R. W. RICHARDSON† AND T. A. SPRINGER*

COMPLEMENTS TO 'THE BRUHAT ORDER ON SYMMETRIC VARIETIES'

ABSTRACT. We give several complements to the paper 'The Bruhat order on symmetric varieties'. Our main result shows that the partial order on the set $\mathscr{I}$ of twisted involutions in the Weyl group W, which was introduced in the earlier paper, agrees with the partial order on $\mathscr{I}$ induced by the usual Bruhat order on W.

In this note, we give several complements to the paper [4], 'The Bruhat order on symmetric varieties', which appeared in Volume 35 of this journal.

1. The partial order on twisted involutions

We follow the notation of [4]. It was stated in the Introduction to loc. cit. that the partial order $\preceq$ (the 'standard order') on the set $\mathscr{I}$ of twisted involutions in the Weyl group W, which was introduced in Section 8 of the paper, was distinct from the order induced on $\mathscr{I}$ by the usual Bruhat order $\leqslant$ on W. In fact, these two partial orders on $\mathscr{I}$ agree. We give a proof below. The proof depends on the following standard lemma on the Bruhat order (see [3, Property $Z(s, w_1, w_2)$]):

THE Z-LEMMA. *Let $s \in S$ and let $x, y \in W$.* (1) *Assume that $x < sx$ and $y < sy$. Then the following three conditions are equivalent:* (i) $y \leqslant x$; (ii) $sy \leqslant sx$; *and* (iii) $y \leqslant sx$. (2) *Assume that $x < xs$ and $y < ys$. Then the following conditions are equivalent:* (i) $y \leqslant x$; (ii) $ys \leqslant xs$; *and* (iii) $y \leqslant xs$.

It follows from Lemma 8.1 and Proposition 5.6 of [4] that if $a, b \in \mathscr{I}$ and if $a \preceq b$, then $a \leqslant b$, so it will suffice to prove that if $a \leqslant b$, then $a \preceq b$. The proof is by induction on $l(b)$, where l denotes the usual length function on W. The result clearly holds if $l(b) = 0$. Assume that $l(b) > 0$ and let $s \in S$ be such that $sb < b$.

CASE 1. Assume $a < sa$; note that this is equivalent to $a < a\theta(s)$. It follows from the Z-lemma that $a \leqslant sb$.

CASE 1.1. Assume that s is real for b, hence that $sb \in \mathscr{I}$. Then $a \leqslant sb < b$. It follows from the inductive hypothesis that $a \preceq sb$ and clearly $sb \prec b$, which shows that $a \prec b$.

†R. W. Richardson died on 15 June 1993.
*Author for correspondence.

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CASE 1.2. Assume that s is complex for b. Then we have $s \circ b = sb\theta(s) < sb$ and $a < a\theta(s)$. It follows from the Z-lemma that $a \leqslant s \circ b$. By the inductive hypothesis, we then obtain $a \preceq s \circ b \prec b$.

CASE 2. Assume that $sa < a$. It then follows from the Z-lemma that $sa \leqslant sb$.

CASE 2.1. s is real for b. Then we have $s \circ a \leqslant sa \leqslant sb = s \circ b < b$. By the inductive hypothesis $s \circ a \preceq s \circ b$. Hence, by 8.2,

$$a = m(s) * (s \circ a) \preceq m(s) * (s * b) = b.$$

CASE 2.2. s is complex for both a and b. Then $s \circ a = sa\theta(s) < sa$ and $s \circ b = sb\theta(b) < sb$. By the Z-lemma, $s \circ a \leqslant s \circ b$. The proof now follows as in Case 2.1.

CASE 2.3. s is real for a and complex for b. Then $s \circ a = sa < sa\theta(s) = a$ and $s \circ b = sb\theta(s) < sb$. By the Z-lemma, $s \circ a \leqslant s \circ b$, and the proof again follows as in Case 2.1.

This completes the proof.

We note that the (incorrect) statement that the two partial orders on $\mathscr{I}$ were different was not used anywhere in [4], and so does not affect the rest of the paper.

The proofs in Section 8 of loc. cit. are somewhat unsatisfactory, since the proof of Proposition 8.5 uses properties of the K-orbits on G/B. If we use the above result that the two partial orders on $\mathscr{I}$ are identical, we can give direct combinatorial proofs of all of the results of Section 8. Propositions 8.5, 8.11, 8.12 and 8.13 of that section all follow from Proposition 8.14 and the general results of Sections 5 and 6. Thus it will suffice to give a combinatorial proof of Proposition 8.14.

PROOF OF PROPOSITION 8.14. Assume that $a < s \circ b$ and $b < s \circ b$. We must prove that either (i) $a \leqslant b$ or (ii) $s \circ a < a$ and $s \circ a < b$. Since $b < s \circ b$, we have $b < sb$ and thus s is either complex or imaginary for b. Again we need to consider several cases.

CASE 1. s imaginary for b and $a < sa$. Then $b < s \circ b = sb$. It follows from the Z-lemma that $a \leqslant b$.

CASE 2. s imaginary for b and $sa < a$. Clearly $s \circ a < a$. Since $sa < a$, $b < sb$ and $a < sb = s \circ b$, it follows from the Z-lemma that $sa < b$. Thus $s \circ a \leqslant sa < b$ and (ii) is satisfied.

CASE 3. s is complex for b and s is imaginary for a. We have $a = sa\theta(s) < sa$ and $sb < sb\theta(s) = s \circ b$. Since $a < s \circ b$, the Z-lemma implies that $a \leqslant sb$. We also have $b < sb$ and $a < sa$. The Z-lemma now implies that $a \leqslant b$.

CASE 4. s is complex for b, $a < a\theta(s)$, and s is complex for a. Then $a < a\theta(s)$ and $sb < sb\theta(s) = s \circ b$. The Z-lemma implies that $a \leqslant sb$. Since $a < sa$ and $b < sb$, another application of the Z-lemma shows that $a \leqslant b$.

CASE 5. s is complex for b and $a\theta(s) < a$. We have $sb < sb\theta(s) = s \circ b$ and $a\theta(s) < a$. By the Z-lemma, $a\theta(s) \leqslant sb$. (1) Assume that s is complex for a. Then $s \circ a = sa\theta(s) < a\theta(s)$ and $b < sb$. The Z-lemma implies that $s \circ a \leqslant b$. It is clear that $s \circ a \neq b$. Thus (ii) is satisfied. (2) Assume that s is real for a. We have $b < sb$ and $s \circ a = sa = a\theta(s) < a$. By the Z-lemma, $s \circ a \leqslant b$ and, again, $s \circ a \neq b$. Thus (ii) holds.

This covers all possible cases and completes the proof of Proposition 8.14.

REMARK. All of our definitions carry over to the case of arbitrary Coxeter groups $W = (W, S)$ and the proofs also carry over if W is finite. Thus the results of this section (and the results 8.5 and 8.11–8.14 of loc. cit.) hold for twisted involutions in finite reflection groups. However, if W is not finite the proof of Lemma 8.1 does not hold, since it makes use of the longest element w_0 of W. Thus it is not clear that our results hold for arbitrary Coxeter groups. We suspect that the results do hold for Coxeter groups.

We would like to clarify another comment in the Introduction to [4]. On line 5 of page 392 in loc. cit, it is stated that 'The map $\varphi: \mathscr{V} \to \mathscr{I}$ defined above is compatible with the respective Bruhat orders'. However, this result is not clearly stated and proved in the body of the paper. We give the appropriate lemma below:

LEMMA 1.1. *If $v', v \in \mathscr{V}$ and if $v' < v$, then $\varphi(v') < \varphi(v)$.*

Proof. We need the following result, which is an easy consequence of Lemmas 7.2 and 7.4:

(1) Let $v \in V$ and $s \in S$, with $v < m(s) \cdot v$. Then $\varphi(v) < \varphi(m(s) \cdot v) = m(s) * \varphi(v)$.

Now let $\vdash$ denote the weak order (on both V and $\mathscr{I}$), as defined in 5.1. It follows from (1) that if $v' \vdash v$, then $\varphi(v') \vdash \varphi(v)$. Since the Bruhat order on V and $\mathscr{I}$ is the 'standard order' defined in 5.2 of [4], the lemma now follows.

2. PRODUCT OF A MINIMAL PARABOLIC AND A DOUBLE COSET

In Section 4 of [4], we analyzed the decomposition of the product $P_s\mathcal{O}_v$ of a minimal parabolic subgroup P_s and an orbit $\mathcal{O}_v$ into $(B \times K)$-orbits. In the 'Case Analysis' of loc. cit. 4.3, no proofs are given and there is only a reference to some parts of [5], from which the relevant results can be extracted, with some effort. In this section, we will give a more satisfactory discussion of decomposition of $P_s\mathcal{O}_v$.

2.1. *Preliminaries*

If $v \in V$ and $s \in S$, then $m(s)\cdot v$ is defined in [4, 4.7] and we write $v \to m(s)\cdot v$ if $v \neq m(s)\cdot v$.

Let $s \in S$, let $x \in \mathscr{V}$ and let $TxK = v \in V$ be the corresponding $(T \times K)$-orbit. Thus $\mathcal{O}_v = BxK$. Let $P_1 = x^{-1}P_s x$, let $B_1 = x^{-1}Bx$ and let $T_1 = x^{-1}Tx$. Let $\Phi_1 = \mathrm{Int}(x^{-1})(\Phi) = \Phi(T_1, G)$, let $\Phi_1^+ = \mathrm{Int}(x^{-1})(\Phi^+)$, let $\Delta_1 = \mathrm{Int}(x^{-1})(\Delta)$ and let $\beta = \mathrm{Int}(x^{-1})(\alpha) \in \Delta_1$. Since $x \in \mathscr{V}$, the maximal torus T_1 is θ-stable and thus θ acts on the root system Φ_1. For each $\gamma \in \Phi_1$, let U_γ be the root subgroup corresponding to γ, let X_γ be a non-zero element of $L(U_\gamma)$, the Lie algebra of U_γ, and let G_γ be the (rank one semisimple) subgroup of G generated by $\{U_\gamma, U_{-\gamma}\}$. Let $s_\gamma \in W(T_1)$ denote the reflection corresponding to γ.

Let $\mathscr{B}(P_1)$ be the variety of Borel subgroups of P_1, let $h: P_1 \to \mathrm{Aut}(\mathscr{B}(P_1)) \cong \mathrm{Aut}(\mathbb{P}^1(F))$ be the obvious homomorphism and let $H = h(P_1 \cap K)$. It is shown in [4, 4.2] that there is a canonical correspondence between the H-orbits on $\mathscr{B}(P_1)$ and the $(B \times K)$-orbits on G. The correspondence goes as follows: Let $B_2 \in \mathscr{B}(P_1)$, so that $B_2 = {}^gB_1$ for some $g \in P_1$. Then the H-orbit of B_2 on $\mathscr{B}(P_1)$ corresponds to the $(B \times K)$-orbit $Bxg^{-1}K$.

Let $n = x\theta(x^{-1})$. Then $n \in N(T)$ and n represents $\varphi(v) = a$. It is clear that $\mathrm{Int}(n) \circ \theta = \mathrm{Int}(x) \circ \theta \circ \mathrm{Int}(x^{-1})$. Hence $\mathrm{Int}(x^{-1})(a\theta(\alpha)) = \theta(\beta)$. Thus α (or s) is complex (respectively real; imaginary) for a if and only if $\theta(\beta) \neq \pm\beta$ (respectively $\theta(\beta) = -\beta$; $\theta(\beta) = \beta$). Note that if s is real (respectively imaginary) for a, then $sa < a$ (respectively $sa > a$).

2.2. *The complex case*

In 2.2, we assume that $\theta(\beta) \neq \pm\beta$. Set

$$\Sigma = \Phi_1^+ \cup \{-\beta\} = \Phi(T_1, P_1) \quad \text{and} \quad \Gamma = \Sigma \cap \theta(\Sigma) = \Phi(T_1, P_1 \cap \theta(P_1)).$$

Let U_Γ be the subgroup of G generated by $\{U_\gamma \mid \gamma \in \Gamma\}$. It follows from [2, §3] that $P_1 \cap \theta(P_1) = T_1 U_\Gamma$.

LEMMA 2.2.1. *U_Γ is a θ-stable unipotent group normalized by T_1.*

Proof. It is clear that U_Γ is θ-stable and is normalized by T_1. If $\theta(\beta) \in \Phi_1^+$ then $\Gamma \subset \Phi_1^+$ and U_Γ is contained in $R_u(B_1)$, the unipotent radical of B_1. If $\theta(\beta) < 0$, then $\Gamma \subset s_\beta(\Phi_1^+)$ and $U_\Gamma \subset R_u(s_\beta B_1 s_\beta)$.

LEMMA 2.2.2. *If U' is a connected θ-stable unipotent subgroup of G, then $L(U')^\theta = L(U'^\theta)$.*

Proof. By [1, Prop. 9.3], the product map $\tau(U') \times U'^\theta \to U'$ is an isomorphism of varieties, so that $L(U') = T_e(\tau(U')) \oplus L(U'^\theta)$, where $T_e(\tau(U'))$ denotes the tangent space at $e \in G$ of $\tau(U')$. Since $\theta(\tau(g)) = \tau(g)^{-1}$ for every $g \in G$, we have $\theta(X) = -X$ for $X \in T_e(\tau(U'))$. Thus $L(U'^\theta)$ is the $+1$ eigenspace of θ on $L(U')$.

PROPOSITION 2.2.3. *If s is complex for a, then H is solvable and $R_u(H) \neq \{0\}$.*

Proof. By Lemma 2.2.1, $P_1 \cap \theta(P_1) = T_1 U_\Gamma$ is a solvable group, and consequently $H = h(P_1 \cap K) \subset h(P_1 \cap \theta(P_1))$ is solvable. Assume that $\theta(\beta) \in \Phi_1^+$. Then $\beta \in \Gamma$ and therefore $X_\beta \in L(U_\Gamma)$. Thus

$$(X_\beta + \theta(X_\beta)) \in L(U_\Gamma)^\theta \subset L(P_1 \cap K).$$

Now $\theta(\beta) \in (\Phi_1^+ \backslash \{\beta\})$ and therefore $\theta(U_\beta) \subset R_u(P_1) \subset \text{kernel}(h)$. Thus $dh(X_\beta) = dh(X_\beta + \theta(X_\beta)) \in L(H)$. Now the differential dh maps $L(G_\beta)$ isomorphically onto $L(\text{Aut}(\mathscr{B}(P_1))$, so that $dh(X_\beta)$ is a non-zero nilpotent element of $L(H)$ and $R_u(H) \neq \{0\}$. If $\theta(-\beta) \in \Phi_1^+$, a similar argument shows that $dh(X_{-\beta})$ is a non-zero nilpotent element of $L(H)$.

LEMMA 2.2.4. *Assume that s is complex for a. Then $P_s\mathcal{O}_v = \mathcal{O}_v \cup \mathcal{O}_{s\cdot v}$ and $s \cdot v \neq v$, so that there are two $(B \times K)$-orbits in $P_s\mathcal{O}_v$. We have $\varphi(s \cdot v) = sa\theta(s)$. (1) If $a < sa$, then $v \to m(s) \cdot v = s \cdot v$, so that $\mathcal{O}_{s\cdot v}$ is open and dense in $P_s\mathcal{O}_v$ and $\mathcal{O}_v$ is closed of codimension one. (2) If $sa < a$, then $s \cdot v \to m(s) \cdot (s \cdot v) = v$, so that $\mathcal{O}_v$ is open dense in $P_s\mathcal{O}_v$ and $\mathcal{O}_{s\cdot v}$ is closed of codimension one.*

Proof. By 2.2.3 and [4, 4.2], there are two orbits. By [4, Lemma 2.1], $\varphi(s \cdot v) = sa\theta(s) \neq a$, so that $v \neq s \cdot v$. Thus $P_s\mathcal{O}_v = \mathcal{O}_v \cup \mathcal{O}_{s\cdot v}$.

(1) Since s is complex for a and $a < sa$, we have $sa < sa\theta(s)$ and consequently

$$\tau(BsBxK) \subset BsBaB\theta(s)B = Bsa\theta(s)B.$$

It is clear that $\tau(\mathcal{O}_{s\cdot v}) \subset Bsa\theta(s)B$ and it follows easily from this that $v \to m(s) \cdot v$ and that $m(s) \cdot v = s \cdot v$. The other conclusions of (1) follow immediately.

(2) We have $\varphi(s \cdot v) = sa\theta(s) < a$. If we interchange the roles of $s \cdot v$ and v, then the conclusions of (2) follows from those of (1).

2.3. *The rank one case*

Before treating the real and imaginary cases, we briefly discuss the case in which G is semisimple of rank one. In this case, G is isomorphic to either

$SL(2, F)$ or $PGL(2, F)$. Every non-trivial involutive automorphism θ of G is inner and any two such automorphisms are conjugate by an inner automorphism. The identity component K^0 of $K = G^\theta$ is a one-dimensional torus. K^0 has three orbits on the flag manifold $\mathscr{B}(G) \cong \mathbb{P}^1(F)$, two fixed points and one open dense orbit. If $G \cong SL(2, F)$, then $K^0 = K$, and the canonical action of the Weyl group W on the orbits permutes the two fixed points. If $G \cong PGL(2, F)$, then K/K^0 is of order 2 and permutes the two fixed points of K^0, so that there are two K-orbits.

2.4. *The real and imaginary cases*

We assume that $a\theta(\alpha) = \pm\alpha$, so that s is either real or imaginary for a. Then $\theta(\beta) = \pm\beta$, so that G_β is a θ-stable rank one semisimple group.

Now $L_1 = T_1G_\beta$ is a θ-stable Levi subgroup of P_1. Let $T_\beta = G_\beta \cap T_1$ and let Z_1 be the identity component of the center of L_1. Then T_β is a θ-stable maximal torus of G_β and $T_1 = Z_1T_\beta$. The group $Z_1R_u(P)$ is the solvable radical of P_1 and $\text{kernel}(h)^0 = Z_1R_u(P_1)$.

LEMMA 2.4.1. *If $\theta(\beta) \neq \pm\beta$, then H^0 is reductive and, if $\theta(\beta) = -\beta$, then H^0 is a torus.*

Proof. The parabolic subgroup $P_1 \cap \theta(P_1)$ is the semi-direct product of the θ-stable subgroups $L_1 = Z_1G_\beta$ and $R_u(P_1 \cap \theta(P_1))$. Thus

$$P_1 \cap K = L_1^\theta R_u(P_1 \cap \theta(P_1))^\theta.$$

Since $R_u(P_1 \cap \theta(P_1))$ is contained in $R_u(P_1)$ and $h(R_u(P_1)) = \{1\}$, we see that $H = h(L_1^\theta)$. The conclusion of Lemma 2.4.1 follows immediately.

LEMMA 2.4.2. *s is compact imaginary for v if and only if $P_s\mathcal{O}_v = \mathcal{O}_v$. In this case, we have $v = s \cdot v = m(s) \cdot v$.*

Proof. It follows from the definitions that s is compact imaginary for v if and only if $G_\beta \subset K$. If $G_\beta \not\subset K$, then $(L_1 \cap K)^0$ is a torus and $h(P_1 \cap K) = h(L_1 \cap K)$ is not transitive on $\mathscr{B}(P_1)$. Thus $P_s\mathcal{O}_v = \mathcal{O}_v$ if and only if $G_\beta \subset K$.

Assume now that the restriction of θ to G_β is non-trivial. Let $K_\beta = G_\beta^\theta$. Then $C = K_\beta^0$ and T_β are maximal tori of G_β and $h(C) = H^0$. Let $K_\beta^* = \{g \in G_\beta \mid h(g) \in H\}$. Then

$$h(K_\beta^*) = H \quad \text{and} \quad C = K_\beta^0 \subset K_\beta \subset K_\beta^* \subset N_{G_\beta}(C).$$

We note that it is not necessarily the case that $K_\beta = K_\beta^*$ or that $K_\beta^* = N_{G_\beta}(C)$. We also note that $T_\beta = C$ (respectively $T_\beta \neq C$) if and only if s is non-compact imaginary (respectively real) for v. Note also that $Z_1 C$ is a maximal torus of G.

LEMMA 2.4.3. *Assume that s is non-compact imaginary for v. Then:* (i) $v \to m(s)\cdot v$ *and* $s\cdot v \to m(s)\cdot(s\cdot v) = m(s)\cdot v$; (ii) $\varphi(m(s)\cdot v) = sa$; *and* (iii) $P_s\mathcal{O}_v = \mathcal{O}_v \cup \mathcal{O}_{s\cdot v} \cup \mathcal{O}_{m(s)\cdot v}$. *There are either two or three $(B\times K)$-orbits in $P_s\mathcal{O}_v$, depending on whether or not $s\cdot v = v$.*

Proof. We have $C = T_\beta$. Let $n_1 \in N_{G_\beta}(C)$ represent s_β. The Borel subgroups B_1 and $B_1' = n_1 B_1 n_1^{-1}$ are the fixed points of $H^0 = h(C)$ on $\mathcal{B}(P_1)$. It is clear that $n' = xn_1x^{-1} \in N(T)$ represents $s = s_\alpha \in W(T)$. Thus $xn_1 = n'x$ represents $s\cdot v$. Using the correspondence between H-orbits on $\mathcal{B}(P_1)$ and $(B\times K)$-orbits on $P_s\mathcal{O}_v$, we see that the H-orbit of B_1' corresponds to $\mathcal{O}_{s\cdot v}$. The conclusions (i) and (iii) now follow from [4, 4.1–4.2].

It remains to prove that $\varphi(m(s)\cdot v) = sa$. Let $y \in G_\beta$ be such that $y^{-1}Cy$ is θ-stable and not equal to C. Then $xy = xyx^{-1}x$ corresponds to $m(s)\cdot v$ and $\tau(y) \in n_1 C$. We have $\tau(xy) = x\tau(y)\theta(x^{-1}) = x\tau(y)x^{-1}\tau(x)$. Since $\tau(y)$ represents $s_\beta \in W(T_1)$, we see that $x\tau(y)x^{-1}$ represents $s = s_\alpha \in W(T)$. Thus $\tau(xy)$ represents sa and consequently $\varphi(m(s)\cdot v) = sa$.

LEMMA 2.4.4. *If s is real for a, then there exists $v' \in V$ such that $P_s\mathcal{O}_v = \mathcal{O}_{v'} \cup \mathcal{O}_{s\cdot v'} \cup \mathcal{O}_v$. There are either two or three orbits, depending on whether or not $s\cdot v' = v'$. Furthermore, s is non-compact imaginary for v' and for $s\cdot v'$ and we have:* (i) $v' \to m(s)\cdot v' = v$ *and* $s\cdot v' \to m(s)\cdot(s\cdot v') = v$; *and* (ii) $\varphi(v') = \varphi(s\cdot v') = sa$.

Proof. Since s is real for v, we see from Lemma 2.4.1 that H^0 is a torus. Hence H^0 has two fixed points, say B_2 and B_2', on $\mathcal{B}(P_1)$. It is clear that B_1 is not fixed by H^0. Choose $y \in G_\beta$ such that $y^{-1}B_1y = B_2$. Then $C' = (B_2 \cap G_\beta)^0$ is a maximal torus of G_β and is contained in K; hence $C' = (G_\beta^\theta)^0$. We note that $y^{-1}T_1y = C'Z_1$ is a θ-stable maximal torus of G, so that $xy \in \mathcal{V}$. Let $v' = TxyK$. Then $\mathcal{O}_{v'} \subset P_s\mathcal{O}_v$ and s is imaginary for v'. Lemma 2.4.4 now follows from Lemma 2.4.3.

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Authors' addresses:

R. W. Richardson,
Centre for Mathematics and its Applications,
Austrialian National University,
Canberra, ACT 2601,
Australia.

T. A. Springer,
Mathematisch Instituut,
Rijksuniversiteit Utrecht,
3508 TA Utrecht,
The Netherlands.

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