USE OF POISSON'S INTEGRAL IN CALCULATING HIGHER VERTICAL DERIVATIVES OF HARMONIC FUNCTIONS – PART 1

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Summary: *The higher vertical derivatives of harmonic functions, expressed by Poisson's integral, are calculated for an infinite plane. The properties of the higher derivatives of the kernel of the integral are investigated and a method of calculation is proposed, which partly eliminates the negative effect caused by their "oscillation".*

1. INTRODUCTION

In recent years considerable attention has been devoted to the analytical continuation of harmonic functions into regions where the solution of boundary problems cannot be applied. Regions in which the harmonicness of the functions can no longer be guaranteed and the problem stops being unique, are usually involved. The solution is then more or less formal and need not correspond to reality. Nevertheless, it is useful to deal with this problem. One of the possible approaches to the solution is the use of a Taylor series which yields a unique solution in the interval where it converges. In this paper we shall deal with the conditions for determining the terms of this series, in particular with calculating the vertical derivatives of harmonic functions.

2. INTRODUCTION OF FUNCTIONS Q_k

The solution of the external Dirichlet problem for a sphere, radius R, is Poisson's integral [1]

$$T(\varrho, \vartheta_0, \lambda_0) = R(4\pi)^{-1} \iint_{\omega} T(R, \vartheta, \lambda)\, r^{-3}(\varrho^2 - R^2)\, \mathrm{d}\omega,\ \varrho > R\,,$$

where T is a function harmonic in the interval $\varrho \geqq R$; $\varrho, \vartheta_0, \lambda_0$ and R, ϑ, λ are spherical co-ordinates (origin at the centre of the sphere) of the investigated and variable point, respectively, and r is their distance; $\mathrm{d}\omega$ is an element of the unit sphere.

By applying the limit $R \to \infty$ we arrive at the Poisson integral for the infinite plane $z = 0$:

$$(1)\qquad T(x_0, y_0, z) = (2\pi)^{-1} \int_0^{\infty}\int_0^{2\pi} z l r^{-3}\, T(x_0 + l\cos\alpha,\, y_0 + l\sin\alpha,\, 0)\, \mathrm{d}l\, \mathrm{d}\alpha,$$

$$r^2 = l^2 + z^2\,,$$

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where l and α are the polar co-ordinates of the variable point in plane $z = 0$ with the origin at point $(x_0, y_0, 0)$, T is a function harmonic in the halfspace $z \geqq 0$ and r is the distance of the investigated and variable point.

The k-th derivative of function T with respect to z at point (x_0, y_0, v) is obtained by differentiating the kernel zl/r^3 of integral (1) k-times. We shall denote this function by $k!\,Q_k$:

$$k!\,Q_k(z, l) = \frac{\partial^k}{\partial z^k}\left(\frac{z}{r^3}\right) l = -\frac{\partial^{k+1}}{\partial z^{k+1}}\left(\frac{1}{r}\right) l \,. \tag{2}$$

It is easy to see that

$$\begin{aligned} 1!\,Q_1 &= r^{-3}(1 - 3\varkappa^2)\, l\,, \\ 2!\,Q_2 &= -r^{-4}\, 3\varkappa(3 - 5\varkappa^2)\, l\,, \\ 3!\,Q_3 &= -r^{-5}\, 3(3 - 30\varkappa^2 + 35\varkappa^4)\, l\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and, in general,

$$k!\,Q_k = a_0^{(k)} l r^{-(k+2)} \sum_{j=1}^{(k+3)/2} a_j^{(k)} \varkappa^{2(j-1)},\; k = 1, 3, 5, \ldots\,; \tag{3}$$

$$k!\,Q_k = a_0^{(k)} l \varkappa r^{-(k+2)} \sum_{j=1}^{(k+2)/2} a_j^{(k)} \varkappa^{2(j-1)},\; k = 2, 4, 6, \ldots, \quad \varkappa = v/r\,.$$

Table 1 gives the coefficients $a_j^{(k)}$ for $k = 1, 2, \ldots, 12$. The sums $\sum_j a_j^{(k)} \varkappa^{2(j-1)}$ are polynomials in terms of $\varkappa^2$ and, therefore, they have $\frac{1}{2}(k + 1)$ (for k odd) or $\frac{1}{2}k$ (for k even) positive roots. It can be proved that all roots lie within the interval $\varkappa^2 \in (0, 1)$,

Table 1.

k	$a_0^{(k)}$	$a_1^{(k)}$	$a_2^{(k)}$	$a_3^{(k)}$	$a_4^{(k)}$	$a_5^{(k)}$	$a_6^{(k)}$	$a_7^{(k)}$
1	1	1	−3					
2	−3	3	−5					
3	−3	3	−30	35				
4	15	15	−70	63				
5	45	5	−105	315				
6	−315	35	−315	693	−429			
7	−315	35	−1260	6930	−12012	6435		
8	2835	315	−4620	18018	−25740	12155		
9	14175	63	−3465	30030	−90090	109395	−46189	
10	−155925	693	−15015	90090	−218790	230945	−88179	
11	−467775	231	−18018	225225	−1021020	2078505	−1939938	679039
12	6081075	3003	−90090	765765	−2771340	4849845	−4056234	1300075

i.e. $|l| \in (0, \infty)$ and that they differ from one another. Therefore, if $l \geqq 0$, the function Q_k has $\frac{1}{2}(k + 3)$ or $\frac{1}{2}(k + 2)$ roots which differ from one another. Table 2 gives the roots of functions Q_k for $v = 1$ and $k = 1, 2, \ldots, 12$. It can be proved that

(4), (5) $$Q_k(v, l) = v^{-(k+1)} Q_k(1, t); \quad t = l/v .$$

Table 2.

k	x_1	x_2	x_3	x_4	x_5	x_6
1	1·4142					
2	0·8165					
3	0·5904	2·7662				
4	0·4667	1·5649				
5	0·3874	1·1325	4·0719			
6	0·3318	0·9048	2·2520			
7	0·2905	0·7587	1·6181	5·3590		
8	0·2585	0·6563	1·2876	2·9174		
9	0·2330	0·5799	1·0799	2·0795	6·6423	
10	0·2121	0·5204	0·9359	1·6465	3·5728	
11	0·1948	0·4726	0·8288	1·3780	2·5280	7·9222
12	0·1800	0·4332	0·7459	1·1932	1·9929	4·2224

This is convenient because we are able to investigate the functions $Q(1, t)$ only, and proceed from them to the functions $Q(v, l)$ using Eq. (4) and substitution (5). We shall put

(6) $$b_k(v) = \int_0^\infty \overline{T}(l)\, Q(v, l)\, \mathrm{d}l\,, \quad k = 1, 2, \ldots,$$

where $\overline{T}(l) = (2\pi)^{-1} \int_0^{2\pi} T(x_0 + l \cos \alpha, y_0 + l \sin \alpha, 0)\, \mathrm{d}\alpha$. Function $b_k(v)$ is thus equal to the k-th derivative of function T with respect to z at point (x_0, y_0, v), divided by $k!$.

3. INTRODUCTION OF FUNCTIONS f_k AND α_k

Due to their "oscillations" functions Q_k are not suitable for calculating the derivatives of b_k. This can be seen in Fig. 1, in which functions Q_1, Q_6 and Q_9 are shown for $v = 1$. If k is roughly larger than 5, the small quantity (6) can be obtained as the difference of several areas, which may be as much as several orders of magnitude larger than b_k itself, and thus the relative error of the caclulated derivative b_k will become several orders of magnitude larger than the relative error of function $\overline{T}$.

From investigating functions Q_k we can see that two successive functions Q_k and Q_{k+1} become increasingly similar (with the exception of the sign) with increasing

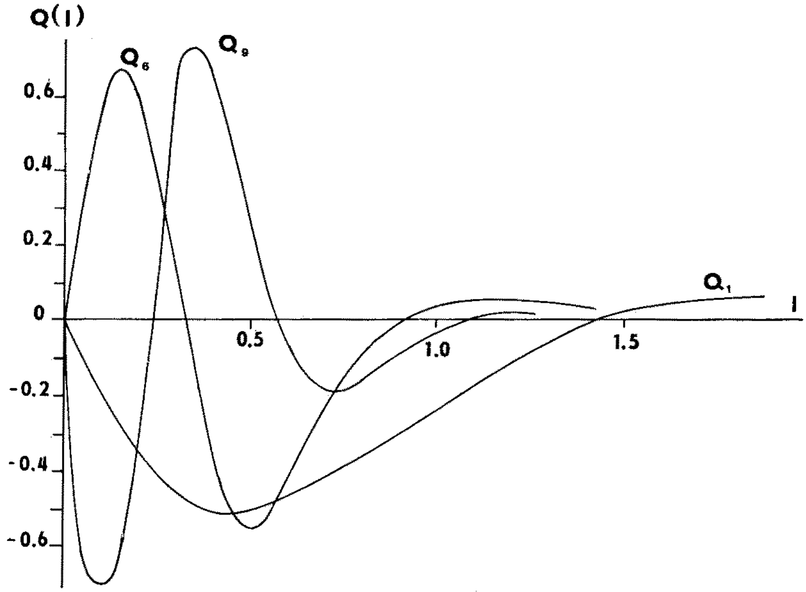

Fig. 1.

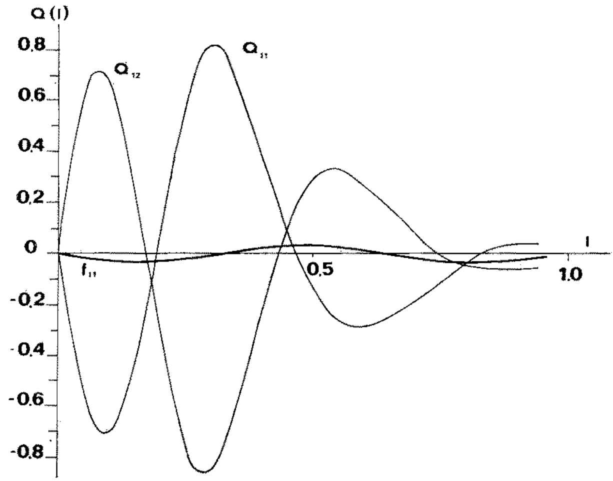

Fig. 2.

k (see functions Q_{11} and Q_{12} in Fig. 2); we are, therefore, offered the idea of creating combinations of the type

$$\sum_{j=1}^{M_k} v^{-1} \lambda_k^{(j)} Q_k(v_k^{(j)}, l) + Q_{k+1}(v, l), \quad k = 1, 2, \ldots, \quad M_k = 1, 2, \ldots, \tag{7}$$

where the difference $\Delta v_k^{(j)} = v - v_k^{(j)}$ is a small number ($|\Delta v| \ll 1$). We shall restrict ourselves just to the case of $M_k = 1$, i.e. to combinations which we shall denote f_k:

$$v^{-(k+2)} f_k(t) = v^{-(k+2)}[\lambda_k Q_k(\varepsilon_k, t) + Q_{k+1}(1, t)] = \tag{8}$$

$$= v^{-1}\lambda_k Q_k(\varepsilon_k v, l) + Q_{k+1}(v, l), \quad t = l/v, \quad k = 1, 2, \ldots .$$

Instead of v_k we have introduced the parameter $\varepsilon_k = v_k/v$, $(\varepsilon_k \approx 1)$. We shall introduce the α_k-integrals by

$$v\,\alpha_k(v) = v^{-(k+2)} \int_0^\infty \overline{T}(vt) f_k(t)\,\mathrm{d}t = \lambda_k b_k(\varepsilon_k v) + v b_{k+1}(v), \quad k = 1, 2, \ldots . \tag{9}$$

Given a suitable choice of parameters λ_k and ε_k, the kernels of the integrals f_k are more suitable for numerical integration than the Q_k-kernels in Eqs (6) (see function f_{11} in Fig. 2). We shall discuss the choice of parameters λ_k and ε_k in Section 5.

4. CALCULATION OF THE DERIVATIVES b_k

In order to obtain the unknowns b_k with the aid of the integrals α_k, we shal create a linear system in terms of b_k from Eqs (9). Assume that we know b_i $(i = 1, 2, \ldots)$. We now write $j + 1$ equations (9) $(j = 1, 2, \ldots)$. For the sake of simplicity we shall put $b_i(v) = b_i$, $\alpha_i(v) = \alpha_i$:

$$\begin{aligned} v\alpha_i &= \lambda_i b_i(\varepsilon_i v) + v b_{i+1}, \\ v\alpha_{i+1} &= \lambda_{i+1} b_{i+1}(\varepsilon_{i+1} v) + v b_{i+2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v\alpha_{i+j} &= \lambda_{i+j} b_{i+j}(\varepsilon_{i+j} v) + v b_{i+j+1}. \end{aligned} \tag{10}$$

We shall develop the terms $b_i(\varepsilon_i v), \ldots, b_{i+j}(\varepsilon_{i+j} v)$ into series in terms of the powers of $v\,\Delta\varepsilon = v - \varepsilon v > 0$:

$$\begin{aligned} v\alpha_i &= b_i\lambda_i + b_{i+1} v[1 - (i+1)\lambda_i \Delta\varepsilon_i] + \lambda_i \sum_{l=2}^{\infty} (-1)^l \frac{(i+l)!}{i!\,l!} b_{i+l}(v\,\Delta\varepsilon_i)^l, \\ v\alpha_{i+1} &= b_{i+1}\lambda_{i+1} + b_{i+2} v[1 - (i+2)\lambda_{i+1} \Delta\varepsilon_{i+1}] + \\ &\quad + \lambda_{i+1} \sum_{l=2}^{\infty} (-1)^l \frac{(i+1+l)!}{(i+1)!\,l!} b_{i+1+l}(v\,\Delta\varepsilon_{i+1})^l, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v\alpha_{i+j} &= b_{i+j}\lambda_{i+j} + b_{i+j+1} v[1 - (i+j+1)\lambda_{i+j} \Delta\varepsilon_{i+j}] + \\ &\quad + \lambda_{i+j} \sum_{l=2}^{\infty} (-1)^l \frac{(i+j+l)!}{(i+j)!\,l!} b_{i+j+l}(v\,\Delta\varepsilon_{i+j})^l. \end{aligned} \tag{11}$$

For this purpose we have used the equations

$$\partial^i b_k(v)/\partial v^i = (i + k)!/(i!\,k!)\, b_{k+1}(v)\,, \quad i, k = 1, 2, \ldots, \tag{12}$$

which follow immediately from Eqs (2) and (6). We shall use Eq. (12) to derive the equations for the improper integrals

$$\begin{aligned} &\int_v^\infty b_j(z)\,\mathrm{d}z = -j^{-1}\, b_{j-1}(v)\,, \\ &\int_v^\infty b_j(z)\, z^i\,\mathrm{d}z = -j^{-1}\, b_{j-1}(v)\, v^i + \frac{i}{j(j+1)}\, b_{j-2}(v)\, v^{i-1} - \\ &\qquad - \frac{i(i-1)}{j(j-1)(j-2)}\, b_{j-3}(v)\, v^{i-2} + \ldots, n = 1, 2, \ldots\,. \end{aligned} \tag{13}$$

We shall integrate the second equation of system (11) within the limits $z \in \langle v, \infty)$, and the third equation twice, first within the limits $z \in \langle z_1, \infty)$ and then within the limits $z_1 \in \langle v, \infty)$; the third equation will be integrate three times, etc., and, finally, the $(j + 1)$-st equation j-times. We now take the terms with b_i over to the l.h.s. and neglect the terms containing $b_{i+j+2}, \ldots$. We thus arrive at a system of $j + 1$ equations for $j + 1$ unknowns $b_{i+1}, \ldots, b_{i+1+j}$:

$$\begin{bmatrix} \frac{1}{\lambda_i} - (i+1)\,\Delta\varepsilon_i S_i^{(0)} & c_i^{(2)}\,\Delta\varepsilon_i^2 S_i^{(0)} \ldots c_i^{(j+1)}\,\Delta\varepsilon_i^{j+1} S_i^{(0)} \\ \frac{i+1}{i+2}\frac{1}{\lambda_{i+1}} - (i+1)\,\Delta\varepsilon_{i+1} S_i^{(1)} & c_i^{(2)}\,\Delta\varepsilon_{i+1}^2 S_i^{(1)} \ldots c_i^{(j+1)}\,\Delta\varepsilon_{i+1}^{j+1} S_i^{(1)} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ \frac{i+1}{i+j+1}\frac{1}{\lambda_{i+j}} - (i+1)\,\Delta\varepsilon_{i+j} S_i^{(j)} & c_i^{(2)}\,\Delta\varepsilon_{i+j}^2 S_i^{(j)} \ldots c_i^{(j+1)}\,\Delta\varepsilon_{i+j}^{j+1} S_i^{(j)} \end{bmatrix} \times \tag{14}$$

$$\times \begin{bmatrix} v b_{i+1} \\ v^2 b_{i+2} \\ \ldots\ldots \\ v^{j+1} b_{i+j+1} \end{bmatrix} = \begin{bmatrix} I_i^{(0)} \\ I_i^{(1)} \\ \ldots \\ I_i^{(j)} \end{bmatrix},$$

$$C_i^{(m)} = (-1)^m\,(i + m)!/(i!\,m!)\,, \quad m = 2, 3, \ldots, j + 1\,, \tag{15}$$

$S_i^{(r)}$ are power series in terms of $\Delta\varepsilon_{i+r} = 1 - \varepsilon_{i+r}$, and it can be proved that

(16) $$S_i^{(r)} = \varepsilon_{i+r}^{-r}, \quad r = 0, 1, \ldots, j,$$

(17) $$I_i^{(r)} = A_i^{(r)} - b_i \left[S_i^{(r)} - \frac{r}{i + 1 + r} \frac{1}{\lambda_{i+r}} \right], \quad r = 1, 2, \ldots, j,$$

$$I_i^{(0)} = \lambda_i^{-1} \alpha_i - b_i,$$

(18) $$A_i^{(r)} = (-1)^r \frac{(i+r)!}{i!} \frac{1}{\lambda_{i+r}} \int_v^\infty \mathrm{d}z_{r-1} \int_{z_{r-1}}^\infty \ldots \int_{z_2}^\infty \mathrm{d}z_1 \int_{z_1}^\infty \alpha_{i+r}(z)\, z \, \mathrm{d}z,$$

$$r = 1, 2, \ldots, j.$$

The elements of the matrix of system (14) e_{kl} diminish with increasing l and, therefore, we shall only calculate the first unknown b_{i+1} (the accuracy of calculating b_{i+l} deteriorates with increasing l). For i we then substitute $m = i + 1$, calculate the unknown b_{m+1}, etc. We shall denote the determinant of system (14) by $D_i^{(j+1)}$ and the sub-determinants of the elements of the first column $[D_i^{(j+1)}]_{r,1}$, $r = 1, 2, \ldots, j + 1$. It then holds that

(19) $$v b_{i+1} = \sum_{k=1}^{j+1} C_i^{(k)} I_i^{(k-1)},$$

(20) $$C_i^{(k)} = (-1)^{1+k} [D_i^{(j+1)}]_{k,1} [D_i^{(j+1)}]^{-1}, \quad k = 1, 2, \ldots, j + 1.$$

5. CHOICE OF THE PARAMETERS λ_k AND ε_k

It is evident that the effectiveness of the method proposed in Section 4 depends on the choice of the parameters λ_k and ε_k. If we draw only on Section 3, we can, e.g., require that function f_k satisfy the condition

(21) $$\int_0^\infty [\lambda_k \, Q_k(\varepsilon_k, t) + Q_{k+1}(1, t)]^2 \, \mathrm{d}t = \min.$$

After differentiating this equation with respect to λ_k and ε_k and modification, we arrive at an implicit equation for ε_k:

(22) $$\int_0^\infty Q_k' Q_{k+1}' \, \mathrm{d}t \int_0^\infty Q_k' Q_{k+1} \, \mathrm{d}t - \int_0^\infty (Q_k')^2 \, \mathrm{d}t \int_0^\infty Q_{k+1}' Q_{k+1} \, \mathrm{d}t = 0,$$

$$Q_l = Q_l(1, t), \quad Q_l' = Q_l(\varepsilon, t),$$

and for λ_k:

(23)
$$-\lambda_k = \int_0^\infty Q_k' Q_{k+1} \, \mathrm{d}t \left[\int_0^\infty (Q_k')^2 \, \mathrm{d}t \right]^{-1} = \int_0^\infty Q_{k+1}' Q_{k+1} \, \mathrm{d}t \left[\int_0^\infty Q_k' Q_{k+1}' \, \mathrm{d}t \right]^{-1}.$$

As an index of "improvement", achieved by introducing functions f_k instead of Q_k, we can adopt the ratio p_k of the standard deviations of both functions:

(24) $$p_k^2 = \int_0^\infty Q_k^2 \, dt \left[\int_0^\infty f_k^2(t) \, dt \right]^{-1} .$$

Table 3 gives the parameters $\lambda_8, \ldots, \lambda_{11}$ and $\varepsilon_8, \ldots, \varepsilon_{11}$, calculated using Eqs (22) and (23), and the values $p_8, \ldots, p_{11}$, calculated using Eq. (24). These values can be considered optimum only with regard to calculating integrals (9), but not with regard to solving the system (14). We shall not deal with the criterion for selecting the parameters suitable for solving the system (14) in this paper. Very roughly speaking, the parameters λ_k and ε_k have to be chosen to render the absolute values of coefficients (20) within the interval $(0, \delta > 1)$, where δ does not differ much from unity.

Table 3.

n	λ_n	ε_n	$\Delta\varepsilon_n$	p_n
8	0·3954	0·8967	0·1033	17·5
9	0·3952	0·9072	0·0928	19·2
10	0·3910	0·9145	0·0855	21·0
11	0·3856	0·9205	0·0795	24·0

However, at the same time the values p_k have to be as large as possible. By calculating the coefficients $C_8^{(1)}$, $C_8^{(2)}$, $C_8^{(3)}$, $C_8^{(4)}$ for the parameters λ_k and ε_k, given in Tab. 3, we shall find that max $|C_8^{(i)}| \approx 45$. This means that the selection of the parameters λ_k and ε_k, optimum for calculating the integrals (9), is not optimum for calculating the integrals (6) with the aid of system (14). Table 4 gives the values of the parameters λ_k and ε_k, determined by approximative methods. The values of the quantity p and coefficients C are also given.

Table 4.

n	λ_n	ε_n	$\Delta\varepsilon_n$	p_n	r	$C_8^{(r)}$
8	0·3432	0·8836	0·1164	13·5	1	0·185417
9	0·4375	0·9167	0·0833	16·0	2	3·986088
10	0·3575	0·9081	0·0919	18·0	3	−2·168847
11	0·4210	0·9270	0·0730	20·7	4	−1·565442

6. CONCLUSION

Introducing the function f_k as a linear combination of functions $Q_k(\varepsilon, t)$ and $Q_{k+1}(1, t)$ improved the conditions of numerical integration expressively and thus will permit the auxiliary plane $z = v$ to approach the initial plane $z = 0$. From the example in Section 5 (Tab. 4) we can see that a suitable choice of the parameters λ_k and ε_k will enable us to achieve very considerable improvement (more accurately speaking, b_k can be calculated with a relative error several times smaller than by direct computation using Eq. (6)). We shall have to deal more rigorously with the problem of optimizing the selection of the parameters λ_k and ε_k with regard to the effectiveness of the proposed method, and to the investigation of the case given by Eq. (7), or possibly of an even more general case.

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