical and algebraic notions we refer to [vB] (Chaps. 1–6) and to [BS]. We recall only those definitions that may be not completely standard. Our modal language contains a denumerable set of propositional variables $\{p_0, p_1, \ldots\}$ and the symbols $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\perp$, L, M (negation, conditional, conjunction, disjunction, biconditional, falsity, necessity, and possibility). Well formed formulas are denoted by α, β, etc. Given a normal modal logic $\mathbf{L}$ we denote by $\Lambda(\mathbf{L})$ the lattice of the extensions of $\mathbf{L}$, i.e., $\Lambda(\mathbf{L}) = \langle \{\mathbf{L}' : \mathbf{L} \subseteq \mathbf{L}'\}, \subseteq \rangle$. We recall that, if we denote by * the well-known correspondence between logics and varieties of modal algebras, then $\Lambda(\mathbf{L})$ and the lattice $\Lambda(\mathbf{L}^*)$ of the subvarieties of $\mathbf{L}^*$ are anti-isomorphic.

A *descriptive frame* is a couple $\langle \mathbf{F}, \mathbf{W} \rangle$ where $\mathbf{F} = \langle W, R \rangle$ is a Kripke frame (i.e., $W \neq \emptyset$ and $R \subseteq W^2$) and $\mathbf{W} \subseteq \mathscr{P}(W)$ satisfies the following conditions: i) it contains the empty set and is closed under boolean operators and L-operator ($L(X) = \{w \in W$: for each $v \in W$, wRv implies $v \in X\}$); (ii) if $w \neq v$ then $w \in X$ and $v \notin X$ for some $X \in \mathbf{W}$; (iii) for each $w, v \in W$, if not wRv then, for some $X \in \mathbf{W}$, $w \in L(X)$ and $v \notin X$; (iv) if $\mathbf{X} \subseteq \mathbf{W}$ has the finite intersection property then $\cap \mathbf{X} \neq \emptyset$. A valuation V on $\mathbf{F}$ is a *valuation on* $\langle \mathbf{F}, \mathbf{W} \rangle$ if, for each propositional variable p, $V(p) \in \mathbf{W}$. Hence $\langle \mathbf{F}, \mathbf{W} \rangle \models \beta[w]$ if $\langle \mathbf{F}, V \rangle \models \beta[w]$ for each valuation V on $\langle \mathbf{F}, \mathbf{W} \rangle$. We denote $\{\alpha : \langle \mathbf{F}, \mathbf{W} \rangle \models \alpha\}$ and $\{\alpha : \mathbf{F} \models \alpha\}$ respectively by $\mathrm{Th}(\langle \mathbf{F}, \mathbf{W} \rangle)$ and $\mathrm{Th}(\mathbf{F})$.

Given a logic $\mathbf{L}$ we set:

$$\mathrm{TF}(\mathbf{L}) = \{\langle \mathbf{F}, \mathbf{W} \rangle : \langle \mathbf{F}, \mathbf{W} \rangle \models \mathbf{L}\},$$

$$\mathrm{KF}(\mathbf{L}) = \{\mathbf{F} : \mathbf{F} \models \mathbf{L}\},$$

$$\mathrm{fTF}(\mathbf{L})\,[\mathrm{fKF}(\mathbf{L})] = \{\mathbf{F} \in \mathrm{TF}(\mathbf{L})\,[\in \mathrm{KF}(\mathbf{L})] : \mathbf{F} \text{ is finite}\},$$

[if $\mathbf{F}$ is finite, then $\langle \mathbf{F}, \mathbf{W} \rangle$ is descriptive iff $\mathbf{W} = \mathscr{P}(W)$, and $\mathrm{Th}(\mathbf{F}) = \mathrm{Th}(\langle \mathbf{F}, \mathbf{W} \rangle)$; thus we can indicate both $\mathrm{fTF}(\mathbf{L})$ and $\mathrm{fKF}(\mathbf{L})$ by $\mathrm{fF}(\mathbf{L})$],

$$\mathrm{GKF}(\mathbf{L})\,[\mathrm{fGF}(\mathbf{L})] = \{\mathbf{F} \in \mathrm{KF}(\mathbf{L})\,[\in \mathrm{fF}(\mathbf{L})] : \mathbf{F} \text{ is generated}\}$$

($\mathbf{F}$ is said to be *generated* if there exists a $w \in W$ such that for each $v \in W$ either $v = w$ or wR^+v, where R^+ is the transitive closure of R.)

Proposition 1.0. *For each logic* $\mathbf{L}$,

$$\mathbf{L} = \cap \{\mathrm{Th}(\langle \mathbf{F}, \mathbf{W} \rangle) : \langle \mathbf{F}, \mathbf{W} \rangle \in \mathrm{TF}(\mathbf{L})\}.$$

A logic $\mathbf{L}$ is said to be: *Kripke complete (K-complete)* if $\mathbf{L} = \cap \{\mathrm{Th}(\mathbf{F}) : \mathbf{F} \in \mathrm{KF}(\mathbf{L})\}$; *with the Finite Model Property* (f.m.p.) if $\mathbf{L} = \cap \{\mathrm{Th}(\mathbf{F}) : \mathbf{F} \in \mathrm{fF}(\mathbf{L})\}$ (in such a case the variety $\mathbf{L}^*$ corresponding to $\mathbf{L}$ is said to be *generated by its finite members*); *tabular* (or *finite*) if $\mathbf{L} = \mathrm{Th}(\mathbf{F})$ for a finite $\mathbf{F}$; *pretabular* if it is not tabular and each of its proper extension is tabular; *canonical* if $\mathbf{L} = \mathrm{Th}(\mathbf{F}_\mathbf{L})$, where $\mathbf{F}_\mathbf{L}$ is the frame of the canonical model for $\mathbf{L}$.

From the Generated Subframe Theorem it follows that

Proposition 1.1. i) *If* $\mathbf{L}$ *is K-complete then* $\mathbf{L} = \{\mathrm{Th}(\mathbf{F}) : \mathbf{F} \in \mathrm{GKF}(\mathbf{L})\}$ *and* ii) *if* $\mathbf{L}$ *has f.m.p. then* $\mathbf{L} = \{\mathrm{Th}(\mathbf{F}) : \mathbf{F} \in \mathrm{fGF}(\mathbf{L})\}$.

Let $\mathbf{F}=\langle W, R\rangle$ (or $\langle \mathbf{F}, \mathbf{W}\rangle=\langle\langle W, R\rangle, \mathbf{W}\rangle$) and let $w \in W$. $S_n^*(\mathbf{F}, w)$ and $S_n(\mathbf{F}, w)$ are defined as follows: $S_0^*(\mathbf{F}, w)=\{w\}$, $S_{m+1}^*(\mathbf{F}, w)=\{v \in W$: for some $u \in S_m^*(\mathbf{F}, w), uRv\}$, $S_n(\mathbf{F}, w)=\bigcup S_m^*(\mathbf{F}, w)$ for $m \leqq n$. A point w is said to be *terminal* if $S_1^*=\emptyset$. Given a formula β, we denote by $d(\beta)$ its *modal degree*, i.e., the maximum number of nested modal operators in β.

Proposition 1.2. *Let β be any formula such that* $d(\beta)=n$, *and let* w, w' *be points respectively of* $\mathbf{F}$ *and* $\mathbf{F}'$ *(or of* $\langle\mathbf{F}, \mathbf{W}\rangle$ *and* $\langle\mathbf{F}', \mathbf{W}'\rangle$*). If* f *is an order-isomorphism from* $\mathbf{F} \upharpoonright S_n(w, \mathbf{F})$ *onto* $\mathbf{F}' \upharpoonright S_n(w', \mathbf{F}')$ *such that* (i) $f(w)=w'$ *and* (ii) $v \in V(p_i)$ *iff* $f(v) \in V'(p_i)$ *for each* p_i *which occours in β and each* $v \in S_n(w, \mathbf{F})$, *then* $\langle\mathbf{F}, V\rangle \models \beta[w]$ *iff* $\langle\mathbf{F}', V'\rangle \models \beta[w']$.

Lemma 1.3. *If* $\langle\mathbf{F}, \mathbf{W}\rangle \in \mathrm{TF}(\mathbf{KAlt}_n)$ *then* $\mathrm{Th}(\langle\mathbf{F}, \mathbf{W}\rangle)=\mathrm{Th}(\mathbf{F})$.

Proof. First we show that

$$\text{if } \langle\mathbf{F}, \mathbf{W}\rangle \in \mathrm{TF}(\mathbf{KAlt}_n) \quad \text{then for each} \quad w \in W\ |S_1^*(\mathbf{F}, w)| \leqq n. \tag{1}$$

Suppose $|S_1^*(\mathbf{F}, w)| \geqq n+1$ and let $X=\{v_0, \ldots, v_n\} \subseteq S_1^*(\mathbf{F}, w)$. Since X is finite, $\mathbf{W} \upharpoonright X =\mathscr{P}(X)$ and hence we can find a valuation V such that, for each $i \in \omega$, $V(p_i) \in \mathbf{W}$ and for each $i \leqq n$ $V(p_i) \cap X=\{v_i\}$. In such a case $\langle\mathbf{F}, V\rangle \not\models \mathrm{Alt}_n[w]$ and hence $\langle\mathbf{F}, \mathbf{W}\rangle \notin \mathrm{TF}(\mathbf{KAlt}_n)$, thus obtaining (1). Now suppose $\mathbf{F} \not\models \beta$, i.e., $\langle\mathbf{F}, V\rangle \not\models \beta[w]$ for a V on $\mathbf{F}$ and $w \in W$. Let $d(\beta)=m$; by (1) we obtain that $|S_m(\mathbf{F}, w)| \leqq \sum n^r$ for $r \leqq m$, i.e., it is finite. So there exists a V' such that, for each $i \in \omega$, $V'(p_i) \in \mathbf{W}$ and $V'(p_i) \cap S_m(\mathbf{F}, w)=V(p_i) \cap S_m(\mathbf{F}, w)$. Therefore, by Proposition 1.2, $\langle\mathbf{F}, \mathbf{W}\rangle \not\models \beta$. Since the converse holds in general, the proof is concluded.

Theorem 1.4. *For each* $n \in \omega$, *if* $\mathbf{KAlt}_n \subseteq \mathbf{L}$ *then* $\mathbf{L}$ *is* (i) *Kripke-complete and* (ii) *canonical.*

Proof. (i) follows from Proposition 1.0 and Lemma 1.3. As regard (ii), let $\mathbf{M}_\mathbf{L}$ be the canonical model for $\mathbf{L}$. It is straightforward to show that $\mathrm{Th}(\mathbf{M}_\mathbf{L})=\mathrm{Th}(\langle\mathbf{F}_\mathbf{L}, \mathbf{W}_\mathbf{L}\rangle)$, where $\mathbf{W}_\mathbf{L}=\{X_\alpha=\{w \in W_\mathbf{L}: \alpha \in w\}$, for $\alpha \in \mathrm{Wff}\}$. Therefore (ii) follows from Lemma 1.3 and the Foundamental Theorem for Modal Logic.

2. The Lattice of Extensions of KAlt$_1$

For each $t \in \omega$ and $\langle n, s\rangle \in \omega^2$ we define $\mathbf{F}_t=\langle W_t, R_t\rangle$ and $\mathbf{F}_{m,r}=\langle W_{m,r}, R_{m,r}\rangle$ as follows (see Fig. 1):

$$W_t=\{w_0, \ldots, w_t\}, R_t=\{\langle w_{n+1}, w_n\rangle: 0 \leqq n \leqq t-1\},$$

$$W_{m,r}=\{w_0, \ldots, w_{m+r}\}, R_{m+r}=\{\langle w_n, w_{n+1}\rangle: 0 \leqq n \leqq m+r-1\} \cup\{\langle w_{m+r}, w_m\rangle\}.$$

We observe that for each $t \in \omega$ and $i \leqq t$ $\mathbf{F}_t \models L^{i+1}\bot \wedge \neg L^i\bot[v]$ iff $v=w_i$.

Lemma 2.0. i) $\mathrm{GKF}(\mathbf{KAlt}_1)=\mathrm{I}(\{\mathbf{F}_t: t \in \omega\} \cup\{\mathbf{F}_{m,r}: \langle m, r\rangle \in \omega^2\} \cup\{\langle\omega, \mathrm{Suc}\rangle\})$, *where* $\mathrm{Suc}=\{\langle n, n+1\rangle: n \in \omega\}$ *and* I(X) *denotes the class of frames isomorphic to a frame of* X.

ii) *For each* $\langle m, r\rangle \in \omega^2, \mathbf{F}_{m,r}$ *is p-morphic image of* $\langle\omega, \mathrm{Suc}\rangle$.

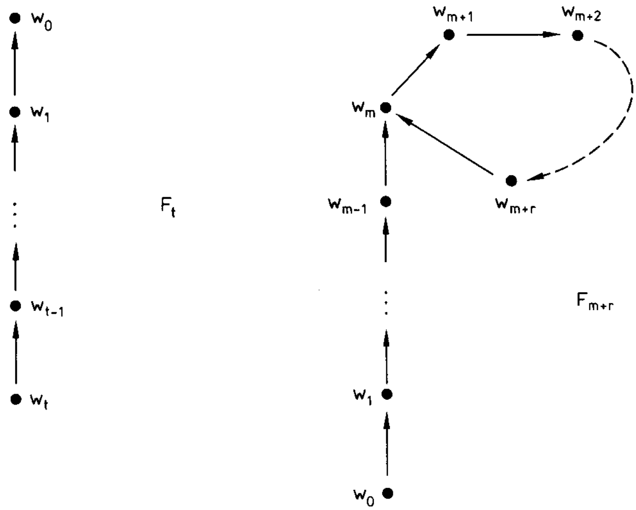

Fig. 1

Proof. i) Straightforward.
ii) The function

$$g(n)=\begin{cases} w_n & \text{if } n\leqq m \\ w_v & \text{otherwise}, \end{cases}$$

where $v\in\{0,\ldots,m+r\}$ and $v\equiv n$ modulo $r+1$, is a p-morphism from $\langle\omega,\mathrm{Suc}\rangle$ onto $\mathbf{F}_{m,r}$.

Theorem 2.1. *If* $\mathbf{L}\in\Lambda(\mathbf{KAlt}_1)$ *then* $\mathbf{L}$ *has the f.m.p.*

Proof. Let $\mathbf{KAlt}_1\subseteq\mathbf{L}$ and $\mathbf{L}\nvdash\beta$. By Theorem 1.4 $\mathbf{L}$ is K-complete and hence, by Proposition 1.1 (i), there exists an $\mathbf{F}\in\mathrm{GKF}(\mathbf{L})$ such that $\mathbf{F}\not\models\beta$. Suppose $\mathbf{F}$ be infinite: by Lemma 2.0 (i) we obtain that $\mathbf{F}\cong\langle\omega,\mathrm{Suc}\rangle$ and then $\langle\omega,\mathrm{Suc}\rangle\in\mathrm{GKF}(\mathbf{L})$ and $\langle\omega,\mathrm{Suc}\rangle\not\models\beta$, i.e., there exists V on ω and $n\in\omega$ such that $\langle\omega,\mathrm{Suc},V\rangle\not\models\beta[n]$. We may suppose, without loss of generality, that $n=0$. Let m,r be such that $m+r\geqq d(\beta)$; by Lemma 2.0 (ii) there is a p-morphism g from $\langle\omega,\mathrm{Suc}\rangle$ onto $\mathbf{F}_{m,r}$, and hence, by p-Morphism Theorem, $\mathbf{F}_{m,r}\in\mathrm{GKF}(\mathbf{L})$. Moreover, $g\restriction\{0,\ldots,m+r\}$ is injective, and so, if we take a V' on $\mathbf{F}$ such that, for each p_i which occours in β and each $q\leqq m+r$, $w_q\in V'(p_i)$ iff $q\in V(p_i)$, we have, by Proposition 1.2, that $\langle\mathbf{F}_{m,r},V'\rangle\not\models\beta[w_0]$

Theorem 2.2. $\mathbf{L}_0=\mathbf{KAlt}_1+(\mathrm{Lp}\to\mathrm{Mp})$ *(or, equivalently,* $\mathbf{KAlt}_1+(\neg\mathrm{L}\bot)$*) is a pretabular logic.*

Proof. It is straightforward to show that

$$\mathbf{L}_0=\mathrm{Th}(\langle\omega,\mathrm{Suc}\rangle). \tag{1}$$

Now, let $\mathbf{L}_0\subseteq\mathbf{L}$. We distinguish into two cases: either there exists an n such that for each $\mathbf{F}_{m,r}\in\mathrm{fGF}(\mathbf{L})$ $|W_{m,r}|\leqq n$ (Case A), or not (Case B). In Case A we have that fGF(L) is, up to isomorphism, finite and so $\mathbf{F}=\sum\{\mathbf{F}_{m,r}:\mathbf{F}_{m,r}\in \mathrm{fGF}(\mathbf{L})\}$ is finite and, by Theorem 2.1 and Disjoint Union Theorem, $\mathbf{L}=\mathrm{Th}(\mathbf{F})$. In Case B, since $\sup\{m+r:\mathbf{F}_{m,r}\in \mathrm{fGF}(\mathbf{L})\}=\omega$, we have that $\langle\omega,\mathrm{Suc}\rangle\in \mathrm{GKF}(\mathbf{L})$; in fact, if I is a set of indices for the frames of fGF(L), then the function g from $\mathbf{F}=\sum\{\mathbf{F}_i:i\in I\}$ onto $\langle\omega,\mathrm{Suc}\rangle$ such that $g(\langle w_n,i\rangle)=n$ for each $i\in I$ is a p-morphism. Hence, by (1), $\mathbf{L}=\mathbf{L}_0$

Description of $\Lambda(\mathbf{KAlt_1})$

By means of the results of this section we may completely describe $\Lambda(\mathbf{KAlt_1})$ (see Fig. 2). The denumerably many extensions of L_0 are of the form $\mathrm{Th}(\sum\{\mathbf{F}_{m,r}:m\in I,r\in J\}$ for I,J finite subsets of ω. On the other hand, if $\mathbf{L}\in\Lambda(\mathbf{KAlt_1})$ and does not extend $\mathbf{L}_0$, then there exists a $t\in\omega$ such that $\mathbf{F}_t\in \mathrm{fGF}(\mathbf{L})$. We distinguish into these cases:

Case A. $\sup\{t:\mathbf{F}_t\in \mathrm{fGF}(\mathbf{L})\}=n$. If $\mathbf{L}$ is not finite (Subcase A1) then $\langle\omega,\mathrm{Suc}\rangle\in \mathrm{GKF}(\mathbf{L})$ and hence, since for each $t\leq n$ $\mathbf{F}_t$ is subframe of $\mathbf{F}_n$, we have that $\mathbf{L}=\mathrm{Th}(\langle\omega,\mathrm{Suc}\rangle)\cap\mathrm{Th}(\mathbf{F}_n)$. Thus $\mathrm{fGF}(\mathbf{L})=\mathrm{fGF}(\mathbf{KAlt_1})-\{\mathbf{F}_t:t\geqq n+1\}$ and then $\mathbf{L}=\mathbf{KAlt_1}+(L^{n+1}\bot\leftrightarrow L^{n+2}\bot)$; in Fig. 2 we have denoted such a logic by $\mathbf{L}_{n+1}$. If L is finite (Subcase A2), then $\mathbf{L}=\mathrm{Th}(\sum\{\mathbf{F}_{m,r}:m\in I,r\in J\})\cap\mathrm{Th}(\mathbf{F}_n)$, for I,J finite subsets of ω.

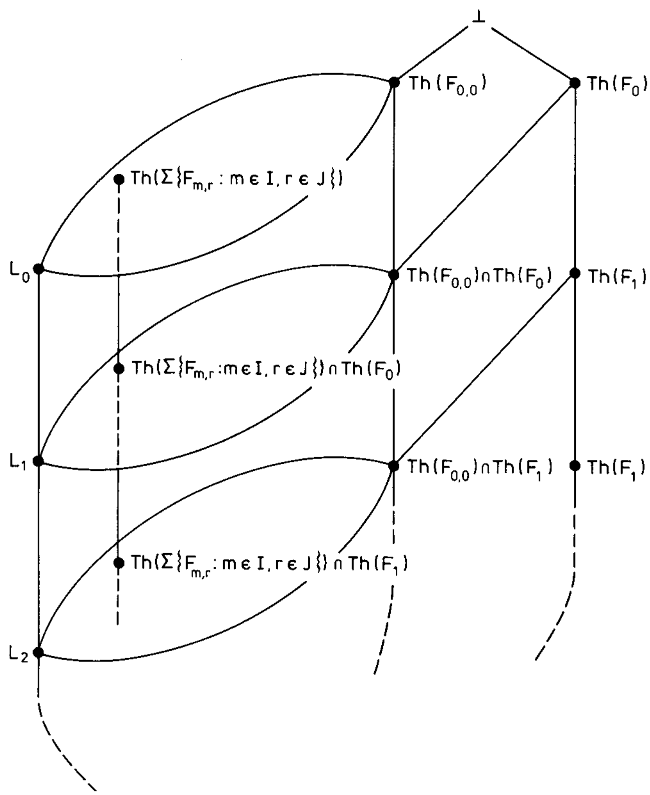

Fig. 2

Case B. $\sup\{t: F_t \in GKF(L)\} = \omega$. In such a case we have that $L = KAlt_1$; it is in fact straightforward to show that $KAlt_1$ is complete with respect to each class $\{F_t: t \in I\}$ such that $\sup(I) = \omega$.

It may be opportune to remark the following characteristic of $\Lambda(KAlt_1)$:

Corollary 2.3. *For each* n *there exists an* **L** *having exactly* n *non-finite proper extensions.*

Proof. $L = L_n = KAlt_1 + (L^n\bot \leftrightarrow L^{n+1}\bot)$. In fact the set of non-finite proper extensions of L_n is $\{L_i : 0 \leqq i \leqq n-1\}$.

3. The Extension of $KAlt_n$, $n \geqq 2$

Because of Theorem 1.4, and since all the logics without f.m.p. or having infinitely many pretabular extensions which appear in the literature are out of $\Lambda(KAlt_n)$, it was plausible to look for the analogous of the results of Sect. 1 for $\Lambda(KAlt_n)$. But, contrarily to that, we have the following negative results:

Theorem 3.0. *There is a continuum of extensions of* $KAlt_2$ *without f.m.p.*

Proof. Let $X = \{x_0, x_1, \ldots, x_m, \ldots\}$ be an infinite subset of ω such that $x_0 = 0, x_1 = 2$, and $x_{i+1} \geqq 2x_i + 1$ for $i \geqq 1$. We define $F_X = \langle W_X, R_X \rangle$ as follows:

$$W_X = \omega \cup \{w^*\},$$

$$R_X = \{n, n+1\rangle : n \in \omega\} \cup \{\langle n, w^*\rangle : n \in X\},$$

(Figure 3 represents F_X where $x_0 = 0$, $x_1 = 2$, and $x_{i+1} = 2x_i + 1$.) Set $\alpha = ML\bot \wedge M^3L\bot$, $\Delta_X = \{M^{x+1}L\bot : x \in X\} \cup \{\neg M^{y+1}L\bot : y \notin X\}$ and $\Gamma_X = \{\alpha \to \beta : \beta \in \Delta_X\}$.

The only point of W_X which satisfies α is 0; in fact, since w^* is the only point which satisfies $L\bot$, we have that $F_X \models ML\bot[w]$ iff $w \in X$, and x_0, x_1 are the only consecutive elements of X whose difference is 2. Moreover there holds that $F_X \models \Delta_X[0]$ and therefore we obtain that $\Gamma_X \subseteqq Th(F_X)$. Now we show that if **F** is finite, then, for each w of W, $F \not\models \Delta_X[w]$ (observe that, since each formula of Δ_X is variable-free, valuations are unessential). Suppose $F \models \Delta_X[w]$. Since for each natural m there exists an m' such that $M^{m'}L\bot \in \Delta_X$, then, as **F** is finite, it follows that, roughly speaking, there exists a loop between w and a terminal point u; in

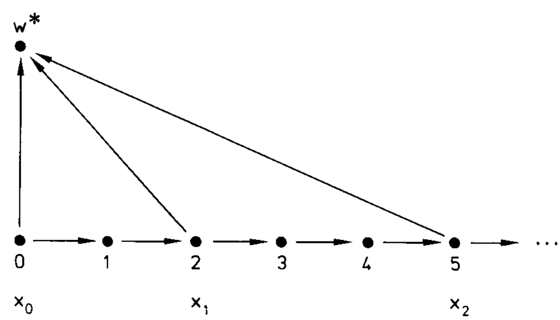

Fig. 3

detail, if $\mathbf{F} \models \Delta_X[w]$ then: (a) there exists a set $\{v_0, \ldots, v_n\} \subseteq W$ $(n \geqq 0)$ such that $v_n R v_0$ and $v_i R v_{i+1}$ for each $i \leqq n-1$; (b) there exist $i \leqq n$ and $q \in \omega$ such that $v_i \in S_q^*(\mathbf{F}, w)$ (we may assume $i=0$); (c) there exist $j \leqq n$, $h \in \omega$ and a terminal point u of W such that $u \in S_h^*(\mathbf{F}, v_j)$. Now, let $r = q+j+h$; we have that $\mathbf{F} \models M^r L\bot[w]$ and also that $\mathbf{F} \models M^{r+k(n+1)} L\bot[w]$ for each $k \in \omega$. But, for each r and n, there exists a k such that $r + k(n+1) \notin X$, i.e., such that $\neg M^{r+k(n+1)} L\bot \in \Delta_X$. Therefore $\mathbf{F} \not\models \Delta_x[w]$ for each w. Now, since $\Gamma_X \in \mathrm{Th}(\mathbf{F}_X)$ we obtain that each $\mathbf{F}$ of $\mathrm{fF}(\mathrm{Th}(\mathbf{F}_X))$ satisfies $\neg\alpha$, which, together with $\mathbf{F}_X \models \alpha[0]$, implies that $\mathrm{Th}(\mathbf{F}_X)$ is without f.m.p. Moreover, if $Y \neq X$, then, from $\mathbf{F}_Y \models \alpha[0]$, it follows that there is a $\beta \in \Delta_X$ such that $\alpha \to \beta \notin \mathrm{Th}(\mathbf{F}_Y)$, and therefore $\mathrm{Th}(\mathbf{F}_X) \neq \mathrm{Th}(\mathbf{F}_Y)$.

Theorem 3.1. *There are infinitely many pretabular extensions of* $\mathbf{KAlt_2}$.

Proof. For each $n \in \omega$ let $\mathbf{G}_n = \langle W_n, R_n \rangle$ be as follows:

$$W_n = \omega \cup \{w^*\},$$

$$R_n = \{\langle m, m+1 \rangle : m \in \omega\} \cup \{\langle s \cdot n, w^* \rangle : s \in \omega\}.$$

The following formulas belong to $\mathrm{Th}(\mathbf{G}_n)$:

$$\neg L\bot \to \bigvee_{1 \leqq m \leqq n} \Big(M^m L\bot \wedge \bigwedge_{1 \leqq r \neq m \leqq n} \neg M^r L\bot\Big), \tag{1}$$

$$\bigwedge_{1 \leqq m \leqq n} (M^m L\bot \to M^{m+n} L\bot), \tag{2}$$

$$(\neg L\bot \wedge \neg M L\bot) \to \mathrm{Alt}_1, \tag{3}$$

$$\mathrm{Alt}_2, \tag{4}$$

$$(M(L\bot \wedge p) \wedge M(L\bot \wedge q)) \to M(L\bot \wedge p \wedge q). \tag{5}$$

Suppose $\mathbf{F} = \langle W, R \rangle \in \mathrm{GKF}(\mathrm{Th}(\mathbf{G}_n))$. From (1) and (2) we have that, for each $w \in W$, either $\mathbf{F} \models L\bot[w]$ or there exists m, $1 \leqq m \leqq n$, such that $\mathbf{F} \models M^r L\bot$ iff $r \equiv m$ modulo n. By (3), we obtain that

$$|S_1(\mathbf{F}, w)| = \begin{cases} 0 & \text{if } \mathbf{F} \models L\bot[w], \\ 2 & \text{if } \mathbf{F} \models ML\bot[w], \\ 1 & \text{otherwise}, \end{cases}$$

and, finally, from (5) and the fact that $\mathbf{F}$ is generated it follows that if $\mathbf{F} \models L\bot[w]$ and $\mathbf{F} \models L\bot[w']$ then $w = w'$. Therefore, if $\mathbf{F}$ is infinite, then $\mathbf{F} \cong \mathbf{G}_n$, while, if it is finite, then either $\mathbf{F} \cong \mathbf{F}_0 = \langle \{w_0\}, \emptyset \rangle$ or $\mathbf{F} \cong \mathbf{F}_{rn}$ for an $r \in \omega$, where

$$W_{rn} = \{0, 1, \ldots, (r \cdot n) - 1\} \cup \{w^*\}$$

and

$$R_{rn} = \{\langle x, x+1 \rangle : 0 \leqq x \leqq (r \cdot n) - 2\} \cup \{\langle r \cdot n - 1, 0 \rangle\} \cup \{\langle s \cdot n, w^* \rangle : 0 \leqq s \leqq r-1\}.$$

(Figure 4 represents the case $r=3$, $n=2$.)

Now let $\mathrm{Th}(\mathbf{G}_n) \subseteq \mathbf{L}$. If $\mathrm{GKF}(\mathbf{L})$ contains an infinite frame, then contains $\mathbf{G}_n$ and hence $\mathbf{L} = \mathrm{Th}(\mathbf{G}_n)$; otherwise we have that $\sup\{|W_{rn}| : \mathbf{F}_{rn} \in \mathrm{GKF}(\mathbf{L})\}$ is finite. In fact, if it is infinite, we have that the function g from $\sum\{\mathbf{F}_i : i \in I\}$ (where I is a set of

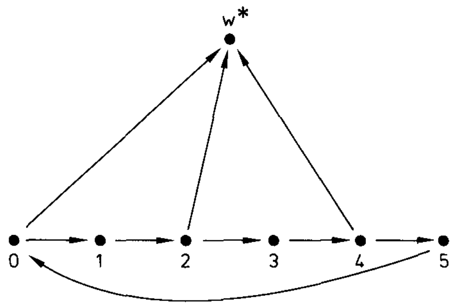

Fig. 4

indices for GKF(**L**)) onto $\mathbf{G}_n$ such that $g(\langle x, i\rangle) = x$ and $g(w^*, i) = w^*$ is a p-morphism, and thus $\mathbf{G}_n \in$ GKF(**L**). Hence suppose $\sup\{|W_{rn}| : \mathbf{F}_{rn} \in \mathrm{GKF}(\mathbf{L})\} = r^* \cdot n$; since for each $r \leqq r^*$ $\mathbf{F}_{rn}$ is p-morphic image of $\mathbf{F}_{r^*n}$ and $\mathbf{F}_0 = \langle\{w_0\}, \emptyset\rangle$ is generated subframe of $\mathbf{F}_{r^*n}$, we have, by Theorem 1.4 and Proposition 1.1(i), that $\mathbf{L} = \mathrm{Th}(\mathbf{F}_{r^*n})$, thus showing that $\mathrm{Th}(\mathbf{G}_n)$ is pretabular. But, if $n \neq n'$ then $\mathrm{Th}(\mathbf{G}_n) \neq \mathrm{Th}(\mathbf{G}_{n'})$, and so the proof is concluded.

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