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The ε-Expansion for the Hierarchical Model

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Abstract. In this paper, the Hierarchical Model is studied near a non-trivial fixed point φ_ε of its renormalization group. Our analysis is an extension of work of Bleher and Sinai. We prove the validity of the ε-expansion for φ_ε. We then show that the renormalization transformations around φ_ε have an unstable manifold which is completely characterized by the tangent map and can be brought to normal form. We then establish relations between this result and the critical behaviour of the model in the thermodynamic limit.

Introduction and Description of Results

This paper brings the ε-expansion of the renormalization group theory for the Hierarchical Model on a sound mathematical footing. The Hierarchical Model is a model on a one-dimensional lattice with ferromagnetic spin interaction whose range depends on a parameter c. As c varies, the behaviour of the model near its critical temperature varies also and actually multicritical points of any degree can occur. The first non-Gaussian critical behaviour occurs when $c=2^{1/2(1-\varepsilon)}$ and then the fixed point of the renormalization group, (which is an exact transformation for this model) can be discussed by the so-called ε-expansion. This model is the simplest model in which an ε-expansion arises [1, 2, 7–9]. The main impetus for the mathematical study of this model comes from the deep work of Bleher and Sinai [3, 4], on which we rely for the existence of a critical spin distribution.

In Section 1, we review the definition and the exact meaning of the ε-expansion for this model (one changes the range of the interaction instead of the dimension). We show that the ε-expansion is the perturbation theory of bifurcation from a simple eigenvalue [5].

Section 2 is the basis of all our results on the validity of the ε-expansion; we show that the fixed point of the renormalization group is differentiable in ε up to any order, provided ε is sufficiently small, and has thus an ε-expansion up to any order. The proofs take up Sections 2–5.

Section 6 is the description of the renormalization group action near the fixed point; this is the theory of the normal form of diffeomorphisms around a fixed point (on Banach spaces [17]).

Section 7 makes the contact of the results of Section 2–6 with the thermodynamic limit in the statistical mechanics of the Hierarchical Model. Similar results have been previously reported by Bleher and Sinai [4], using a different point of view. We find validity of the ε-expansion for the anomalous dimensions of the relevant scaling fields. For the free energy per degree of freedom, the scaling field equals the thermodynamic limit of the corresponding quantity. Such a result is not shown in the case of the susceptibility, because we have no good bounds on thermodynamic limit, and we discuss only the scaling limit. Our methods allow for analogous results for odd functions (magnetization) and for the case of multicritical points of the Hierarchical Model.

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1. Formalism for the Hierarchical Model

We recall the definition of the Hierarchical Model and we put its ε-expansion around a certain Gaussian fixed point in perspective. On the one-dimensional finite lattice consisting of the points $1,\dots 2^N$, with spins $s_1,\dots s_{2^N}$, one considers the Hamiltonian

$$\mathscr{H}_N = -\sum_{k=1}^{N} \frac{c^k}{2^{2k+1}} \sum_{j=0}^{2^{N-k}-1} \left(\sum_{l=1}^{2^k} s_{j2^k+l}\right)^2 , \tag{1.1}$$

(with the notation of Bleher and Sinai [3]). This is an interaction with potential $\sim \mathrm{dist}^{\log_2 c-2}$, i.e. the range of the interaction depends on c. The critical dimension (for the possibility of a non-Gaussian fixed point) predicted by the Landau theory is $d_{\mathrm{crit}} = 2(1-\log_2 c)$, so that for

$$c = c_\varepsilon = 2^{1/2(1-\varepsilon)} , \tag{1.2}$$

the critical dimension is $1+\varepsilon$ which is by ε above the actual dimension. Therefore the expansion in ε of the critical spin distribution is analogous to the usual ε-expansion. We shall now derive carefully a non-linear equation for the mean spin distribution which we then discuss. The recursive equations for the model described by (1.1) at inverse temperature β are

$$f_N(z,\beta) = L_N(\beta,c) \int dz_1 dz_2 \cdot \delta\left(\frac{z_1+z_2}{2} - \frac{z}{\sqrt{c}}\right) f_{N-1}(z_1,\beta) f_{N-1}(z_2,\beta) e^{1/2\frac{c}{4}\beta(z_1+z_2)^2} . \tag{1.3}$$

Here, $f_N(z,\beta)$ is the rescaled mean spin distribution for the model in "volume" 2^N. Henceforth, we shall omit the normalization $L_N(z,\beta)$. The corresponding fixed point equation for the distribution is thus

$$\mathring{f}(z,\beta) = e^{1/2\beta z^2} \int_{-\infty}^{+\infty} du\, \mathring{f}(zc^{-1/2}+u,\beta)\, \mathring{f}(zc^{-1/2}-u,\beta) . \tag{1.4}$$

In the beautiful work of Bleher and Sinai [3, 4] it is shown that certain initial single spin distributions are attracted by the iteration of (1.3) to a non-Gaussian fixed point of (1.4), if $\varepsilon > 0$. We shall discuss these fixed points and their neighborhoods. For this, we introduce the following change of variables. A straightforward calculation shows that φ_ε is a solution of

$$\begin{aligned}\varphi(z) &= \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-u^2} \varphi(zc^{-1/2}+u)\varphi(zc^{-1/2}-u)du \\ &= \mathcal{N}_\varepsilon(\varphi)(z)\,,\end{aligned} \tag{1.5}$$

with $c_\varepsilon = 2^{1/2(1-\varepsilon)}$ if and only if

$$\mathring{f}(z,\beta) = \left[\pi^{1/2}\left(\frac{2-c}{\beta c}\right)^{1/2}\right]^{-1} e^{-\frac{\beta}{2}\frac{c}{2-c}z^2} \varphi_\varepsilon\left(\left(\frac{\beta c}{2-c}\right)^{1/2} z\right), \tag{1.6}$$

is a solution of (1.4). It suffices thus to study (1.5). In particular, $\varphi_\varepsilon = 1$ corresponds to the Gaussian solution of (1.4). We now study bifurcations from this solution for small ε. Since we intend to discuss the ε-expansion of φ_ε, we first state the algorithm for producing this expansion systematically. In fact we show that this is a typical situation of a "bifurcation from a simple eigenvalue" (cf. Crandall and Rabinowitz [5]). Set $F(\varepsilon, \psi) = \mathcal{N}_\varepsilon(\psi+1) - (\psi+1)$, and let $\mathcal{L}_{\psi,\varepsilon} = \partial_\psi F(\varepsilon, \psi)$ be the tangent map to F at ψ. By definition $\mathcal{L}_{\psi,\varepsilon}$ is the linear map given by

$$\mathcal{L}_{\psi,\varepsilon}(\varphi)(z) = 2\pi^{-1/2} \int_{-\infty}^{+\infty} e^{-u^2}(1+\psi(zc_\varepsilon^{-1/2}+u))\varphi(zc_\varepsilon^{-1/2}-u)du - \varphi(z)\,. \tag{1.7}$$

As we have seen, $F(\varepsilon, \psi) = 0$ has the "Gaussian" solution $\psi = 0$, so we concentrate on $\mathcal{L}_{0,\varepsilon}$. On the space of even functions, $\mathcal{L}_{0,\varepsilon}$ has spectrum $2c_\varepsilon^{-k} - 1$, $k = 0, 1, 2, \ldots$ with eigenvectors

$$H_{2k,\varepsilon}(z) = e^{x^2} \partial_x^{2k} e^{-x^2}\big|_{x=\gamma_\varepsilon^{1/2} z}\,, \tag{1.8}$$

where

$$\gamma_\varepsilon = 1 - c_\varepsilon^{-1}\,. \tag{1.9}$$

We shall write $\gamma = \gamma_{\varepsilon=0} = 1 - 2^{-1/2}$, and $H_{2k} = H_{2k,\varepsilon=0}$.

The functions $H_{2k,\varepsilon}$ are the Hermite polynomials, and the functions $2^{-k/2}(k!)^{-1/2} H_k$ form an orthonormal basis on $L_2(\mathbb{R}, \exp(-\gamma_\varepsilon z^2)dz)$.

The important fact is now that $2c_\varepsilon^{-2} - 1 = 0$ for $\varepsilon = 0$. (1.10)

Furthermore $\partial_\varepsilon \partial_\psi F(0,0) \cdot H_4 \notin \text{Range}\,(\partial_\psi F(0,0))$. (1.11)

Therefore one can expect a bifurcation in the H_4 "direction" and it can be found, as a formal power series, as follows:

Let $\psi_\perp(\alpha) = \sum_{j \neq 2} a_j(\alpha) H_{2j}$ and define

$$f(\alpha, \varepsilon(\alpha), \psi_\perp(\alpha)) = \begin{cases} \dfrac{1}{\alpha} F(\varepsilon(\alpha), \alpha H_4 + \alpha\psi_\perp(\alpha)), & \alpha \neq 0 \\ 0\,, & \alpha = 0 \end{cases}. \tag{1.12}$$

The conditions (1.10), (1.11) ensure that the implicit function theorem for formal power series [18] can be applied to the equation $f(\alpha, \varepsilon(\alpha), \psi_\perp(\alpha))=0$ and yields a nontrivial formal solution. We do this explicitly in the Appendix.

Unfortunately, it seems that the topological conditions, which are needed for the *existence of a solution* of $f(\alpha, \varepsilon, \psi_\perp)=0$ *in some function space* are not satisfied for the non-linear Equation (1.5). We have in fact been unable to verify the hypotheses of any of the strong implicit function theorems on Banach spaces or Frechet spaces [11] (Nash-Moser type theorems) [cf. Lemma 3.1 and Eq. (A7)]. Therefore, although the formalism (1.10)–(1.12) is very elegant, we are forced to use the direct calculations of Bleher and Sinai [4] to insure the existence of the solution. We hope however that (1.10)–(1.12) systematizes suitably ε-expansions and we shall use such ideas in the sequel.

2. The Fixed Point

We discuss the properties of the non-trivial solution which the Equation (1.5) has for ε satisfying $\varepsilon_0>\varepsilon>0$. We view $\varphi_\varepsilon(.)$ as an element in $[0,\varepsilon_0)\times L_{2,\sigma}=M_{2,\sigma,\varepsilon_0}$, where $L_{s,\sigma}=L_s(\mathbb{R}, e^{-\sigma z^2}\sigma^{1/2}\pi^{-1/2}dz)$. Let $\gamma=1-2^{-1/2}$. Our first result is then

Theorem 2.1. *For all* $N\in\mathbb{Z}^+$ *and all* $\sigma\in(0,\gamma]$ *there is an* $\varepsilon_0(N,\sigma)$ *such that for* $0\leqq\varepsilon<\varepsilon_0(N,\sigma)$ *a solution* $\varphi_\varepsilon(z)$ *of* $\mathcal{N}_\varepsilon(\varphi_\varepsilon)=\varphi_\varepsilon$ *satisfies*

i) φ_ε *is a* C^N *function of* ε *and* z *as an element of* $M_{2,\sigma,\varepsilon_0(N,\sigma)}$[1].

ii) *The derivatives with respect to* ε *at* $\varepsilon=0$ *are given by the* ε*-expansion, cf. Appendix* A.

iii) *In particular,*

$$\varphi_\varepsilon(z)=1-\varepsilon\vartheta H_4(z)+\mathcal{O}_{L_{2,\sigma}}(\varepsilon^2)\,, \tag{2.1}$$

with $\vartheta=(\log 2)/(144(2^{1/2}-1)^2)$.

The main input to our result is the analysis of Bleher and Sinai [4], whose result we state with a minor generalization which is easy to incorporate in their proof.

Theorem 2.2. *For* $\varepsilon\geqq 0$ *sufficiently small, the equation* $\mathcal{N}_\varepsilon(\varphi)=\varphi$ *has a solution* $\varphi=\varphi_\varepsilon$ *which is not constant when* $\varepsilon>0$ *and which satisfies*

i) $|\varphi_\varepsilon(z)|, |\partial_z\varphi_\varepsilon(z)|<2.$ (2.2)

ii) *For fixed* z, $\varphi_\varepsilon(z)$ *is continuous in* $\varepsilon\geqq 0$.

iii) *For every* $d>0$ *there is an* $\varepsilon_0(d)>0$ *such that for* $0\leqq\varepsilon<\varepsilon_0(d)$, *the function* φ_ε *has the following representation for* $|z|<(d\ln\varepsilon^{-1})^{1/2}$:

$$\varphi_\varepsilon(z)=1-\varepsilon\vartheta H_{4,\varepsilon}(z)+\varepsilon^{5/3}\mathring{R}_\varepsilon(z)\,, \tag{2.3}$$

with

$$|\mathring{R}_\varepsilon(z)|, |\partial_z\mathring{R}_\varepsilon(z)|<1\,. \tag{2.4}$$

Remark. Presumably $\varepsilon_0(d)\to 0$ as $d\to\infty$.

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[1] The topology of $M_{2,\sigma,\varepsilon_0}$ is given through the norm $\sup_{0\leqq\varepsilon<\varepsilon_0}\|\varphi_\varepsilon\|_{2,\sigma}$, where $\|\cdot\|_{2,\sigma}$ is the norm of $L_{2,\sigma}$

It is our aim to work in the more convenient class of $L_{s,\sigma}$ spaces instead of using the regions $|z|<(d\ln\varepsilon^{-1})^{1/2}$. However, as we shall see below, the remainder $\mathring{R}_\varepsilon$ and the linearization $\mathscr{A}_{\varphi_\varepsilon}$ of $\mathscr{N}$ around $\varphi=\varphi_\varepsilon$ are not sufficiently well controlled on all $L_{s,\sigma}$-spaces simultaneously for fixed ε, and we will have to work on a scale of spaces.

For $\varepsilon>0$, let

$$d_\varepsilon=\sup\{d|\varepsilon_0(d)\geqq\varepsilon, d\leqq\varepsilon^{-1/6}\} . \tag{2.5}$$

Note that for ε sufficiently small, one has $d_\varepsilon\geqq 1$, (2.6)
and furthermore $d_\varepsilon\leqq d_{\varepsilon'}$ if $\varepsilon'\leqq\varepsilon$, and $d_\varepsilon\to\infty$ as $\varepsilon\to 0$.

Define R_ε by

$$R_\varepsilon=\varepsilon^{-5/3}(\varphi_\varepsilon-1+\varepsilon\vartheta H_4), \quad \text{if} \quad \varepsilon>0 ,$$
$$R_0=0 . \tag{2.7}$$

Proposition 2.3. *If* $0<\sigma\leqq\gamma$, $s\geqq 1$, $\sigma/s\geqq 10/(3d_\varepsilon)$, *then* $\|R_{\varepsilon'}\|_{s,\sigma}$ *and* $\|\partial_z R_{\varepsilon'}\|_{s,\sigma}$ *are uniformly bounded for* $\varepsilon'\in[0,\varepsilon)$.

Proof. On

$$D_{\varepsilon'}=\{z\in\mathbb{R}||z|<(d_{\varepsilon'}\ln(1/\varepsilon'))^{1/2}\} , \tag{2.8}$$

one has $|H_4(z)-H_{4,\varepsilon'(z)}|\leqq\mathcal{O}(\varepsilon'\cdot\varepsilon'^{-1/5})$, by the definition of $H_{4,\varepsilon'}$, H_4, and $d_{\varepsilon'}$. Therefore, on $D_{\varepsilon'}$,

$$\begin{aligned} R_{\varepsilon'}&=\varepsilon'^{-5/3}(\varphi_{\varepsilon'}-1+\varepsilon'\vartheta H_{4,\varepsilon'}-\vartheta\varepsilon'(H_{4,\varepsilon'}-H_4)) \\ &=\mathring{R}_{\varepsilon'}+\mathcal{O}(\varepsilon'^{2/15}) , \end{aligned}$$

so that $|R_{\varepsilon'}(z)|<2$ for sufficiently small $\varepsilon'>0$, and $z\in D_{\varepsilon'}$. Therefore

$$\left(\int_{D_{\varepsilon'}}|R_{\varepsilon'}(z)|^s e^{-\sigma z^2}dz\right)^{1/s}=\mathcal{O}(1) .$$

On the complement $D^c_{\varepsilon'}$ of $D_{\varepsilon'}$ one has $|\varphi_{\varepsilon'}(z)|<2$, by Equation (2.2) and therefore

$$\begin{aligned} (\sigma/\pi)^{-1/2}\int_{D^c_{\varepsilon'}}|\varphi_{\varepsilon'}(z)|^s e^{-\sigma z^2}dz &\leqq(\sigma/\pi)^{-1/2}2^{s+1}\int_{(d_{\varepsilon'}\ln 1/\varepsilon')^{1/2}}^{\infty} e^{-\sigma z^2}dz \\ &\leqq e^{-\frac{\sigma}{2}d_{\varepsilon'}\ln\frac{1}{\varepsilon'}}2^{1/2+s}=2^{s+1/2}\varepsilon'^{\sigma d_{\varepsilon'}/2} . \end{aligned}$$

Similarly,

$$(\sigma/\pi)^{-1/2}\int_{D^c_{\varepsilon'}}|1-\varepsilon'\vartheta H_{4,\varepsilon'}(z)|^s e^{-\sigma z^2}dz\leqq e^{\sigma d_{\varepsilon'}/2}C(\sigma,s) .$$

It follows that with characteristic functions χ, one has

$$\begin{aligned} \|R_{\varepsilon'}\|_{s,\sigma}&\leqq\|\chi_{D_{\varepsilon'}}R_{\varepsilon'}\|_{s,\sigma}+\varepsilon'^{-5/3}(\|\chi_{D^c_{\varepsilon'}}\varphi_{\varepsilon'}(\,.\,)\|_{s,\sigma}+\|\chi_{D^c_{\varepsilon'}}(1-\varepsilon'\vartheta H_{4,\varepsilon'})\|_{s,\sigma}) \\ &\leqq\mathcal{O}(1)+\mathcal{O}(\varepsilon'^{-5/3}\varepsilon'^{\sigma d_{\varepsilon'}/(2s)})\leqq C(\sigma,s) , \end{aligned}$$

provided $\sigma d_{\varepsilon'}/(2s)\geqq 5/3$, which follows from $\sigma d_\varepsilon/(2s)\geqq 5/3$.

This proves the assertion for R_ε, the case $\partial_z R_\varepsilon$ is similar.

3. Differentiability in z

Our inductive bounds are based on the following

Lemma 3.1. *Let* $s,t,r \geqq 1$; $\sigma,\tau>0$, $s \leqq t$. *If*

$$1-\sigma/s-\tau/t>0\,, \tag{3.1}$$

and

$$\frac{\varrho}{r}\left(1-\frac{\sigma}{s}-\frac{\tau}{t}\right)-\frac{1}{c}\left(\frac{\sigma}{s}+\frac{\tau}{t}\right)+\frac{4}{c}\frac{\sigma}{s}\frac{\tau}{t}>0\,, \tag{3.2}$$

then one has for $f \in L_{s,\sigma}$ *and* $g \in L_{t,\tau}$,

$$\int e^{-u^2}|f(zc^{-1/2}+u)g(zc^{-1/2}-u)|du=h(z)\in L_{r,\varrho} \tag{3.3}$$

and

$$\|h\|_{r,\varrho} \leqq \text{const.}\,\|f\|_{s,\sigma}\|g\|_{t,\tau}\,. \tag{3.4}$$

Furthermore, the map $f \to h$, *defined by* (3.3) *for fixed* g *is compact.*

Proof. If $K(z,u)$ is the kernel of an operator K from $L_s(\mathbb{R},dx)$ to $L_r(\mathbb{R},dx)$, then the operator K is compact if

$$|K|_{r,s}=\{\int du[\int dz|K(z,u)|^r]^{|s'/r}\}^{1/s'}<\infty\,,\qquad s'=s/(s-1)\,,$$

(an easy generalisation of [6, p. 518]).

Since we work on $L_{s,\sigma}$, we reduce the situation to $L_s(\mathbb{R},dx)$ by setting $\mathring{f}(x)=f(x)\exp(-\sigma x^2/s)$, and similarly for g and h. Then the kernel corresponding to the map $\mathring{f}\to\mathring{h}$, is

$$K_g(z,u)=e^{-\varrho/rz^2}\left\{e^{-(zc^{-1/2}-u)^2}\left(e^{\frac{t}{\tau}(2zc^{-1/2}-u)^2}\mathring{g}(2zc^{-1/2}-u)\right)\right\}e^{\sigma/su^2}\,.$$

Using the Hölder inequality in z, we bound $|K_g|^{s'}_{r;s}$ by

$$\mathcal{O}(1)\int du\left\{\int dz\left[\exp\left(-\varrho z^2-\mathrm{r}(\mathrm{zc}^{-1/2}-u)^2+r\frac{\tau}{t}(2zc^{-1/2}-u)^2\right)\right]^{\frac{t}{t-r}}\right\}^{\frac{s'(t-r)}{tr}}$$

$$\cdot\,\|g\|^{s'}_{t,\tau}\exp\left(s'\frac{\sigma}{s}u^2\right).$$

It is now a straightforward matter to evaluate the Gaussian integrals (first the z integral), and this yields the conditions

$$\frac{\varrho}{r}>\frac{1}{c}\left[4\frac{\tau}{t}-1\right],\ 1-\frac{\tau}{t}-\frac{\sigma}{s}>\frac{r}{c}\left(1-\frac{2\tau}{t}\right)^2\bigg/\left(\varrho+\frac{r}{c}\left(1-\frac{4\tau}{t}\right)^2\right),$$

which after some transformations can be seen to be equivalent to (3.1), (3.2). This proves the assertion.

Note that (3.3) does *not* define a continuous map $L_{s,\sigma}\times L_{s,\sigma}\to L_{s,\sigma}$, whatever $s\geqq 1$, $\sigma>0$ may be. We shall use the following special cases later:

The map (3.3) is compact and continuous on the spaces

$$L_{2,\sigma}\times L_{2,\sigma}\to L_{2,3\sigma/c}\,; \tag{3.5}$$
$$L_{s,\sigma}\times L_{t,\tau}\to L_{s,\sigma}\,,$$

provided $c\geqq 6/5$, $\sigma/s\leqq 1/8$, $\tau/t\leqq\sigma/(20s)$, $s\leqq t$.

In particular one can choose

$$L_{2,\sigma}\times L_{4,\tau}\to L_{2,\sigma}\,, \tag{3.6}$$

provided $c\geqq 6/5$, $\sigma\leqq 1/4$, $\tau\leqq\sigma/10$, and

$$L_{4/3,\sigma}\times L_{2,\tau}\to L_{4/3,\sigma}\,, \tag{3.7}$$

provided $c\geqq 6/5$, $\sigma\leqq 1/6$, $\tau\leqq 3\sigma/40$.

Remark. Lemma 3.1 holds with $e^{-u^2}du$ replaced by $ue^{-u^2}du$ [with a slight change of the constants, but not of (3.1), (3.2)].

Theorem 3.2. *For every $N\in\mathbb{Z}^+$ and every $\gamma\geqq\sigma>0$, there is an $\varepsilon_1(N,\sigma)$, such that φ_ε is N times continuously differentiable in z as an element of $L_{2,\sigma}$ if $0\leqq\varepsilon<\varepsilon_1(N,\sigma)$.*

Proof. Fix $\varepsilon_1=\varepsilon_1(N,\sigma)$ such that $\sigma\cdot 9^{N+1}/2=10/(3d_{\varepsilon_1})$, where d_ε is defined by (2.5). Note that $\varepsilon_1(N,\sigma)$ is monotonically decreasing as a function of N and σ. Since Hermite polynomials are differentiable in all $L_{s,\sigma}$-spaces, the case $N=0,1$ is an immediate consequence of Proposition 2.3. We proceed by induction, and we suppose the result holds for $f_j=\partial_z^j\varphi_\varepsilon$, $j=0,\ldots,N-1$. By the equation $\mathcal{N}_\varepsilon(\varphi_\varepsilon)=\varphi_\varepsilon$, we have with $c=c_\varepsilon$,

$$f_{N-1}(z) = c^{-N/2}\pi^{-1/2}\sum_{j=0}^{N-1}\binom{N-1}{j}\int e^{-u^2}f_j(zc^{-1/2}+u)f_{N-1-j}(zc^{-1/2}-u)du\,. \tag{A_{N-1}}$$

Since $c\sim 2^{1/2}$ this equality holds on $L_{2,\sigma/9\cdot 3/c}\subset L_{2,\sigma/3}$, by Equation (3.5) and by the relation $\varepsilon_1(N-1,\sigma/9)=\varepsilon_1(N,\sigma)$. (This relation implies $f_j\in L_{2,\sigma/9}$ for $j=1,\ldots N-1$.) Define also

$$\begin{aligned}g_N={}&c^{-N/2}\pi^{-1/2}\sum_{k=1}^{N-1}\binom{N}{k}\int e^{-u^2}f_k(zc^{-1/2}+u)f_{N-k}(zc^{-1/2}-u)du\\&+2c^{-(N-1)/2}\pi^{-1/2}\int e^{-u^2}udu(f_{N-1}(zc^{-1/2}+u)f_0(zc^{-1/2}-u)\\&+f_0(zc^{-1/2}+u)f_{N-1}(zc^{-1/2}-u))\\&-c^{-N/2}\pi^{-1/2}\int e^{-u^2}(f_{N-1}(zc^{-1/2}+u)f_1(zc^{-1/2}-u)\\&+f_1(zc^{-1/2}+u)f_{N-1}(zc^{-1/2}-u))du\,.\end{aligned} \tag{B_N}$$

By the inductive assumption and by (3.5), g_N is defined on $L_{2,\sigma/9\cdot 3/c}\subset L_{2,\sigma/3}$ and bounded uniformly in $0\leqq\varepsilon<\varepsilon_1(N,\sigma)$. Using an integration by parts formula, it is easy to see that g_N is a candidate for $f_N=\partial_z^N\varphi_\varepsilon$. By the inductive assumption and partial integration, g_N is the derivative of g_{N-1} with respect to z on $L_{2,\sigma/3}$. Also $g_{N-1}=f_{N-1}$ on $L_{2,\sigma/3}$, since the corresponding r.h.s. of A_{N-1} and B_{N-1} coincide on this space. Therefore $g_N=\partial_z g_{N-1}=\partial_z f_{N-1}$, i.e. f_{N-1} is differentiable on $L_{2,\sigma/3}$ and in fact continuously differentiable as can be seen by a change of variables $u \to u \mp zc^{-1/2}$, its derivative is then equal to the r.h.s. of A_N, as an element of $L_{2,\sigma} \supset L_{2,\sigma/9\cdot 3/c}$. The induction step is complete.

4. Bounds on the Linearization

For a suitable function g, to be specified below, we define the operator $\mathscr{A}_{g,\varepsilon}$ by the equation

$$\mathscr{A}_{g,\varepsilon}(f)(z) = 2\pi^{-1/2} \int_{-\infty}^{+\infty} e^{-u^2} g(zc_\varepsilon^{-1/2} + u) f(zc_\varepsilon^{-1/2} - u) du \,. \tag{4.1}$$

[$\mathscr{A}_{g,\varepsilon} = \mathscr{L}_{g-1,\varepsilon} + \mathbb{1}$, cf. Equation (1.7).]

For $g = \varphi_\varepsilon$ we shall note $\mathscr{A}_\varepsilon = \mathscr{A}_{\varphi_\varepsilon,\varepsilon}$.

The following facts from Bleher and Sinai [4] are used later.

Theorem 4.1. *For $0 \leqq \varepsilon$ sufficiently small the operator $\mathscr{A}_\varepsilon$ is bounded on $L_{2,1-c_\varepsilon^{-1}}$, and it has the following properties:*

i) *$\mathscr{A}_\varepsilon$ has an eigenvalue λ of the form*

$$\lambda = 1 - \varepsilon(\log 2) + \mathcal{O}(\varepsilon^{3/2})^2 \tag{4.2}$$

with eigenvector g_ε satisfying

$$\sup_{|z| \leqq 10(\ln \varepsilon^{-1})^{1/2}} |g_\varepsilon(z) - H_{4,\varepsilon}(z)| \leqq \varepsilon^{4/5} \,, \tag{4.3a}$$

and

$$|g_\varepsilon(z)| \leqq |z|^5 \exp(-\mathcal{O}(\varepsilon |z|^4)) \quad \textit{for} \quad |z| \geqq 10(\ln \varepsilon^{-1})^{1/2} \,. \tag{4.3b}$$

ii) *The remainder of the spectrum of $\mathscr{A}_\varepsilon$ is bounded away from 1, uniformly in $\varepsilon \geqq 0$.*

Let $\sigma, 0 < \sigma \leqq 1/4$ be given and define ε_0 by $c_{\varepsilon_0} = 6/5$. For $0 < \tau \leqq \sigma/10$, and $0 < \varepsilon \leqq \varepsilon_0$, the numbers $\sigma, \tau, c_{\varepsilon_0}$ satisfy (3.6). Fix now $0 < \varepsilon_1 \leqq \varepsilon_0$ such that $\sigma/2, \tau/4 > 10/(3d)$. Then we can improve Theorem 4.1 to

Theorem 4.2. *For all $\sigma, 0 < \sigma \leqq 1/4$, there is an $\varepsilon_2(\sigma) = \varepsilon_2 < \varepsilon_1$, $\varepsilon_2 > 0$, such that for $0 < \varepsilon \leqq \varepsilon_2$ one has*

i) *$\mathscr{A}_\varepsilon - \mathbb{1}$ is a bounded, invertible operator on $L_{2,\sigma}$. Its inverse is a norm continuous function of $\varepsilon > 0$, and it is bounded in norm by $\mathcal{O}(\varepsilon^{-1})$.*

ii) *Let $\|g - \varphi_\varepsilon\|_{4,\tau} = \mathcal{O}(\varepsilon^{5/4})$. Then $(\mathscr{A}_{g,\varepsilon} - \mathbb{1})^{-1}$ is a bounded operator of norm $\mathcal{O}(\varepsilon^{-1})$ on $L_{2,\sigma}$ and it is norm continuous as a function of $g \in L_{4,\tau}$.*

Proof. By the condition $\tau/4 > 10/(3d_{\varepsilon_1})$, by Equation (2.7), and by Proposition 2.3 we have $\varphi_\varepsilon \in L_{4,\tau}$. Therefore, since σ, τ satisfy (3.6),

$$\mathscr{A}_\varepsilon \text{ is compact from } L_{2,\sigma} \text{ to } L_{2,\sigma} \,. \tag{4.4}$$

For small $\varepsilon \geqq 0$ we have $L_{2,\sigma} \subset L_{2,1-c_\varepsilon-1}$, since $\sigma \leqq 1/2$. Therefore the spectrum of $\mathscr{A}_\varepsilon - \mathbb{1}$ consists of a point with multiplicity one near $\mathcal{O}(\varepsilon)$ and a remainder bounded

[2] The coefficient can be found from Equation (A5) by perturbation theory

away from zero. The bound on $(\mathscr{A}_\varepsilon - \mathbb{1})^{-1}$ is complicated by the fact that $\mathscr{A}_\varepsilon$ is *not symmetric*. Let P_ε be the orthogonal projection onto φ_ε in $L_{2,\sigma}$, $P_\varepsilon^\perp = \mathbb{1} - P_\varepsilon$. Then one has

Lemma 4.3. *The operator $P_\varepsilon^\perp(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp$ is invertible on $P_\varepsilon^\perp L_{2,\sigma}$ and the norm of its inverse is uniformly bounded for $\varepsilon > 0$ sufficiently small.*

We postpone the proof of this lemma and continue the proof of Theorem 4.2. Consider the "matrix"

$$\mathscr{A}_\varepsilon - \mathbb{1} = \begin{pmatrix} P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon & P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp \\ 0 & P_\varepsilon^\perp(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp \end{pmatrix} \tag{4.5}$$

on

$$P_\varepsilon L_{2,\sigma} \oplus P_\varepsilon^\perp L_{2,\sigma} .$$

The element $P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon$ is invertible and its inverse is bounded by $\mathcal{O}(\varepsilon^{-1})$ on $P_\varepsilon L_{2,\sigma}$, as a consequence of Theorem 4.1. The operator $P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp$ is rank 1 on $L_{2,\sigma}$ and its norm is bounded as a function of small $\varepsilon \geqq 0$ for fixed σ, as can be seen by explicitly calculating the Hilbert-Schmidt norm of $\mathscr{A}_\varepsilon$ on $L_{2,\sigma}$. Therefore the inverse of (4.5), which is

$$\begin{pmatrix} (P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon)^{-1} & -(P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon)^{-1}(P_\varepsilon(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp)(P_\varepsilon^\perp(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp)^{-1} \\ 0 & (P_\varepsilon^\perp(\mathscr{A}_\varepsilon - \mathbb{1})P_\varepsilon^\perp)^{-1} \end{pmatrix} \tag{4.6}$$

is bounded in norm by $\mathcal{O}(\varepsilon^{-1})$, (the sum of the norms of the matrix elements). This proves i), up to the norm continuity.By Lemma 3.1, $\|\mathscr{A}_{g,\varepsilon} - \mathscr{A}_{\varphi_\varepsilon,\varepsilon}\|_{2,\sigma} \leqq \mathcal{O}(\varepsilon^{5/4})$. The assertion follows now by i) and standard perturbation theory [14, IV, Theorem 1.16]. This completes the proof of Theorem 4.2, ii). By the continuity of φ_ε in $L_{4,\tau}$ the remainder of Theorem 4.2, i) follows.

Proof of Lemma 4.3. We first note that $\|\varphi_\varepsilon - 1\|_{4,\tau} \leqq \mathcal{O}(\varepsilon)$, by (2.7) and Proposition 2.3, so that by Lemma 3.1, $\|\mathscr{A}_{\varphi_\varepsilon,\varepsilon} - \mathscr{A}_{1,0}\|_{2,\sigma} = \mathcal{O}(\varepsilon)$. It suffices thus to show (by [14, IV, Theorem 1.16]) that $P_\varepsilon^\perp(\mathscr{A}_{1,0} - \mathbb{1})P_\varepsilon^\perp$ has a bounded inverse. Similarly, we note that

$$\|\varphi_\varepsilon - H_{4,0}\|_{2,\sigma} \leqq \|\varphi_\varepsilon - H_{4,\varepsilon}\|_{2,\sigma} + \|H_{4,\varepsilon} - H_{4,0}\|_{2,\sigma} < \mathcal{O}(\varepsilon^{4/5}) + \mathcal{O}(\varepsilon) ,$$

by (4.3a), (4.3b) and the definition (1.8) of Hermite polynomials. Therefore $\|P_\varepsilon^\perp - P_{H_4}^\perp\|_{2,\sigma} = \|P_{H_4} - P_\varepsilon\|_{2,\sigma} = \mathcal{O}(\varepsilon^{4/5})$ and it suffices to show the bounded invertibility of $P^\perp(\mathscr{A}_{1,0} - \mathbb{1})P^\perp$, where $P^\perp = \mathbb{1} - P_{H_4}$ on $L_{2,\sigma}$. Now $\mathscr{A}_{1,0}$ is compact on $L_{2,\sigma}$, by (4.4) and hence 1 is at most an isolated eigenvalue of $P^\perp \mathscr{A}_{1,0} P^\perp$. Suppose ψ is in the nullspace of $P^\perp(\mathscr{A}_{1,0} - \mathbb{1})P^\perp$. Then there is a λ such that $(\mathscr{A}_{1,0} - \mathbb{1})P^\perp\psi = \lambda H_4$. Since $L_{2,\sigma} \subset L_{2,\gamma}$ this equality holds on $L_{2,\gamma}$, so that (by the selfadjointness of $\mathscr{A}_{1,0}$ on $L_{2,\gamma}$), $\lambda = 0$ and $P^\perp\psi = \lambda' H_{4,0}$. Going back to $L_{2,\sigma}$, we see that $\lambda' = 0$. Hence $P^\perp(\mathscr{A}_{1,0} - \mathbb{1})P^\perp$ is invertible on $P^\perp L_{2,\sigma}$ and has a bounded inverse (for fixed σ). This proves Lemma 4.3.

5. Proof of Theorem 2.1

We proceed in several steps. We first show in Lemma 5.1 that φ_ε is C^N for $\varepsilon > 0$ sufficiently small (depending on N). Then we show in Lemma 5.2 that $\partial_z^k \varphi_\varepsilon(z)$ is bounded as $\varepsilon \to 0$. In Theorem 5.4 we establish an asymptotic expansion in ε for $\varphi_\varepsilon(z)$ from which we finally deduce the differentiability of φ_ε.

Lemma 5.1. *For all $N \geqq 0$, $\sigma > 0$ there is an $\varepsilon_2 = \varepsilon_2(N, \sigma) > 0$ such that for $0 < \varepsilon < \varepsilon_2$ the function $\varphi_\varepsilon(z)$ is C^N in ε and z as an element of $M_{2,\sigma,\varepsilon_2}$* (cf. Theorem 2.1).

Proof. As in the proof of Theorem 3.2, we work with a sequence of $\varepsilon_2(N, \sigma)$ satisfying $\varepsilon_2(N-1, \sigma/27) > \varepsilon_2(N, \sigma)$ and $\varepsilon_2(N, \sigma)$ is such that $27\sigma/2 > 10/(3d_{\varepsilon_2})$. We shall show recursively the following properties.

P'_N: For $k = 0, 1, 2, \ldots$, $\partial_z^k \partial_\varepsilon^N \varphi_\varepsilon$ is in $L_{2,\sigma}$ for $0 < \varepsilon < \varepsilon_2(N+k, \sigma)$ and it is continuous in ε.

P_N: $(\partial_\varepsilon^N \varphi_\varepsilon)(z)$

$$= \pi^{-1/2} \sum_{j=0}^{N} \int e^{-u^2} \binom{N}{j} \partial_\varepsilon^j(\varphi_\varepsilon(zc_\varepsilon^{-1/2}+u)) \partial_\varepsilon^{N-j}(\varphi_\varepsilon(zc_\varepsilon^{-1/2}-u))du\,.$$

Note that P_0 is a trivial consequence of Theorem 3.2 and Lemma 3.1. Also P'_0 expresses the fact that $\varphi_\varepsilon \in L_{2,\sigma/3}$ solves $\mathcal{N}_\varepsilon(\varphi_\varepsilon) = \varphi_\varepsilon$. Suppose now that P_j, P'_j hold for $j \leqq N$. In particular, we have on $L_{2,\sigma}$,

$$\begin{aligned}(\partial_\varepsilon^N \varphi_\varepsilon)(z) &= \pi^{-1/2} \sum_{j=1}^{N-1} \int e^{-u^2} \binom{N}{j} \partial_\varepsilon^j(\varphi_\varepsilon(zc_\varepsilon^{-1/2}+u)) \\ &\quad \cdot \partial_\varepsilon^{N-j}(\varphi_\varepsilon(zc_\varepsilon^{-1/2}-u))du \\ &\quad + 2\pi^{-1/2} \int e^{-u^2} \partial_\varepsilon^N(\varphi_\varepsilon(zc_\varepsilon^{-1/2}+u))\varphi_\varepsilon(zc_\varepsilon^{-1/2}-u)du \\ &= g_\varepsilon^{(N)}(z) + 2\pi^{-1/2} \int e^{-u^2} \varphi_\varepsilon(zc_\varepsilon^{-1/2}+u) \\ &\quad \cdot (\partial_\varepsilon^N \varphi_\varepsilon)(zc_\varepsilon^{-1/2}-u)du\,. \end{aligned} \tag{5.1}$$

By the chain rule, we find

$$\begin{aligned} g_\varepsilon^{(N)}(z) &= \pi^{-1/2} \sum_{j=1}^{N-1} \int e^{-u^2} \binom{N}{j} \partial_\varepsilon^j(\varphi_\varepsilon(zc_\varepsilon^{-1/2}+u)) \partial_\varepsilon^{N-j}(\varphi_\varepsilon(zc_\varepsilon^{-1/2}-u))du \\ &\quad + 2\pi^{-1/2} \sum_{\substack{j+\sum l n_l = N \\ j<N}} \frac{N!}{j! \prod_l n_l! \, l!^{n_l}} \prod_l \{z \partial_\varepsilon^l c_\varepsilon^{-1/2}\}^{n_l} \\ &\quad \cdot \int e^{-u^2} (\partial_\varepsilon^j \partial_z^{\sum n_l} \varphi_\varepsilon)(zc_\varepsilon^{-1/2}+u)\varphi_\varepsilon(zc_\varepsilon^{-1/2}-u)du\,. \end{aligned} \tag{5.2}$$

These expressions are well defined on $L_{2,\sigma}$ by the inductive assumption P_N. We can now form for $0 < \varepsilon, \varepsilon' < \varepsilon_2(N+1, \sigma)$ on $L_{2,\sigma}$,

$$\begin{aligned} (\partial_\varepsilon^N \varphi_\varepsilon)(z) - (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(z) &= g_\varepsilon^{(N)}(z) - g_{\varepsilon'}^{(N)}(z) \\ &\quad + 2\pi^{-1/2} \int e^{-u^2} \varphi_\varepsilon(zc_\varepsilon^{-1/2}+u) [(\partial_\varepsilon^N \varphi_\varepsilon)(zc_\varepsilon^{-1/2}-u) - (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(zc_\varepsilon^{-1/2}-u)]du \\ &\quad + 2\pi^{-1/2} \int e^{-u^2} \varphi_\varepsilon(zc_\varepsilon^{-1/2}+u) [(\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(zc_\varepsilon^{-1/2}-u) - (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(zc_{\varepsilon'}^{-1/2}-u)]du \\ &\quad + 2\pi^{-1/2} \int e^{-u^2} [\varphi_\varepsilon(zc_\varepsilon^{-1/2}+u) - \varphi_{\varepsilon'}(zc_{\varepsilon'}^{-1/2}+u)] (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(zc_{\varepsilon'}^{-1/2}-u)du \\ &= g_\varepsilon^{(N)}(z) - g_{\varepsilon'}^{(N)}(z) + \sum_{k=1}^{3} g_k^{(N)}(\varepsilon, \varepsilon', z)\,. \end{aligned}$$

Solving for $(\partial_\varepsilon^N \varphi_\varepsilon)(z) - (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(z)$, we get

$$(\mathbb{1} - \mathscr{A}_{\varphi_\varepsilon,\varepsilon})(\partial_\varepsilon^N \varphi_\varepsilon - \partial_{\varepsilon'}^N \varphi_{\varepsilon'}) = g_\varepsilon^{(N)} - g_{\varepsilon'}^{(N)} + \sum_{k=2,3} g_k^{(N)}(\varepsilon, \varepsilon', \cdot)\,, \tag{5.3}$$

and this is well defined on $L_{2,\sigma}$. Since $\partial_\varepsilon^N \varphi_\varepsilon$ and $\partial_{\varepsilon'}^N \varphi_{\varepsilon'}$ are continuously differentiable as functions of z, by P'_N, we may rewrite $g_2^{(N)} + g_3^{(N)}$ as "derivatives plus remainder", i.e.

$$\begin{aligned} g_2^{(N)}(\varepsilon, \varepsilon', z) = (\varepsilon - \varepsilon') \cdot 2\pi^{-1/2} z \partial_\varepsilon c_\varepsilon^{-1/2} \int e^{-u^2} \varphi_\varepsilon(z c_{\varepsilon'}^{-1/2} + u) \\ (\partial_z \partial_\varepsilon^N \varphi_\varepsilon)(z c_\varepsilon^{-1/2} - u) du + \mathcal{O}((\varepsilon - \varepsilon')^2)\,, \end{aligned} \tag{5.4}$$

on $L_{2,\sigma}$ provided $0 < \varepsilon, \varepsilon' < \varepsilon_2(N+1, \sigma) \leqq \varepsilon_2(N, \sigma/27)$.

We similarly have

$$\begin{aligned} g_3^{(N)}(\varepsilon, \varepsilon', z) &= (\varepsilon - \varepsilon') \cdot 2\pi^{-1/2} \int e^{-u^2} (\partial_\varepsilon \varphi_\varepsilon)(z c_\varepsilon^{-1/2} + u) \\ &\quad \cdot (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(z c_{\varepsilon'}^{-1/2} - u) du \\ &\quad + (\varepsilon - \varepsilon') \cdot 2\pi^{-1/2} z \partial_\varepsilon c_\varepsilon^{-1/2} \int e^{-u^2} (\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} + u) \\ &\quad \cdot (\partial_{\varepsilon'}^N \varphi_{\varepsilon'})(z c_{\varepsilon'}^{-1/2} - u) du + \mathcal{O}((\varepsilon - \varepsilon')^2)\,, \end{aligned} \tag{5.5}$$

under the same conditions as before, if $N \geqq 1$. For $N = 0$, Equation (5.3) is replaced by

$$\begin{aligned} &[(\mathbb{1} - \tfrac{1}{2}\mathscr{A}_{\varphi_\varepsilon,\varepsilon} - \tfrac{1}{2}\mathscr{A}_{\varphi_{\varepsilon'},\varepsilon})(\varphi_\varepsilon - \varphi_{\varepsilon'})](z) \\ &= \pi^{-1/2} \int e^{-u^2} \{\varphi_{\varepsilon'}(z c_\varepsilon^{-1/2} + u) \varphi_{\varepsilon'}(z c_\varepsilon^{-1/2} - u) - \varphi_{\varepsilon'}(z c_{\varepsilon'}^{-1/2} + u) \varphi_{\varepsilon'}(z c_{\varepsilon'}^{-1/2} - u)\} du \\ &= 2\pi^{-1/2}(\varepsilon - \varepsilon') \int e^{-u^2} \{(\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} + u) \varphi_\varepsilon(z c_\varepsilon^{-1/2} - u) z \partial_\varepsilon c_\varepsilon^{-1/2}\} du \\ &\quad + \mathcal{O}((\varepsilon - \varepsilon')^2)\,. \end{aligned} \tag{5.3'}$$

Henceforth we only discuss Equation (5.3), the case (5.3′) is analogous. By Theorem 4.2, $\mathscr{A}_{\varphi_\varepsilon,\varepsilon} - \mathbb{1} = \mathscr{A}_\varepsilon - \mathbb{1}$ is invertible on $L_{2,\sigma}$ for sufficiently small $\varepsilon > 0$, and its inverse is bounded in norm by $\mathcal{O}(\varepsilon^{-1})$, and continuous in $\varepsilon > 0$. Therefore

$$\begin{aligned} \partial_\varepsilon(\partial_\varepsilon^N \varphi_\varepsilon) &= \lim_{\varepsilon' \to \varepsilon} (\varepsilon - \varepsilon')^{-1} (\partial_\varepsilon^N \varphi_\varepsilon - \partial_{\varepsilon'}^N \varphi_{\varepsilon'}) \\ &= (\mathbb{1} - \mathscr{A}_\varepsilon)^{-1} \left(\partial_\varepsilon g_\varepsilon^{(N)} + \sum_{k=2,3} \partial_{\varepsilon'} g_2(\varepsilon, \varepsilon', \cdot)|_{\varepsilon' = \varepsilon} \right) \end{aligned} \tag{5.6}$$

is continuous in $\varepsilon > 0$, on $L_{2,\sigma}$.

Multiplying (5.6) by $\mathbb{1} - \mathscr{A}_\varepsilon$ on both sides, one gets the relation P'_{N+1}. Next we show that $\partial_\varepsilon^{N+1} \varphi_\varepsilon$ is differentiable in z. Using the relation P'_{N+1} and the inductive assumption P_N, it is clear from Equation (5.1) that it suffices to show the differentiability of

$$\int e^{-u^2} (\partial_\varepsilon^{N+1} \varphi_\varepsilon)(z c_\varepsilon^{-1/2} + u) \varphi_\varepsilon(z c_\varepsilon^{-1/2} - u) du\,.$$

But this equals

$$\int e^{-(u - z c_\varepsilon^{-1/2})^2} \partial_\varepsilon^{N+1} \varphi_\varepsilon(u) \varphi_\varepsilon(2 z c_\varepsilon^{-1/2} - u) du\,, \tag{5.7}$$

and the assertion follows now by Theorem 3.2 and by the bound P'_{N+1} on $\partial_\varepsilon^{N+1} \varphi_\varepsilon$.

We now work towards differentiability at $\varepsilon=0$. Our first result is

Lemma 5.2. *For all $\sigma>0$ and $K\in\mathbb{Z}^+$, the function $\partial_z^K\varphi_\varepsilon$ is uniformly bounded in $0\leqq\varepsilon\leqq\varepsilon_1(K,\sigma)$, as an element of $L_{2,\sigma}$.*

Proof. This follows immediately from Theorem 3.2 and the fact that the assertion is true for $K=0,1$, by (2.7) and Proposition 2.3.

Lemma 5.3. *For all $k\geqq 2$ and $\varepsilon\geqq 0$ there is a (unique) polynomial $P_k(\varepsilon,x)$ of degree less than 2^k [3] in x and $k-1$ in ε such that*

i) $P_k(\varepsilon,x)=1-\varepsilon\vartheta H_4(x)+\text{higher orders in }\varepsilon.$ (5.8)

ii) $|\mathcal{N}_\varepsilon(P_k(\varepsilon,x))-P_k(\varepsilon,x)|\leqq C_k\varepsilon^k(|x|^{2^{k+1}}+1)\,.$ (5.9)

Remark. The lemma asserts the existence of the ε-expansion for φ_ε as a formal power series. We shall prove the statement in Appendix A.

We next show that φ_ε has an asymptotic expansion at $\varepsilon=0$ (this does not necessarily imply its differentiability).

Theorem 5.4. *For all $k\geqq 2$ and for all $\sigma>0$ there is an $\varepsilon_3(k,\sigma)$ such that for $0\leqq\varepsilon<\varepsilon_3(k,\sigma)$ one has*

$$\varphi_\varepsilon=P_{k+2}(\varepsilon,\cdot)+\mathcal{O}(\varepsilon^k) \tag{5.10}$$

in $L_{2,\sigma}$.

Proof. Setting $R_{k,\varepsilon}=\varepsilon^{-4/3}(\varphi_\varepsilon-1+\varepsilon\vartheta H_4)-\varepsilon^{-4/3}(P_{k+2}(\varepsilon,\cdot)-1+\varepsilon\vartheta H_4)$, we have by Proposition 2.3 and Lemma 5.2, that

$$\varphi_\varepsilon=P_{k+2}(\varepsilon,\cdot)+\varepsilon^{4/3}R_{k,\varepsilon}\,, \tag{5.11}$$

$R_{k,\varepsilon}=\mathcal{O}(1)$ in $L_{4,\tau}$ provided $\tau/4>10/(3d_\varepsilon)$. In addition $R_{k,\varepsilon'}\to 0$ as $\varepsilon'\to 0$ in $L_{2,\sigma}$ for $\sigma/2>10/(3d_\varepsilon)$. We now use that φ_ε is a fixed point of $\mathcal{N}_\varepsilon$. Therefore

$$\begin{aligned}&P_{k+2}(\varepsilon,\cdot)+\varepsilon^{4/3}R_{k,\varepsilon}\\&=\mathcal{N}_\varepsilon(P_{k+2}(\varepsilon,\cdot))+\varepsilon^{4/3}\mathcal{A}_{P_{k+2}(\varepsilon,\cdot),\varepsilon}(R_{k,\varepsilon})+\varepsilon^{8/3}\mathcal{N}_\varepsilon(R_{k,\varepsilon})\\&=\mathcal{N}_\varepsilon(P_{k+2}(\varepsilon,\cdot))+\varepsilon^{4/3}\mathcal{A}_{P_{k+2}(\varepsilon,\cdot)+1/2\varepsilon^{4/3}R_{k,\varepsilon},\varepsilon}(R_{k,\varepsilon})\,,\end{aligned}$$

and by rearranging we get the crucial identity

$$\begin{aligned}\varepsilon^{4/3}(\mathcal{A}_{P_{k+2}(\varepsilon,\cdot)+1/2\varepsilon^{4/3}R_{k,\varepsilon},\varepsilon}-\mathbb{1})R_{k,\varepsilon}&=P_{k+2}(\varepsilon,\cdot)-\mathcal{N}_\varepsilon(P_{k+2}(\varepsilon,\cdot))\\&=\varepsilon^{k+1}Q_k(\varepsilon,\cdot)\,,\end{aligned} \tag{5.12}$$

where Q_k is a polynomial. If we fix $\sigma>0$, $\tau>0$ sufficiently small, then (3.1), (3.2) hold for $\varepsilon\geqq 0$ sufficiently small. Therefore, by Proposition 2.3,

$$P_{k+2}(\varepsilon,\cdot)+1/2\varepsilon^{4/3}R_{k,\varepsilon}-\varphi_\varepsilon=\mathcal{O}(\varepsilon^{4/3})$$

in $L_{4,\tau}$ by (5.11).

[3] The correct bound is $4k-4$ for $k\geqq 1$

Since $\varepsilon^{4/3} < \varepsilon^{5/4}$, we may apply Theorem 4.2ii) with $g = P_{k+2}(\varepsilon, \cdot) + 1/2\varepsilon^{4/3}R_\varepsilon$, so that

$$R_{k,\varepsilon} = \varepsilon^{-4/3}(\mathscr{A}_g - \mathbb{1})^{-1}\varepsilon^{k+1}Q_k(\varepsilon, \cdot)$$

and

$$\varepsilon^{4/3}\|R_{k,\varepsilon}\|_{2,\sigma} \leqq \mathcal{O}(1)\varepsilon^{-1}\varepsilon^{k+1}\|Q_k(\varepsilon, \cdot)\|_{2,\sigma} \leqq \mathcal{O}(\varepsilon^k) .$$

The assertion follows from (5.11).

Lemma 5.5. *For all $k, n \geqq 0$, and $\sigma > 0$ one has for sufficiently small $\varepsilon \geqq 0$ the representation*

$$\partial_z^n \varphi_\varepsilon = \partial_z^n P_{k+3}(\varepsilon, \cdot) + R'_{k,\varepsilon,n} , \tag{5.13}$$

with

$$\|R'_{k,\varepsilon,n}\|_{2,\sigma} \leqq \varepsilon^k C(n, k, \sigma) .$$

Proof. The case $n=0$ is covered in Theorem 5.4. To prove the case $n=1$, we write first Equation (5.10) for $k+1$

$$\varphi_\varepsilon = P_{k+3}(\varepsilon, \cdot) + \varepsilon^k R'_\varepsilon , \tag{5.14}$$

with $\|R'_\varepsilon\|_{2,\sigma/3} = \mathcal{O}(\varepsilon)$, for sufficiently small $\varepsilon \geqq 0$. Since φ_ε and $P_{k+3}(\varepsilon, \cdot)$ are differentiable in z on $L_{2,\sigma/3}$ (for ε sufficiently small), we find

$$\partial_z \varphi_\varepsilon = \partial_z P_{k+3}(\varepsilon, \cdot) + \varepsilon^k \partial_z R'_\varepsilon . \tag{5.15}$$

Using now

$$\partial_z \varphi_\varepsilon(z) = 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} \varphi_\varepsilon(z c_\varepsilon^{-1/2} + u)(\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} - u) du , \tag{5.16}$$

we find on $L_{2,\sigma}$ the identity

$$\begin{aligned} &\partial_z P_{k+3}(\varepsilon, z) + \varepsilon^k \partial_z R'_\varepsilon(z) \\ &\quad = 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} P_{k+3}(\varepsilon, z c_\varepsilon^{-1/2} + u)(\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} - u) du \\ &\qquad + 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} \varepsilon^k R'_\varepsilon(z c_\varepsilon^{-1/2} + u)(\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} - u) du , \end{aligned}$$

which becomes upon integrating by parts

$$\begin{aligned} \varepsilon^k \partial_z R'_\varepsilon(z) = & -\partial_z P_{k+3}(\varepsilon, z) \\ & + 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} \{(\partial_z P_{k+3})(\varepsilon, z c_\varepsilon^{-1/2} + u) \\ & + 2u P_{k+3}(\varepsilon, z c_\varepsilon^{-1/2} + u)\} \varphi_\varepsilon(z c_\varepsilon^{-1/2} - u) du \\ & + 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} \varepsilon^k R'_\varepsilon(z c_\varepsilon^{-1/2} + u)(\partial_z \varphi_\varepsilon)(z c_\varepsilon^{-1/2} - u) du . \end{aligned} \tag{5.17}$$

By Theorem 5.4, Lemma 3.1, Theorem 3.2, and Equation (2.7), the second integral is $\mathcal{O}(\varepsilon^{k+1})$ in $L_{2,\sigma}$. In the first integral of (5.17), we split φ_ε according to (5.14) and the term coming from R'_ε is bounded by $\mathcal{O}(\varepsilon^{k+1})$ in $L_{2,\sigma}$. The other term is equal to

$$\begin{aligned} & 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} \{(\partial_z P_{k+3})(\varepsilon, z c_\varepsilon^{-1/2} + u) + 2u P_{k+3}(\varepsilon, z c_\varepsilon^{-1/2} + u)\} \\ & \quad \cdot P_{k+3}(\varepsilon, z c_\varepsilon^{-1/2} - u) du \\ & = 2\pi^{-1/2} c_\varepsilon^{-1/2} \int e^{-u^2} P_{k+3}(\varepsilon, z c_\varepsilon^{-1/2} + u)(\partial_z P_{k+3})(\varepsilon, z c_\varepsilon^{-1/2} - u) du . \end{aligned} \tag{5.18}$$

By perturbation theory, (5.18) is equal to $\partial_z P_{k+3}(\varepsilon, z)$ up to a polynomial which is of order $k+3$ in ε. Going back to (5.17) we see that $\|\partial_z R'_\varepsilon\|_{2,\sigma} = \mathcal{O}(\varepsilon)$ so that it is in particular uniformly bounded in $\varepsilon \geqq 0$, and well defined for $\varepsilon = 0$. This proves (5.13) for $n=1$. The cases $n \geqq 2$ follows by induction as in Theorem 3.2.

End of Proof of Theorem 2.1. We show that for all $k, N, l, k \geqq N+3$, and $\sigma > 0$ one has for $\varepsilon \geqq 0$ sufficiently small, depending on k, N, l, σ, the representation

$$\partial_z^l \partial_\varepsilon^N \varphi_\varepsilon = \partial_z^l \partial_\varepsilon^N P_k(\varepsilon, \cdot) + \mathcal{O}(\varepsilon^{k-N-3}) , \tag{5.19}$$

on $L_{2,\sigma}$. This obviously implies Theorem 2.1. We prove (5.19) by induction on N. For $N=0$ it is the content of Lemma 5.5. Suppose now (5.19) is true for $N \leqq n$ on $L_{2,\sigma/3}$. By the property P'_{n+1} of Lemma 5.1, we have for $\varepsilon \geqq 0$ on $L_{2,\sigma/3}$ the identity

$$\begin{aligned}(\mathbb{1} - \mathscr{A}_{\varphi_\varepsilon,\varepsilon})&(\partial_\varepsilon^{n+1}\varphi_\varepsilon)(z)\\ &= \pi^{-1/2} \sum_{j=1}^{n} \int e^{-u^2} \binom{n+1}{j} (\partial_\varepsilon^j \varphi_\varepsilon)(zc_\varepsilon^{-1/2} + u)(\partial_\varepsilon^{n+1-j}\varphi_\varepsilon)(zc_\varepsilon^{-1/2} - u) du ,\end{aligned} \tag{5.20}$$

so that by the induction hypothesis and the definition of P_k we have on $L_{2,\sigma/3}$

$$(\mathbb{1} - \mathscr{A}_\varepsilon)(\partial_\varepsilon^{n+1}\varphi_\varepsilon) = (\mathbb{1} - \mathscr{A}_\varepsilon)\partial_\varepsilon^{n+1} P_k(\varepsilon, \cdot) + \mathcal{O}(\varepsilon^{k-n-3}) . \tag{5.21}$$

Applying Theorem 4.2, we get (5.19) for $N = n+1$ and $l=0$, and therefore the derivatives with respect to ε of φ_ε are bounded and can be extended to $\varepsilon = 0$. By the induction hypothesis, the terms on the r.h.s. of (5.20) are l times differentiable in z so that $(\mathbb{1} - \mathscr{A}_\varepsilon)\partial_z^l(\partial_\varepsilon^{n+1}\varphi_\varepsilon)(z)$ can be defined as a suitable sum of $\int e^{-u^2}$ times derivatives of the form

$$\partial_z^{l'} \partial_\varepsilon^j \varphi_\varepsilon , \quad j = 0, \dots n ; \quad l' = 0, \dots l ;$$

$$\partial_\varepsilon^p c_\varepsilon^{-1/2}, u$$

[apply the Eqs. (5.1), (5.2) to $\partial_\varepsilon^{n+1}\varphi_\varepsilon$, solve for $(\partial_\varepsilon^{n+1}\varphi_\varepsilon)$, differentiate both sides l times with respect to z]. We can now use the induction hypothesis (5.19) for $l' \leqq l$, $N \leqq n$ on $L_{2,\sigma/3}$ and then (5.19) follows for $l' \leqq l$, $N \leqq n+1$ on $L_{2,\sigma}$, since no further powers of ε are lost by differentiating with respect to z. This completes the induction proof of (5.19) and hence the proof of Theorem 2.1.

6. The Normal Form around a Fixed Point

So far, we have only regarded the equation $\mathscr{N}_\varepsilon(\varphi) = \varphi$. We shall now discuss the "flow" around the fixed point φ_ε, $\varepsilon > 0$. In order to be able to talk about a flow, one either needs a differential equation or at least a diffeomorphism. This means that one has to abandon some of the generality of the equation $\mathscr{N}_\varepsilon(\varphi) = \varphi$, and one must introduce an iteration scheme. As has been pointed out by Jona-Lasinio in his careful analysis [13], there is some arbitrariness as to the formulation of such an iteration scheme. We may take, e.g.

$$\varphi_{n+1} = \mathscr{N}_\varepsilon(\varphi_n) \tag{6.1}$$

or

$$\varphi_{n+1} = \mathscr{N}_\varepsilon \circ \dots \circ \mathscr{N}_\varepsilon(\varphi_n) \tag{6.2}$$

or

$$\varphi_{n+1} = 2\mathcal{N}_\varepsilon(\varphi_n) - \varphi_n\,. \tag{6.3}$$

Jona-Lasinio shows that in the free case (infinite temperature), the first choice is somewhat more natural from an intuitive point of view and we shall therefore discuss the "flow" associated with (6.1). The normal form of diffeomorphisms is discussed in the mathematics literature (especially on $\mathbb{R}^n$), and we shall use such considerations for the case at hand. The main point of the ensueing analysis is that we linearize only the unstable manifold by the Sternberg analysis [17][4], while the stable manifold is not linearized, but handled by contractions, using an idea of Mather [12].

We work on $L_\infty(\mathbb{R},dz)$, and we start by controlling the spectrum of the tangent map to $\mathcal{N}_\varepsilon(\varphi)$ at $\varphi = \varphi_\varepsilon$.

Lemma 6.1. *For sufficiently small $\varepsilon > 0$, the map $\mathcal{A}_\varepsilon = \mathcal{A}_{\varphi_\varepsilon,\varepsilon}$ is compact on L_∞.*

Proof. We show that the unit ball of L_∞ is mapped by $\mathcal{A}_\varepsilon$ onto a set of equicontinuous functions which tend uniformly to zero as $|z| \to \infty$. Compactness follows then by the theorem of Arzela-Ascoli. Let $f \in L_\infty$, $\|f\|_\infty \leqq 1$. Then

$$\begin{aligned} |\mathcal{A}_\varepsilon(f)(z)| &= 2\pi^{-1/2} \left| \int e^{-u^2} f(zc_\varepsilon^{-1/2} - u)\varphi_\varepsilon(zc_\varepsilon^{-1/2} + u)du \right| \\ &\leqq 2\pi^{-1/2} \int e^{-\mathcal{O}(\varepsilon)(zc_\varepsilon^{-1/2}+u)^4} e^{-u^2} du\,, \end{aligned} \tag{6.4}$$

by Bleher-Sinai [4, Theorem 1]. The right hand side of (6.4) tends uniformly to zero as $|z| \to \infty$, so that the same is true for the left hand side. To prove continuity, we consider

$$\begin{aligned} &\mathcal{A}_\varepsilon(f)(z) - \mathcal{A}_\varepsilon(f)(z') \\ &= 2\pi^{-1/2} \int \{ e^{-(zc_\varepsilon^{-1/2}+u)^2} \varphi_\varepsilon(2zc_\varepsilon^{-1/2} + u) \\ &\quad - e^{-(z'c_\varepsilon^{-1/2}+u)^2} \varphi_\varepsilon(2z'c_\varepsilon^{-1/2} + u)\} f(u)du\,. \end{aligned} \tag{6.5}$$

The function $\exp(-(zc_\varepsilon^{-1/2} + u)^2)$ is continuous in z, and so is φ_ε (by Theorem 2.2). Since both functions tend to zero at infinity, and since they are in $L_1(dz)$, the integral is equicontinuous. This proves Lemma 6.1.

Lemma 6.2. *The operator $\mathcal{A}_\varepsilon$ has discrete spectrum on L_∞^{even} for $\varepsilon > 0$. If $\lambda_j(\varepsilon)$ is the sequence of eigenvalues in the order of decreasing absolute value, then the following is true:*

For $M, N \geqq 0$ there is an $\varepsilon(M,N) > 0$ such that for $0 \leqq \varepsilon \leqq \varepsilon(M,N)$, the eigenvalues $\lambda_j(\varepsilon)$ are C^M in ε for $j = 0, 1, \ldots, N$[5], and they are simple. For $j > N$ the eigenvalues $\lambda_j(\varepsilon)$ satisfy $|\lambda_j(\varepsilon)| < 2/c_\varepsilon^{N-1}$.

Proof. The discreteness of the spectrum follows from Lemma 6.1 and the Riesz-Schauder theorem. Since $L_\infty(dz)$ is dense in $L_{4/3,\sigma}$, it suffices to show the assertions on $L_{4/3,\sigma}$. We claim that $\mathcal{A}_{g,\varepsilon}$ is an analytic compact-operator valued function of $g \in L_{2,\tau}$ on the space $L_{4/3,\sigma}$ provided $c_\varepsilon \geqq 6/5$, $\sigma \leqq 1/6$, $\tau \leqq 3\sigma/40$. This follows at once

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[4] M. Droz has pointed out to us that a similar analysis has been done by Wegner [19]

[5] Here, we define $\lambda_j(0) = 2^{1-j/2}$

from (3.7) and the linearity of $\mathscr{A}_{g,\varepsilon}$ in g. The isolated eigenvalues of $\mathscr{A}_{g,\varepsilon}$ are thus C^N functions of g and of ε since $\mathscr{A}_{g,\varepsilon}$ depends in a C^N fashion on ε. The first part of Lemma 6.2 follows thus from Theorem 2.1i), while the last assertion of Lemma 6.2 follows from [14, IV. §3], and the fact that it is true for $\varepsilon=0$.

Our next problem is to make sure that the part of the spectrum of $\mathscr{A}_\varepsilon=\mathscr{A}_{\varphi_\varepsilon,\varepsilon}$ corresponding to the eigenvalues of modulus <1 is a contraction. This is shown in the following theorem of Mather [12].

Lemma 6.3. *Let A be a linear continuous map from a Banach space E onto itself, with a finite number of eigenvalues of modulus above 1 and the remainder of the spectrum in a circle of radius less than 1 around the origin. Then there is a spectral decomposition $E=E_1\oplus E_2$, invariant under A, and there are norms on E_1, E_2 equivalent to the original ones such that $A|_{E_2}$ is a contraction and $(A|_{E_1})^{-1}$ is a contraction.*

(The norm is defined as follows: Let ϱ be the radius of the circle, $\varrho'=(\varrho+1)/2$, $A_2=A|_{E_2}$. One has for sufficiently large n the inequality $\|A_2^n\|^{1/n}<\varrho'$. Therefore $\|A_2^n\|<C\cdot(\varrho')^n$ for some C. Let p be such that $C\cdot(\varrho')^p<1$. Then the new norm is defined by

$$|||f|||=\sum_{q=0}^{p}\|A_2^q f\|$$

for $f\in E_2$.)

We shall use $|\cdot|_i$ as the symbol for the new norm $|||\cdot|||$ defined through Lemma 6.3 on E_i, $i=1,2$. We now discuss the non-linear map

$$T_\varepsilon(\psi)=\mathscr{N}_\varepsilon(\varphi_\varepsilon+\psi)-\mathscr{N}_\varepsilon(\varphi_\varepsilon)=\mathscr{A}_\varepsilon(\psi)+\mathscr{N}_\varepsilon(\psi), \tag{6.6}$$

and we introduce some notation. Let E_1, E_2 be the two subspaces corresponding to the decomposition of $L_\infty(dz)$ according to the operator $DT_\varepsilon(0,\psi)=\mathscr{A}_\varepsilon(\psi)$ in the sense of Lemma 6.3. Let $A_i=\mathscr{A}_\varepsilon|_{E_i}$, N_{ij} = projection onto E_i parallel to E_j of $\mathscr{N}_\varepsilon|_{E_j}$, i,j = 1, 2. By the quadratic nature of $\mathscr{N}_\varepsilon$ we have obviously (for $i,j=1,2$),

$$|N_{ij}(\psi)|_i\leqq\mu\cdot|\psi|_j^2\,. \tag{6.7}$$

Such an inequality is not true on $L_{s,\sigma}$-spaces and this has dictated our choice of L_∞-spaces.

We shall now trade in continuity for boundedness. For this we introduce the notion of Lipschitz continuity; $\Phi:\mathscr{B}\to\mathscr{B}'$ ($\mathscr{B},\mathscr{B}'$ are Banach spaces) is called ν-Lipschitz if $\Phi(0)=0$ and $\|\Phi(\psi)\|'\leqq\text{const.}\,\|\psi\|^\nu$. The ν-Lipschitz maps on the ball $\mathscr{B}(a)=\{\|\psi\|\leqq a\}\subset\mathscr{B}$, form a Banach space $\mathscr{L}_{(\nu)}(\mathscr{B}(a)\to\mathscr{B}')$ with norm $|\Phi|_{(\nu)}$

$$=\sup_{\|\psi\|\leqq a}\|\Phi(\psi)\|'/\|\psi\|^\nu \text{ if } \nu>0\text{, as is easily verified.}$$

Lemma 6.4. *If Φ is ν-Lipschitz on $\mathscr{B}(b)$ and of ν-Lipschitz norm N_Φ then it is ν'-Lipschitz for $0<\nu'\leqq\nu$ on $\mathscr{B}(b)$ and its ν'-Lipschitz norm is bounded by $b^{\nu-\nu'}N_\Phi$. The composition $\Phi\Phi'$ of a ν-Lipschitz map Φ and a ν'-Lipschitz map ν' from $\mathscr{B}(b)$ to $\mathscr{B}'(b')$ and from $\mathscr{B}'(b')$ to $\mathscr{B}''(b'')$ is $\nu\cdot\nu'$-Lipschitz from $\mathscr{B}(b)$ to $\mathscr{B}''(b'')$.*

The proof is a trivial consequence of the definition. We now diagonalize T_ε up to a 3/2-Lipschitz part $\mathscr{L}$.

Theorem 6.5. *There is a C^∞ diffeomorphism S defined on $\mathscr{B}(a) \subset L_\infty$ for $a > 0$ sufficiently small and a C^∞ map $\mathscr{L}$ from $\mathscr{B}(a)$ to L_∞ such that*

(i) $\mathscr{L}E_1 \subset E_1$, $\mathscr{L}E_2 \subset E_2$,

(ii) $\mathscr{L}|_{E_j}$ *is 3/2-Lipschitz for* $j = 1, 2$,

(iii) $(\mathscr{A}_\varepsilon + \mathscr{L})S = ST_\varepsilon (= S(\mathscr{A}_\varepsilon + \mathscr{N}_\varepsilon))$. (6.8)

Proof. Write $L_\infty = E = E_1 \oplus E_2$ according to Lemma 6.3 applied to $\mathscr{A}_\varepsilon$. We look for an operator S of the form

$$S = \begin{pmatrix} \mathbb{1} & S_{12} \\ S_{21} & \mathbb{1} \end{pmatrix}$$

on $E_1 \oplus E_2$, with S_{ij} 3/2-Lipschitz, while $\mathscr{L}$ will be of the form

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_1 & 0 \\ 0 & \mathscr{L}_2 \end{pmatrix}.$$

Then the Equation (6.8) follows from the relations

$$A_1 + \mathscr{L}_1 = A_1 + N_{11} + S_{12}N_{21}, \tag{6.9}$$

$$(A_1 + \mathscr{L}_1)S_{12} = N_{12} + S_{12}(A_2 + N_{22}), \tag{6.10}$$

$$(A_2 + \mathscr{L}_2)S_{21} = N_{21} + S_{21}(A_1 + N_{11}), \tag{6.11}$$

$$A_2 + \mathscr{L}_2 = A_2 + N_{22} + S_{21}N_{12}. \tag{6.12}$$

Given S_{12}, Equation (6.9) determines $\mathscr{L}_1$, and substituting into (6.10), one finds

$$A_1 S_{12} = N_{12} + S_{12}(A_2 + N_{22}) - N_{11}S_{12} - S_{12}N_{21}S_{12}, \tag{6.13}$$

or

$$\begin{aligned} S_{12} = {} & A_1^{-1}N_{12} + A_1^{-1}S_{12}(A_2 + N_{22}) \\ & - A_1^{-1}N_{11}S_{12} - A_1^{-1}S_{12}N_{21}S_{12}. \end{aligned} \tag{6.14}$$

A similar equation follows from (6.11) and (6.12). We show now the existence and smoothness of a unique solution S_{12} of (6.14). By Lemma 6.4 and Equation (6.7), N_{ij} is 3/2-Lipschitz on $\mathscr{B}_j(a) \subset E_j$ and of 3/2-Lipschitz norm $\mu a^{1/2}$. Consider now the r.h.s. of (6.14) as a map $S_{12} \to F(S_{12})$ on the Banach space $\mathscr{L}_{(3/2)}(\mathscr{B}_2(a) \to E_1)$. If $|R|_{(3/2)} = \alpha < 1$, and since $|A_2|_2 \leqq \beta < 1$ and $|A_1^{-1}|_1 \leqq \beta < 1$ by Lemma 6.3, one has by Lemma 6.4 and Equation (6.7) the inequality

$$\begin{aligned} |F(R)|_{(3/2)} & \leqq \beta\alpha(\beta + \mu a)^{3/2} + \beta\mu a^{1/2} \\ & \quad + \beta\mu\alpha^2 a^{3/2} + \beta\alpha\mu^{3/2}\alpha^3 a^3 \\ & \leqq \alpha, \end{aligned} \tag{6.15}$$

provided $a = a(\alpha, \beta, \mu)$ is sufficiently small. In the same way, one shows

$$|F(R) - F(R')|_{(3/2)} \leqq \tfrac{1}{2}(1 + \beta^{5/2})|R - R'|_{(3/2)},$$

provided $|R|_{(3/2)}$, $|R-R|_{(3/2)}$, α, a, are sufficiently small. Hence F is a contraction on a ball $\mathscr{B}_\alpha$ of the Banach space $\mathscr{L}_{(3/2)}(\mathscr{B}_2(a)\to E_1)$ and posses a unique fixed point $F(S_{12})=S_{12}$.

We shall now bound the derivatives of S_{12}. If we view $N=N_{ij}$, which is quadratic, as a bilinear function, then we see by iteration that S_{12} can be written as $S_{12}=\sum_{n=1}^{\infty} r_n$, where r_n is 2^n-linear. We shall show inductively that

$$|r_n(x_1,\ldots x_{2^n})|_1<\gamma_n\prod_j |x_j|_2\,, \tag{6.16}$$

with $\gamma_n\leqq K^{2^n-3/2}$ for some $K>\mu$ sufficiently large. It then follows that for $a<(2K)^{-1}$ sufficiently small, one has for $\psi\in\mathscr{B}_2(a)$ and all p,

$$\begin{aligned}&|D^pS_{12}(\psi,\delta\psi_1,\ldots\delta\psi_p)|_1\\ &\quad\leqq\prod_j|\delta\psi_j|_2\sum_{n=1}^{\infty}K^{2^n-3/2}a^{2^n-p}\binom{2^n}{p}p'\,,\end{aligned} \tag{6.17}$$

which shows that S_{12} is C^∞ on $\mathscr{B}_2(a)$.

To show (6.16), we first observe that it is true for some γ_1 for $n=1$, because

$$r_1=\sum_{k=0}^{\infty}A_1^{-k-1}N_{12}A_2^k.$$

We proceed by introduction. Suppose the result is true up to $n-1$, we show it for n. By (6.14), for $n\geqq 2$.

$$r_n-A_1^{-1}r_nA_2=A_1^{-1}r_{n-1}N_{22}-A_1^{-1}N_{12}r_{n-1}-\sum_{\substack{p+q=n-1\\ p,q\geqq 1}}A_1^{-1}r_pN_{21}r_q=s_n\,,$$

and the norm of s_n is bounded by

$$\begin{aligned}&\beta K^{2^{n-1}-3/2}\mu^{2^{n-1}}\\ &\quad+\beta\mu(K^{2^{n-1}-3/2})^2+\sum_{p+q+1=n}\beta K^{2^p-3/2}[\mu(K^{2^q-3/2})^2]^{2^p}\\ &=K^{2^n-3/2}\left\{\beta K^{-2^{n-1}}\mu^{2^{n-1}}+\beta\mu K^{-3/2}+\sum_{\substack{p+q+1=n\\ p,q\geqq 1}}\beta K^{-2^{p+1}}\mu^{2^p}\right\}\\ &\leqq(1-\beta)K^{2^n-3/2}\end{aligned}$$

provided K is sufficiently large.

Now $r\to A_1^{-1}rA_2$ is a linear map of norm β^{2^n+1} on the space of 2^n-linear operators r, and hence $r_n-A_1^{-1}r_nA_2^1=s_n$ has a solution of norm $\leqq(1-\beta)^{-1}\|s_n\|$. Thus r_n exists and the induction step is complete. This proves (6.16) and hence the differentiability of S_{12} follows from (6.17) and we have shown the existence of a C^∞ map S_{12}, satisfying (6.14) and $|S_{12}|_{(3/2)}\leqq\alpha$ and there is an identical argument for S_{21}. Going back to S, we see that $S=\mathbb{1}+\mathcal{O}(\alpha)$, so that S is a C^∞ diffeomorphism. The bound on $\mathscr{L}$ follows from Equations (6.9) and (6.12). This completes the proof of Proposition 6.5.

Let now $T'_\varepsilon = ST_\varepsilon S^{-1}$, on $\mathscr{B}(a)$. Note that by construction $DT'_\sigma(0,\psi) = \mathscr{A}_\varepsilon(\psi)$, since $DS(0,\psi)=\psi$, $DT_\varepsilon(0,\psi)=\mathscr{A}_\varepsilon(\psi)$. We now complete the diagonalization of $A_1 + \mathscr{L}_1$ (the dilating part of $\mathscr{A}_\varepsilon + \mathscr{L}$). For this we verify the so called Sternberg conditions [17] for the eigenvalues $\lambda_i(\varepsilon)$, which have absolute value >1 on $L_\infty \subset L_{2,\sigma}$[6]. By Bleher, Sinai [4, Theorem 3.1], one has

$$|\lambda_i(\varepsilon) - 2/c^i| < \varepsilon^{4/5}, \quad i=0,1. \tag{6.18}$$

Also

$$\lambda_2(\varepsilon) = 1 - \varepsilon \cdot \log 2 + \mathcal{O}(\varepsilon^{3/2}), \tag{6.19}$$

by (4.2), and

$$|\lambda_j(\varepsilon)| < 2^{-1/2}\, 8/7, \quad j=3,4,\ldots, \tag{6.20}$$

provided $\varepsilon \geqq 0$ is sufficiently small, by the continuity of $\mathscr{A}_\varepsilon$ (Section 4). Therefore the expanding subspace E_1 is of dimension 2, and the eigenvalue conditions of Sternberg are for this case the conditions

$$\begin{aligned} &\lambda_0(\varepsilon) \neq \lambda_1^m(\varepsilon) \quad \text{for} \quad m=0,1,2,\ldots, \\ &\lambda_1(\varepsilon) \neq \lambda_0^m(\varepsilon) \quad \text{for} \quad m=0,1,2,\ldots. \end{aligned} \tag{6.21}$$

In Appendix B, we show that (6.21) is true, and we do the proof also for all "relevant" eigenvalues in the m-critical case. The condition (6.21) allows us to apply the theorem of Sternberg [17, pp. 38, 46, 53], and it asserts the existence of a C^∞ diffeomorphism U of a neighborhood of the origin of E_1 such that

$$A_1 = U(A_1 + \mathscr{L}_1)U^{-1}, \quad DU(0,\delta\psi) = \delta\psi. \tag{6.22}$$

Theorem 6.6. *Normal form of T_ε. For sufficiently small $\varepsilon > 0$ there is a neighborhood $\mathscr{V}_\varepsilon$ of the origin in L_∞, a C^∞ diffeomorphism $\mathscr{U}_\varepsilon$ and a 3/2-Lipschitz contraction L_ε such that*

$$D\mathscr{U}_\varepsilon(0,\delta\psi) = \delta\psi$$

and

$$\mathscr{U}_\varepsilon(\mathscr{A}_\varepsilon + \mathscr{N}_\varepsilon)\mathscr{U}_\varepsilon^{-1} = A_1 \oplus A_2 + L_\varepsilon \mathscr{P}_\varepsilon .$$

where $\mathscr{P}_\varepsilon$ is the projection onto E_2 parallel to E_1.

Proof. On $E_1 \oplus E_2$, set $\mathscr{U}_\varepsilon(U \oplus I)S$, where U is defined by (6.22) and S is defined by Theorem 6.5. Set $L_\varepsilon = \mathscr{L}$, and the result follows.

Having found the normal form of the map $T_\varepsilon = \mathscr{A}_\varepsilon + \mathscr{N}_\varepsilon$, we now look for the normal form of the diffeomorphism defined by Equation (6.1).

Corollary 6.7. *Normal form of $\varphi \to \mathscr{N}_\varepsilon(\varphi)$ around φ_ε. For sufficiently small $\varepsilon > 0$ there are on the neighborhood $\mathscr{V}_\varepsilon$ two transverse C^∞ foliations $\mathscr{F}_u$, $\mathscr{F}_s$, invariant under T_ε such that*

(i) $\dim \mathscr{F}_u = 2$, $\operatorname{codim} \mathscr{F}_s = 2$.

[6] We work from now on on the even subspace of L_∞, but our analysis could also be extended to include the odd subspace

(ii) *The sheets of* $\mathscr{F}_u$ *(resp.* $\mathscr{F}_s$*) passing through the origin* $(=\varphi_\varepsilon)$ *are the unstable (resp. stable) manifolds* W_u *(resp.* W_s*) of* T_ε*, and they are tangent to* E_1 *(resp.* E_2*).*

(iii) $T_\varepsilon|_{\mathscr{F}_s}$ *contracts to* W_u*, and* $T_\varepsilon|_{\mathscr{F}_u}^{-1}$ *contracts to* W_s.

Proof. Since $\mathscr{U}_\varepsilon$ is a C^∞ diffeomorphism, it suffices to show the assertions for $\mathscr{A}_\varepsilon + L_\varepsilon \mathscr{P}_\varepsilon$. But this is a standard fact ([12], pp. 139, 141).

7. Critical Indices

The analysis of Section 6 has shown us that the first two eigenvalues of $\mathscr{A}_\varepsilon$ determine the unstable part of the "flow" (="relevant scaling fields"). We now specialize the situation even further by showing that there are Hierarchial Models (defined by their free spin distributions = free one point spin distribution) such that the critical indices of the thermodynamic limits of these models are related in the standard fashion to the eigenvalues of $\mathscr{A}_\varepsilon$. In particular, this shows (by Theorems 4.2, Theorem 2.1. and the fact that $\mathscr{A}_\varepsilon$ is linear in φ_ε and hence analytic) that the critical indices have standard perturbation theory as their asymptotic expansion (up to arbitrary order).

We begin by constructing the models in question. For this we shall need single spin distributions φ satisfying the following conditions:

c1) $\varphi \in W_s \cap \mathscr{V}_\varepsilon$,
c2) $\varphi > 0$,
c3) $\varphi \in C^1$,
c4) $z\partial_z\varphi(z)/(\varphi(z))^{1/2} \in L_\infty$,
c5) $\log\varphi(z) z\partial_z\varphi(z)/(\varphi(z))^{1/2} \in L_\infty$,
c6) $\|\partial_z\varphi - \partial_z\varphi_\varepsilon\|_\infty$ is small.

Lemma 7.1. *There exist functions* φ *satisfying* c1-c6.

Proof. By Bleher-Sinai [4, Theorem 8.1] there are functions $\varphi > 0$ such that $\psi_n = \mathscr{N}_\varepsilon^n(\varphi)$ tends to φ_ε, and this convergence takes place in $L_\infty \cap C^1$. Therefore one can satisfy c1, c2. The other conditions are then easily satisfied by modifying φ slightly near infinity.

This lemma allows the following definition. Let $\varepsilon > 0$ be given, let $c = c_\varepsilon$ and let φ be a function satisfying c1-c6. Choose furthermore a constant $\alpha > 0$ such that

$$\frac{4\pi}{c}\left(\frac{2-c}{\alpha c}\right) \neq \frac{1}{e}. \tag{7.1}$$

Then we define a function

$$f_\varphi(z) = f(z) = \left[\varphi\left(z\left(\frac{2-c}{\alpha c}\right)^{-1/2}\right) e^{-\frac{z^2\alpha c}{(2(2-c))}} \left[\frac{4\pi}{c}\left(\frac{2-c}{\alpha c}\right)\right]^{-1/2}\right]^{1/\alpha}, \tag{7.2}$$

which will be called an *admissible single spin distribution.*

Let

$$\mathscr{H}_{N,f}(s) = -\sum_{j=1}^{2^N} \log f(s_j) + \mathscr{H}_N, \tag{7.3}$$

where $\mathscr{H}_N = \mathscr{H}_N(s)$ is the Hierarchical Hamiltonian defined in (1.1). Thus $\log f$ plays the rôle of a single spin distribution. By (7.1), (7.2), and because $\varphi \in W_s$, the model defined by $\mathscr{H}_{N,f}$ is critical at $\beta_{\text{crit}} = \alpha$.

According to the general ideas of renormalization group theory, the anomalous dimensions (logarithms of the eigenvalues of $\mathscr{A}_\varepsilon$) describe the behaviour of physical quantities close to the fixed point φ_ε, under the transformation 1.7. The aim of this section is to show

Theorem 7.2. *The thermodynamic limit of the free energy and the scaling limit of the susceptibility behave according to the "flow" described in Section 6.*

We shall show below the existence of the thermodynamic limit for the free energy. A similar statement could be made for the two point function. The question of the finiteness of the susceptibility outside the critical temperature is quite a different matter and not known for most models. Technically, our calculations show that the *scaling-limit* can be taken for such quantities.

The above theorem connects in a rigorous fashion aspects of the "geometrical" theory of Section 6 to the theory of the thermodynamic limit in the statistical mechanics of the Hamiltonian $\mathscr{H}_{N,f}$. *From this point of view, universality means that our results hold for many different choices of f and do not depend on them.* The combination of the results (shown below) with Lemma 6.2 implies that the *ε-expansion is valid* (to arbitrary order) *for the critical indices of the models defined by the thermodynamic limit of Hierarchical Models.*

We first establish the existence of the thermodynamic limits for the free energy and the two-point function.

Theorem 7.3. *For $\varepsilon > 0$ sufficiently small and for $0 \leqq f \leqq \exp(-\mathcal{O}(\varepsilon)|x|^4)$ the limits*

$$F_{\beta,f} = \lim_{N\to\infty} \frac{1}{2^N} \log Z_{N,\beta,f}, \tag{7.4a}$$

where $Z_{N,\beta,f} = \int \prod_{j=1}^{2^N} ds_j e^{-\beta\mathscr{H}_{N,f}(s)}$,

and

$$\langle s_i^2 \rangle_{\beta,f} = \lim_{N\to\infty} Z_{N,\beta,f}^{-1} \int \prod_{j=1}^{2^N} ds_j e^{-\beta\mathscr{H}_{N,f}(s)} s_i^2, \tag{7.4b}$$

exist, and are uniformly bounded in β.

Proof. Consider the generating function

$$Z_{N,\beta,f,\mu} = \int \prod_{j=1}^{2^N} (ds_j e^{\mu s_j^2} f^\beta(s_j)) e^{-\beta\mathscr{H}_N(s)}, \tag{7.5}$$

and let $\langle\rangle_{N,\beta,f,\mu}$ be the expectation with respect to the corresponding measure. We claim

$$C_1^{2^N} \leqq Z_{N,\beta,f,\mu} \leqq C_2^{2^N}, \quad C_1 > 0. \tag{7.6}$$

Indeed, the upper bound follows from the bound $\left(\sum_1^M s_i\right)^2 \leqq M \sum_{i=1}^M s_i^2$, and using (1.1), from

$$|\mathscr{H}_N(s)| \leqq 2 \sum_{j=1}^{2^N} s_j^2 \sum_{k=1}^{\infty} c^k 2^{-k-1},$$

with $C_2 = \int e^{\mu s^2} e^{\left(1-\frac{c}{2}\right)^{-1} \beta s^2} f(s) ds$. The lower bound follows because $\exp(-\beta \mathscr{H}_N) \geqq 1$, with $C_1 = \int e^{\mu s^2} f(s) ds$.

From the first Griffiths inequality [18], one deduces that $Z_{N,\beta,f,\mu}$ is monotone in N, and by the nature of the logarithm,

$$\liminf_{N \to \infty} 2^{-N} \log Z_{N,\beta,f,\mu} = \sup_N 2^{-N} \log Z_{N,\beta,f,\mu}$$

and hence the limit $\lim_{N \to \infty} 2^{-N} \log Z_{N,\beta,f,\mu}$ exists and is bounded by $\log C_2$, using (7.6). (These arguments are fairly standard cf. e.g. [15].)

We now turn to the 2-point function. Although the Hierarchical Model is not translation invariant, the quantity $\langle s_i^2 \rangle_{N,\beta}$ does not depend on i and one can therefore define it by

$$\langle s_i^2 \rangle_{N,\beta,f} = 2^{-N} \partial_\mu \log Z_{N,\beta,f,\mu|\mu=0} . \tag{7.7}$$

A straightforward calculation shows that $F_{N,\beta,f,\mu} = 2^{-N} \log Z_{N,\beta,f,\mu}$ is a convex function of μ, and therefore Equations (7.6), (7.7) imply

$$\langle s_i^2 \rangle_{N,\beta,f} \leqq (F_{N,\beta,f,\mu} - F_{N,\beta,f,0})/\mu \leqq 2 \log C_2 . \tag{7.8}$$

This implies uniform boundedness. The existence of the thermodynamic limit for $\langle s_i^2 \rangle_{N,\beta,f}$ follows now from the second Griffiths inequality [18].

Corollary 7.4. *The function* $\langle s_i s_j \rangle_{\beta,f}$ *defined in analogy with* (7.4b) *exists and is bounded.*

Proof. From the Schwarz inequality and the bound (7.8) we have:

$$|\langle s_i s_j \rangle_{N,\beta,f}| \leqq (\langle s_i^2 \rangle_{N,\beta,f})^{1/2} (\langle s_j^2 \rangle_{N,\beta,f})^{1/2} \leqq 2 \log C_2 .$$

Moreover, $\langle s_i s_j \rangle_{N,\beta,f}$ is an increasing function of N by the second Griffiths inequality [18], hence the existence of the thermodynamic limit.

Remark. Theorem 7.3 and Corollary 7.4 imply that $F_{\beta,f}$ and $\langle s_i s_j \rangle_{\beta,f}$ are lower semi-continuous functions of the inverse temperature β.

We now discuss the precise action of the renormalization group. It follows by direct computation from the definitions (1.1), (7.2) of a model with admissible single spin distribution f, that

$$Z_{N,\beta,f} = Z_{N-1,\beta,\mathscr{N}^{(\beta)}(f^\beta)^{1/\beta}} =: Z_{N-1,\beta,\mathscr{G}_\beta(f)} , \tag{7.9}$$

where $c = c_\varepsilon$ and

$$\mathscr{N}^{(\beta)}(g)(z) = 2c^{-1/2} \int ds' ds'' g(s') g(s'') \delta(s' + s'' - 2zc^{-1/2}) e^{\beta z^2/2} . \tag{7.10}$$

Let now

$$\mathscr{T}_\beta(f)(z) = f^\beta \left(z \left(\frac{2-c}{\beta c} \right)^{1/2} \right) e^{z^2/2} \left(\left(\frac{2-c}{\beta c} \right) \frac{4\pi}{c} \right)^{1/2} . \tag{7.11}$$

By construction, $\mathscr{T}_\alpha(f_\varphi)(z) = \varphi(z)$, cf. Equation (7.2), and one verifies that

$$\mathscr{T}_\beta \mathscr{G}_\beta \mathscr{T}_\beta^{-1} = \mathscr{N} , \tag{7.12}$$

where $\mathcal{N} = \mathcal{N}_\varepsilon$ is the nonlinear operator defined in Equation (1.5) and discussed throughout the paper.

Suppose ϕ satisfies c1–c6, and α satisfies (7.1). We call the set of functions $\phi(\beta,.) = \mathcal{T}_\beta(f_\phi)$, β near α, the *temperature trajectory of ϕ with inverse critical temperature α.* Note that $\phi(\alpha, z) = \phi(z)$.

Lemma 7.5. *The curve $\phi(\beta,.)$ is differentiable in L_∞, and its derivative $\partial_\beta \phi(\beta,.)|_{\beta=\alpha} = \Phi_2$ satisfies $P_{e_0}\Phi_2 \neq 0$, where P_{e_0} it the projection onto e_0 parallel to e_1 and E_2 (cf. Theorem 6.6, e_j is the eigenvector corresponding to λ_j).*

Proof. From the definitions (7.2) and (7.11) we deduce:

$$\begin{aligned}\phi(\beta, z) &= f_\phi^\beta\left(z\left(\frac{2-c}{\beta c}\right)^{1/2}\right)e^{z^2/2}\left(\left(\frac{2-c}{\beta c}\right)\frac{4\pi}{c}\right)^{1/2}\\ &= \phi^{\beta/\alpha}\left(z\left(\frac{\alpha}{\beta}\right)^{1/2}\right)\cdot\left[\frac{4\pi}{c}\left(\frac{2-c}{\alpha c}\right)\right]^{-\beta/2\alpha}\left[\left(\frac{2-c}{\beta c}\right)\frac{4\pi}{c}\right]^{1/2}.\end{aligned}$$

This formula together with the hypotheses on the function ϕ proves the differentiability. Moreover, we have

$$\begin{aligned}\partial_\beta\phi(\alpha, z) = &-\frac{1}{2\alpha}\left\{1+\log\left[\frac{4\pi}{c}\left(\frac{2-c}{\alpha c}\right)\right]\right\}\phi(z)\\ &+\frac{1}{\alpha}\left\{\phi(z)\log\phi(z) - \frac{z}{2}\partial_z\phi(z)\right\}.\end{aligned}$$

In L_∞ we have

$$\phi = \varphi_\varepsilon + \mathcal{O}(\varepsilon), \qquad \partial_z\phi = \partial_z\varphi_\varepsilon + \mathcal{O}(\varepsilon),$$

so that in L_{2,γ_ε} the following representations are valid;

$$\phi = 1 + \mathcal{O}(\varepsilon), \qquad \partial_z\phi = \mathcal{O}(\varepsilon),$$

and we deduce

$$\partial_\beta\phi(\alpha,.) = -\frac{1}{2\alpha}\left\{1+\log\left[\frac{4\pi}{c}\left(\frac{2-c}{\alpha c}\right)\right]\right\} + \mathcal{O}(\varepsilon).$$

By the assumptions on ϕ and α, $\partial_\beta\phi(\beta, z)$ has a non-negligeable projection onto $\psi_{0,\varepsilon}$. But from standard perturbation theory [14] it then follows that the same is true for the projection onto e_0 in L_∞.

Lemma 7.5 shows that the tangent to the curve $\phi(\beta,.)$ has a component in the direction of the first eigenvector of $\mathcal{A}_\varepsilon$. In view of Theorem 6.6 and Corollary 6.7, this implies that $\mathcal{U}_\varepsilon\phi(\beta,.)$ has the same property, and that *the curve $\phi(\beta,.)$ is transversal to the stable* (critical) *manifold W_s*. Thus the coefficient g_0 of the "field" $\psi_{0,\varepsilon}$ can be used as a parameter on the curve. By the implicit function theorem β is a C^2 function of g_0, and we shall use the symbol $g_0(\beta-\alpha)$ [with $g_0(0)=0$] to indicate the dependence of g_0 on the reduced inverse temperature.

We now discuss the scaling properties of the free energy

$$F_{\beta,f} = \lim_{N\to\infty} 2^{-N}\log Z_{N,\beta,f}.$$

By (7.9), we get

$$F_{\beta,f}=1/2F_{\beta,\mathscr{G}_\beta(f)}, \tag{7.13}$$

or, going over to the space $E_1 \oplus E_2$, by the transformation $\hat{\phi}_\beta=\mathscr{U}_\varepsilon(\mathscr{T}_\beta(f_\phi)-\varphi_\varepsilon)+\varphi_\varepsilon$, we have with obvious notation,

$$\hat{F}_{\beta,\hat{\phi}_\beta}=1/2\hat{F}_{\beta,(A_1\oplus A_2+L_\varepsilon\mathscr{P}_\varepsilon)(\hat{\phi}_\beta-\varphi_\varepsilon)+\varphi_\varepsilon}, \tag{7.14}$$

cf. Theorem 6.7.

Given $\varrho>0$ sufficiently small, define β_n by $\lambda_0^n|\beta_n-\alpha|=\varrho$ and $\operatorname{sign}(\beta_n-\alpha)=\text{const}$. According to Lemma 7.5. $\hat{\phi}_\beta$ is of the form $\hat{\phi}_\beta=\varphi_\varepsilon+g_0(\beta-\alpha)e_0+g_1(\beta-\alpha)e_1+r$, with $e_0, e_1\in E_1$, $r\in E_2$ and hence

$$(A_1\oplus A_2+{}_\varepsilon\mathscr{P}_\varepsilon)^n(\hat{\phi}_{\beta_n}-\varphi_\varepsilon)=\lambda_0^n g_0(\beta_n-\alpha)e_0+\lambda_1^n g_1(\beta_n-\alpha)e_1+r_n.$$

It follows from (7.9) and the fact that $g_0'(0)\neq 0$ (Lemma 7.5) that

$$\hat{F}_{\beta_n,\hat{\phi}_{\beta_n}}=2^{-n}\hat{F}_{\beta_n,\varphi_\varepsilon+\varrho g_0'(0)e_0+\lambda_1^n g_1(\beta_n-\alpha)e_1+r_n+\mathscr{O}(\beta_n-\alpha)},$$

so that we find for the critical index $2-\alpha$,

$$\begin{aligned}2-\alpha&=\lim_{n\to\infty}\frac{\log F_{\beta_n,f_\phi}}{\log(\beta_n-\alpha)}=\lim_{n\to\infty}\frac{\log\hat{F}_{\beta_n,\hat{\phi}_{\beta_n}}}{\log(\beta_n-\alpha)}\\&=\lim_{n\to\infty}\frac{-n\log 2+\mathscr{O}(1)}{-n\log\lambda_0(\varepsilon)+\mathscr{O}(1)}=1.\end{aligned}$$

This agrees with the heuristical discussion of Gallavotti and Knops [10], and provides thus a rigorous proof of their considerations.

Now things are not much different for the correlation length but we do not know the existence of the thermodynamic limit. We shall take the point of view adopted by Gallavotti and Knops [10, Eqs. (5.17) and (5.18)] and show that γ, *defined* by

$$\sum_{1\leqq i,j\leqq 2^N}\langle s_i s_j\rangle_\beta 2^{-N}\sim\mathscr{O}((\beta-\alpha)^{-\gamma}),$$

is the "correct" critical index. So let

$$M_N=s_1+\ldots+s_{2^N}.$$

We change now slightly the definition of the non-linear map $\mathscr{N}$, eliminating the first unstable direction. Using again the definitions (1.1), (7.2), we find

$$2^{-N}\langle\!\langle M_N^2\rangle\!\rangle_{N,\beta,f}=\frac{2}{c}2^{-N+1}\langle\!\langle M_{N-1}^2\rangle\!\rangle_{N-1,\beta,\tilde{\mathscr{G}}_\beta(f)},$$

where

$$\langle\!\langle g(.)\rangle\!\rangle_{N,\beta,f}=\int\prod_{j=1}^{2^N}ds_j e^{-\beta\mathscr{H}_{N,f}(s)}g(s)\Big/\int\prod_{j=1}^{2^N}ds_j e^{-\beta\mathscr{H}_{N,f}(s)},$$

and where the "normalized" transformation is

$$\tilde{\mathscr{G}}_\beta(f)(z)=\left[\frac{\mathscr{N}^{(\beta)}(f^\beta)(z)}{\int dy\mathscr{N}^{(\beta)}(f^\beta)(y)}\right]^{1/\beta}.$$

From Equation (7.12) we deduce:

$$\mathcal{T}_\beta \tilde{\mathcal{G}}_\beta \mathcal{T}_\beta^{-1}(f) = \mathcal{N}(f)/\int dy \mathcal{N}^{(\beta)}[[\mathcal{T}_\beta^{-1}(f)]^\beta](y)$$
$$= \mathcal{N}(f)\cdot\left[\left(\frac{c}{4\pi^2}\right)^{1/2} \iint du dz e^{-\frac{z^2}{2}-u^2} f(zc^{-1/2}+u) f(zc^{-1/2}-u)\right]^{-1}$$
$$=: \tilde{\mathcal{N}}(f).$$

Let now

$$a^{-1} = \left(\frac{c}{4\pi^2}\right)^{1/2} \iint du dz e^{-\frac{z^2}{2}-u^2} \varphi_\varepsilon(zc^{1/2}+u)\varphi_\varepsilon(zc^{-1/2}-u),$$

it is easy to see that if $\tilde{\varphi}_\varepsilon = a\varphi_\varepsilon$ we have

$$\tilde{\mathcal{N}}(\tilde{\varphi}_\varepsilon) = \tilde{\varphi}_\varepsilon.$$

Moreover, if ε is sufficiently small, $\tilde{\mathcal{N}}$ is defined and continuous on a ball in L_∞, of center $\tilde{\varphi}_\varepsilon$ and radius ε.

We consider now the map $\tilde{T}_\varepsilon$ defined by

$$\tilde{T}_\varepsilon(\psi) = \tilde{\mathcal{N}}(\tilde{\varphi}_\varepsilon + \psi) - \tilde{\varphi}_\varepsilon,$$

and we find

$$\tilde{T}_\varepsilon(\psi) = \mathcal{A}_{\varphi_\varepsilon}(\psi) - 2a\varphi_\varepsilon \theta(\varphi_\varepsilon, \psi)$$
$$+ (\mathcal{N}(\psi) - a\varphi_\varepsilon\theta(\psi,\psi) - 2a\theta(\varphi_\varepsilon,\psi)\mathcal{A}_{\varphi_\varepsilon}(\psi) - \theta(\psi,\psi)\mathcal{A}_{\varphi_\varepsilon}(\psi)$$
$$+ 4a^2\varphi_\varepsilon\theta^2(\varphi_\varepsilon,\psi) + 2a\varphi_\varepsilon\theta(\psi,\psi)\theta(\varphi_\varepsilon,\psi))$$
$$\cdot (a + 2a\theta(\varphi_\varepsilon,\psi) + \theta(\psi,\psi))^{-1},$$

where

$$\theta(\psi,\psi') = \left(\frac{c}{4\pi^2}\right)^{1/2} \iint dz du e^{-\frac{z^2}{2}-u^2} \psi(zc^{-1/2}+u)\psi'(zc^{-1/2}-u).$$

The differential of this map at the origin (the new fixed point) is given by

$$D\tilde{T}_\varepsilon(0,\psi) = \mathcal{A}_{\varphi_\varepsilon}(\psi) - 2a\varphi_\varepsilon\theta(\varphi_\varepsilon,\psi).$$

From Lemma 6.1 we deduce that $D\tilde{T}_\varepsilon(0,.)$ is compact, its spectrum is given by

$$\mathrm{Sp}(D\tilde{T}_\varepsilon(0,.)) = \mathrm{Sp}(\mathcal{A}_{\varphi_\varepsilon}) \cup \{0\} \backslash \{2\}$$

and the first eigenvector is $\tilde{e}_1 = e_1 - 2\frac{a}{\lambda_1}\theta(e_0, e_1)e_0$.

It is easy to verify that the analysis of Chapter 6 can be applied to the operator $\tilde{T}_\varepsilon$ and the fixpoint $\psi = 0$. We have again a normal form of $\tilde{T}_\varepsilon$ but with only one "relevant" direction: with this modofication, the conclusions of Theorem 6.6 and Corollary 6.7 are valid, and we call $\tilde{\mathcal{U}}_\varepsilon$ the "diagonalisation map".

In order to repeat for the susceptibility $\lambda_{N,\beta,f} = 2^{-N}\langle\langle M_N^2\rangle\rangle_{N,\beta,f}$ the argument used above to calculate the critical index of the free energy, one has to know the existence of the thermodynamic limit. This is an open question in the neighbourhood of the critical temperature (see [16] for a similar problem).

We now investigate the susceptibility in the scaling limit. We have

$$\tilde{\chi}_{N,\beta,\tilde{\phi}_\beta}=\frac{2}{c}\tilde{\chi}_{N-1,\beta,\mathcal{N}(\tilde{\phi}_\beta)}, \tag{7.15}$$

where $\tilde{\phi}_\beta=\tilde{\mathcal{U}}_\varepsilon(\mathcal{T}_\beta(f_\phi)-\tilde{\varphi}_\varepsilon)$ and $\tilde{\chi}_{N,\beta,\phi_\beta}=\chi_{N,\beta,\phi}$. Let p be a fixed positive integer, given $\varrho>0$ sufficiently small, define β_n by $\lambda_1^n|\beta_n-\alpha|=\varrho$ and $\mathrm{sign}(\beta_n-\alpha)=\mathrm{const}$. As in Lemma 7.5, $\tilde{\phi}_\beta$ is of the form

$$\tilde{\phi}_\beta=\tilde{\varphi}_\varepsilon+\tilde{g}_1(\beta-\alpha)\tilde{e}_1+\tilde{r},\quad \text{and}\quad \partial_\beta\tilde{g}_1(0)\neq 0,$$

where $\tilde{r}$ is an "irrelevant vector". Now

$$\tilde{\chi}_{n+p,\beta_n,\tilde{\phi}_{\beta_n}}=\left(\frac{2}{c}\right)^n\tilde{\chi}_{p,\beta_n,\tilde{\varphi}_\varepsilon+\lambda_1^n\tilde{g}_1(\beta_n-\alpha)\tilde{e}_1+(\tilde{\mathcal{U}}_\varepsilon\tilde{\mathcal{N}}\tilde{\mathcal{U}}_\varepsilon^{-1})^n\tilde{r}},$$

and we conclude as for the free energy, that the critical index γ is given by

$$\gamma=\lim_{n\to\infty}\frac{\log\tilde{\chi}_{n+p,\beta_n,\tilde{\phi}_{\beta_n}}}{\log(\beta_n-\alpha)}=\frac{\log c/2}{\log\lambda_1}.$$

One may remark that this value does not depend on p. The value of γ thus found coincides with that of the literature [4,10]. This proves Theorem 7.2.

Using the thermodynamic limit of Corollary 7.4 one can show in exactly the same way, that in the scaling limit one has $1-\eta=\dfrac{\log c}{\log\lambda_1}$.

Appendix A

Perturbation Expansion. We collect here only some considerations and give some of the intermediate formulae without proofs. We recall first the definitions:

$$c_\varepsilon=2^{1/2(1-\varepsilon)}, \tag{A1}$$

$$\gamma_\varepsilon=1-c_\varepsilon^{-1}, \tag{A2}$$

$$H_{n,\varepsilon}=(-1)^n e^{x^2}\partial_x^n e^{-x^2}|_{x=\gamma_\varepsilon^{1/2}z}. \tag{A3}$$

The functions

$$\psi_{n,\varepsilon}(z)=H_{n,\varepsilon}(z)\cdot 2^{-n/2}n!^{-1/2} \tag{A4}$$

are orthonormalized Hermite polynomials on L_{2,γ_ε}. We shall expand below φ_ε, the solution of $\mathcal{N}_\varepsilon(\varphi_\varepsilon)=\varphi_\varepsilon$, in these polynomials. Basic to this is the following

Lemma A1. *For* $\tilde{\mathcal{N}}_\varepsilon(f,g)(z)=\pi^{-1/2}\int e^{-u^2}f(zc_\varepsilon^{-1/2}+u)g(zc_\varepsilon^{-1/2}-u)du$ *we have*

$$(\psi_{2k,\varepsilon},\tilde{\mathcal{N}}_\varepsilon(\psi_{2n,\varepsilon},\psi_{2n',\varepsilon}))_{2,\gamma_\varepsilon}$$
$$=\begin{cases}\dfrac{1}{c_\varepsilon^k}\dfrac{\left(\frac{2}{c_\varepsilon}-1\right)^{n+n'-k}}{(n+n'-k)!}\dbinom{2k}{k+|n-n'|}\left(\dfrac{2n!2n'!}{2k!}\right)^{1/2} & \text{if}\quad |n-n'|\leqq k,\ n+n'\geqq k,\\ 0, & \text{otherwise}.\end{cases} \tag{A5}$$

This is shown using the orthogonality relations between the $\psi_{n,\varepsilon}$ and integration by parts.

We next set

$$\varrho_{j,\varepsilon}(z)=\psi_{2j,\varepsilon}(z)(2j)!^{-1/2}\left(\frac{2}{c_\varepsilon}-1\right)^{-j},$$

and make the ansatz [5], [cf. (1.12), with a change of normalization],

$$\varphi_{\varepsilon(\alpha)}(z)=\sum_j a_j(\alpha)\varrho_{j,\varepsilon(\alpha)}(z), \tag{A6}$$

with $a_j(0)=\delta_{j0}$, $a_2(\alpha)=\alpha$. If we represent a function φ by its series $\{a_i\}$, φ' by $\{a'_j\}$ and $\mathcal{N}(\varphi,\varphi')$ by $\{b_j\}$, then it follows from (A5) that

$$b_k=\sum_{\substack{|n-n'|\leqq k\\ n+n'\geqq k}} a_n a'_{n'}\binom{2k}{k+|n-n'|}\frac{1}{c_\varepsilon^k(n+n'-k)!}. \tag{A7}$$

Incidentally, Equation (A7) exhibits the unboundedness of $\mathcal{N}$ (or $\tilde{\mathcal{N}}$) cf. the terms coming from $n=n'$, $n+n'=k$.

Setting

$$a_j(\alpha)=\sum_{n\geqq 0} a_{jn}\alpha^n, \qquad \varepsilon(\alpha)=\sum_{n\geqq 1}\varepsilon_n\alpha^n, \tag{A8}$$

one expands

$$c_{\varepsilon(\alpha)}^{-1}=2^{-1/2}(1+\alpha 1/2\varepsilon_1\log 2+\alpha^2(1/2\varepsilon_2\log 2+1/8(\varepsilon_1\log 2)^2))+\mathcal{O}(\alpha^3). \tag{A9}$$

Substituting now (A6)–(A8) in the equation $\tilde{\mathcal{N}}(\varphi_\varepsilon,\varphi_\varepsilon)=\varphi_\varepsilon$, it is easy to see that this can be solved inductively by solving for increasing powers of α. The result is

$$\begin{aligned}
&a_0(\alpha)=1-\alpha^2/24+\mathcal{O}(\alpha^3),\\
&a_1(\alpha)=-\alpha^2(3(2-2^{1/2}))^{-1}+\mathcal{O}(\alpha^3),\\
&a_3(\alpha)=+\alpha^2\cdot 10(2^{1/2}-1)^{-1}+\mathcal{O}(\alpha^3),\\
&a_4(\alpha)=+\alpha^2\cdot 35+\mathcal{O}(\alpha^3),\\
&a_k(\alpha)=\mathcal{O}(\alpha^3), \quad k=5,6,7,\ldots,\\
&\varepsilon(\alpha)=-\alpha\cdot 3(2\log 2)^{-1}-\alpha^2((17+18\,2^{1/2})(3\log 2)^{-1})+\mathcal{O}(\alpha^3).
\end{aligned}$$

It remains now to solve for ε (by inversion of the power series), and to express each $\varrho_{j,\varepsilon}$ as a formal power series in ε and $H_{2k,0}$, $k=1,\ldots j$. This is done easily, using the definition of Hermite polynomials. One notes here that only a finite number of $a_j(\alpha(\varepsilon))\varrho_{j,\varepsilon}$ contribute to a term $\varepsilon^l H_{2k,0}$. This shows that Equations (5.8), (5.9) hold; the bound on the highest power in x follows simply from the fact that $\mathcal{N}$ at most doubles the degree of a polynomial.

Summarizing, we get in particular

$$\begin{aligned}
H_{4,\varepsilon}(x)=H_{4,0}(x)-\varepsilon(2^{1/2}-1)^{-1}\log 2(H_{4,0}(x)+6H_{2,0}(x))\\
+\mathcal{O}(\varepsilon^2)
\end{aligned}$$

and putting everything together

$$\varphi_\varepsilon(x)=1-\varepsilon\log 2/(144(2^{1/2}-1)^2)\cdot H_{4,0}$$
$$+\varepsilon^2\left\{-\frac{(\log 2)^2}{54}+H_{2,0}\left(-\frac{(\log 2)^2}{2^{1/2}27(2^{1/2}-1)^2}+\frac{(\log 2)^2}{24(2^{1/2}-1)^3}\right)\right.$$
$$+H_{4,0}\left(-\frac{(\log 2)^2}{(2^{1/2}-1)^2}\frac{17+18\cdot 2^{1/2}}{972}+\frac{(2^{1/2}+1)(\log 2)^2}{144(2^{1/2}-1)^3}\right)$$
$$\left.+H_{6,0}\frac{(\log 2)^2}{(2^{1/2}-1)^4 1296}+H_{8,0}\frac{(\log 2)^2}{(2^{1/2}-1)^4 41472}\right\}+\mathcal{O}(\varepsilon^3).$$

Appendix B

Let $\lambda_0^{(m)}(\varepsilon),\ldots,\lambda_{m-1}^{(m)}(\varepsilon)$ be the m "relevant" eigenvalues, (i.e. those >1) at the m-critical point $c_\varepsilon^{(m)}$ which we parametrize as $c_\varepsilon^{(m)}=(2/(1+\varepsilon))^{1/m}$. *Then there is for sufficiently small $\varepsilon>0$ no relation of the form*

$$\lambda_i^{(m)}(\varepsilon)=\prod_{j=0}^{m-1}\{\lambda_j^{(m)}(\varepsilon)\}^{k_j},\quad k_j\in\mathbb{Z}^+,\quad i\leqq m-1,\tag{B1}$$

except the trivial one.

Proof. If the relation (B1) is to hold, it has to hold in particular up to first order in ε for small $\varepsilon>0$ since the eigenvalues have asymptotic expansions. One finds, according to Appendix A,

$$\varphi_\varepsilon^{(m)}=1-\varepsilon\theta_m\psi_{2m,\varepsilon}^{(m)}+\mathcal{O}(\varepsilon^2),\tag{B2}$$

with

$$\theta_m=\left[1/2\frac{(2^{\frac{m-1}{m}}-1)^m}{m!}\binom{2m}{m}(2m!)^{1/2}\right]^{-1},\tag{B3}$$

and

$$\psi_{k,\varepsilon}^{(m)}(z)=(-1)^k e^{x^2}\partial_x^k e^{-x^2}|_{x=(1-1/c_\varepsilon^{(m)})^{1/2}z}2^{-k/2}k!^{-1/2}.\tag{B4}$$

Therefore, since $\mathscr{A}_g$ is linear in g,

$$\lambda_j^{(m)}(\varepsilon)=\frac{2}{c_\varepsilon^{(m)j}}$$
$$-\varepsilon\theta_m 2(\psi_{2j,\varepsilon}^{(m)},\pi^{-1/2}\int e^{-u^2}\psi_{2m,\varepsilon}^{(m)}(\cdot c_\varepsilon^{(m)^{-1/2}}+u)\psi_{2j,\varepsilon}^{(m)}(\cdot c_\varepsilon^{(m)^{-1/2}}-u)du)_{2,1-c^{(m)-1}}$$
$$+\mathcal{O}(\varepsilon^2)$$
$$=2^{\frac{m-j}{m}}\left(1+\varepsilon\left(\frac{j}{m}-2\frac{\binom{2j}{m}}{\binom{2m}{m}}\right)\right)+\mathcal{O}(\varepsilon^2),\tag{B5}$$

where we have used (A5).

Therefore (B1) holds up to first order only if

$$m-i=\sum_{j=0}^{m-1} k_j(m-j) \tag{B6}$$

and

$$\frac{i}{m}-2\frac{\binom{2i}{m}}{\binom{2m}{m}}=\sum_{j=0}^{m-1} k_j\left(\frac{j}{m}-2\frac{\binom{2j}{m}}{\binom{2m}{m}}\right). \tag{B7}$$

Using (B6), we can replace (B7) by the more convenient

$$\binom{2m}{m}-2\binom{2i}{m}=\sum_{j=0}^{m-1} k_j\left(\binom{2m}{m}-2\binom{2j}{m}\right). \tag{B8}$$

We now claim that for $j>i, m>j\geqq \frac{m+i}{2}$ [this value of j must occur due to (B6)] one has

$$\frac{\binom{2m}{m}-2\binom{2j}{m}}{\binom{2m}{m}-2\binom{2i}{m}}>\frac{m-j}{m-i}. \tag{B9}$$

Now (B9) excludes that (B6) and (B7) hold simultaneously in a non-trivial fashion, which proves the assertion.

To prove (B9), we show first that

$$\frac{m-j}{m-i}\leqq\frac{m-(m+i)/2}{m-i}=1/2.$$

On the other hand the l.h.s. of (B9) is bounded below by

$$\frac{\binom{2m}{m}-2\binom{2(m-1)}{m}}{\binom{2m}{m}}=1-\frac{m-1}{2m-1}>1/2,$$

so that (B9) follows.

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Note Added in Proof

The bounds we have given for the derivatives of ${}_1S_2$ in the proof of Theorem 6.5 are incorrect. We thank D. Chillingworth and L. Guimaraez for pointing out this error to us. The corrected version will be given in a Lecture Note volume on the subject (in preparation). This will also contain a new proof of existence of ϕ_ε.