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DUALITY, SECTIONS AND PROJECTIONS OF CERTAIN EUCLIDEAN TILINGS

ABSTRACT. Pegged tilings localize the defining property of Voronoï or Laguerre tilings, and, like them, admit a natural duality (corresponding to the Delaunay tilings of Voronoï tilings). It can thus be shown that the projection method, which is generally used to construct quasi-periodic tilings related to Voronoï tilings of higher dimensional lattices, applies to this wider class of tilings. Of further importance is that pegged tilings are just those which can be lifted to the graphs of convex functions with a certain strong locally polyhedrality property. The context of convex functions also provides a direct way of viewing the projection method, and leads to alternative pictures of special cases such as various grid methods.

1. INTRODUCTION

The development of what is called the *projection method* for constructing quasi-periodic tilings owes about as much to non-mathematicians as to mathematicians, as a glance at the extensive literature makes clear. (See [7], [10], [12], [15], [18] for samples of this.) In what might be called the *classical* case, when it is applied to the ordinary tiling of $\mathbb{E}^n$ by unit cubes (the *cubic* tiling), the underlying theory is transparent enough to be dealt with by hand. But, more recently, the method has been generalized to tilings of $\mathbb{E}^n$ by Voronoï regions of a lattice, and here the theoretical underpinnings are, perhaps, less obvious (but see also the recent paper [20]). One of our intentions in this paper is to show that the intuitive picture held by those who use this generalized method is correct, and, in fact, will work in a yet wider class of tilings of $\mathbb{E}^n$.

The tilings we treat, which we call *pegged*, are obtained by localizing the conditions for a Voronoï domain of a point-set. However, we set things in an even more general context, by working with *partial* tilings of (part of) $\mathbb{E}^n$. The global conditions for a generalized Voronoï or *Laguerre* tiling will clearly imply ours. The projection property arises because pegged tilings admit a *duality* in a natural way. Dualizing a pegged tiling, taking a *general* section of the result, and then dualizing again, yield the projection. In one sense, it is then better to think of the projection as the inverse of an injection back into the original tiling; indeed, it is possible that a misperception of this projection has led to some of the underlying principles being overlooked hitherto. (It is worth mentioning here the *block* or *Klotz* construction of [12], which is designed to overcome these conceptual problems.)

Our theory has implications for the construction of *quasi-periodic* tilings.

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One wants here not only an essential non-periodicity, but also inflation and deflation operations, like those of the famous *Penrose* tilings. This aim can be achieved when the projection method is applied to a tiling whose dual is the Voronoï tiling of a lattice with a suitable projection. For example, new three-dimensional quasi-periodic tilings were found by McMullen [13] in this way (and independently a little later in [3], [11]; [14] also contains a more detailed account than [13] of this as well as higher dimensional examples).

An important tool in our later investigations is the relation of our tilings to the graphs of convex functions with a strong local polyhedrality property. As a result, we can show that pegged tilings coincide with Laguerre tilings in a generalized sense. In particular, the local property of their definition extends to a global property. Various operations on convex functions correspond to operations on tilings; the special cases of the projection method which are known as *grid* methods then admit a perspicuous geometric interpretation.

We should note here an incomplete earlier attempt in [2] (in the context of finite tilings, although this feature is unimportant) to establish some of our results. We shall discuss this further in Section 7.

The paper falls naturally into two parts. In the first, we shall treat tilings entirely within one space. In the second, we shall lift tilings to convex functions, and this will enable us to perform various operations on them. Certain results proved in the first part make it self-contained; however, equivalent results would still have had to be proved in the second.

Part of the research in this paper was done during the Study Group and Workshop on Quasicrystals held at the Zentrum für interdisziplinäre Forschung in Bielefeld in March 1991. The meetings of the Study Group were very stimulating; particularly so were several discussions with Martin Schlottmann, whose own investigations provided a catalyst (in a somewhat different context; see [20]) to the approach to tilings using convex functions.

2. Pegged tilings

While our main interest is in studying tilings of the whole of n-dimensional Euclidean space $\mathbb{E}^n$, there is a more natural context for much of our investigation. A *partial tiling* of $\mathbb{E}^n$, or a tiling *in* (rather than *of*) $\mathbb{E}^n$, consists of a family $\mathscr{T}$ of convex n-polytopes called *tiles*, whose *domain* dom $\mathscr{T} := \cup\mathscr{T}$ is homeomorphic to $\mathbb{E}^n$, and which is such that the interiors of two different tiles do not meet. As usual, a *face* of $\mathscr{T}$ is a face of a tile of $\mathscr{T}$.

Concepts such as *local finiteness* and *discreteness* for a partial tiling $\mathscr{T}$ will always be taken relative to dom $\mathscr{T}$; thus local finiteness means that each compact set in dom $\mathscr{T}$ meets only finitely many tiles of $\mathscr{T}$. (For the background on tilings in general, see [9].) We shall see later that, even for tilings of $\mathbb{E}^n$ itself, a stronger condition on $\mathscr{T}$, such as *normality*, meaning that there are fixed bounds for the in- and circumradii of tiles (see [9, p. 122]), would be inappropriate. Unless stated to the contrary, we shall always assume our partial tilings to be locally finite, and we shall then usually refer to them just as tilings.

We call a tiling $\mathscr{T}$ *pegged* if with each tile $P \in \mathscr{T}$ is associated a point $v^*(P) \in \mathbb{E}^n$, the *peg* of P, such that, if the tile $P' \in \mathscr{T}$ is *adjacent* to P, and so meets it on a facet, then $v^*(P') - v^*(P)$ is an outer normal vector to P at that facet. We denote by $V^* := V^*(\mathscr{T})$ a *peg-set* of the tiling $\mathscr{T}$. (The notation anticipates the duality properties we shall establish later.) If we indicate a tile in $\mathscr{T}$ by P_i, then we write $v_i^* := v^*(P_i)$ for its peg, and so on.

In an obvious way, the definition of a pegged tiling generalizes that of a Laguerre tiling (which in turn generalizes that of a Voronoĭ tiling). Indeed, an appropriate alternative name might be a *local* Laguerre tiling. However, as mentioned in Section 1, it will be shown that (apart from the question of the domain) a pegged tiling is, in fact, a Laguerre tiling (see Section 7 below).

Recall that two sets S, S' in E_n are said to be *homothetic* if $S' = \lambda S + t$ for some positive number λ and some translation vector $t \in \mathbb{E}^n$. An obvious remark is

LEMMA 2.1. *Let $\mathscr{T}$ be a pegged tiling in E^n. Then any set homothetic to a peg-set $V^*(\mathscr{T})$ of $\mathscr{T}$ is also a peg-set of $\mathscr{T}$, and any two homothetic pegged tilings have the same peg-sets.*

In calling a tiling $\mathscr{T}$ *face-to-face* we mean that any two tiles of $\mathscr{T}$ meet on a common (possibly empty) face of each. We begin by proving

THEOREM 2.2. *A pegged tiling is face-to-face.*

Suppose to the contrary that $\mathscr{T}$ is a pegged tiling which is not face-to-face. We can thus find a tile $P_0 \in \mathscr{T}$, which has a facet F whose relative interior relint F meets two or more other tiles in $\mathscr{T}$. Then some $(n-2)$-face G of such a tile also meets relint F. Using the local finiteness of $\mathscr{T}$, we can then find a point $z \in \operatorname{relint} F \cap \operatorname{relint} G$ which lies in no face of $\mathscr{T}$ of dimension less than $n-2$. Then tiles that surround z then form a sequence $P_0, P_1, \ldots, P_{k-1}, P_k = P_0$, such that P_{i-1} and P_i are adjacent for each $i = 1, \ldots, k$. The corresponding pegs $v_i^* = v^*(P_i)$ (with $i = 0, \ldots, k$) must lie in a plane orthogonal to the affine hull aff G of G, and, indeed, form successive vertices of a convex polygon. But this is impossible, since $v_1^* - v_0^*$ and $v_{k-1}^* - v_0^*$ are both outer normal vectors to P_0 at its facet F, and so are parallel. This contradiction establishes the theorem. □

3. Duality

We shall now see that pegged tilings admit a natural duality. (Our rationale for working with partial tilings appears here, since the dual of a tiling of $\mathbb{E}^n$ need not itself tile $\mathbb{E}^n$.) For each face F of a pegged tiling $\mathscr{T}$, let

$$\hat{F} := \operatorname{conv}\{v^*(P) \mid P \in \mathscr{T} \quad \text{and} \quad F \subseteq P\}.$$

We also write $\hat{v}$ instead of $\widehat{\{v\}}$ when $v \in \operatorname{vert} \mathscr{T}$, the set of vertices of $\mathscr{T}$.

THEOREM 3.1. *Let $\mathscr{T}$ be a pegged tiling in E^n. Then the polytopes $\hat{v}$, with $v \in \operatorname{vert} \mathscr{T}$, form a pegged tiling $\mathscr{T}^*$ in E^n, whose peg-set is* $\operatorname{vert} \mathscr{T}$. *Furthermore, $\mathscr{T}^{**} = \mathscr{T}$.*

We need to show first that the polytopes $\hat{v}$ form a locally finite partial tiling of E^n; it will then be easy to prove that this tiling is pegged.

We first show that $\mathscr{T}^*$ packs $\mathbb{E}^n$. Let $P \in \mathscr{T}$ be a tile, F a non-empty face of P (possibly $F = P$), and $N(F, P)$ the normal cone to P at F. If $\mathscr{T}_v$ is the set of tiles adjacent to P which contain the vertex v of P, and $v^* := v^*(P)$, then

$$N(v, P) = \operatorname{pos}\{v^*(P') - v^* \mid P' \in \mathscr{T}_v\},$$

where pos denotes the positive hull. Now, we always have

$$\cap \{N(v, P) \mid v \in \operatorname{vert} F\} = N(F, P),$$

and since (up to translation) $N(F, P)$ is the cone generated by $\hat{F}$ at its vertex v^*, it easily follows that no two of these polytopes $\hat{v}$ can overlap. We thus have a local packing property.

The same argument also shows that the polytopes $\hat{v}$ with $v \in \operatorname{vert} P$ fit together nicely around the vertex $\hat{P}$, and, indeed, surround $\hat{P}$; the local finiteness of $\mathscr{T}$ ensures that their union contains a neighbourhood of $\hat{P}$, since only finitely many tiles P' meet P. We now use a standard argument to show that no two n-polytopes in $\mathscr{T}^*$ can overlap–such an overlap leads to a loop in dom $\mathscr{T}^*$ which meets no face of $\mathscr{T}^*$ of dimension less than $n - 1$, and such a loop can be contracted over dom $\mathscr{T}^*$ while only passing through $(n - 2)$-faces because $\mathbb{E}^n$ is simply connected, implying eventually that the overlap was in fact a coincidence. It is now clear that, topologically, $\mathscr{T}^*$ is a tiling in dom $\mathscr{T}^*$ dual to $\mathscr{T}$, and hence it will be locally finite. Thus $\mathscr{T}^*$ is a tiling–that is, a locally finite partial tiling.

In fact, the argument above shows that if v and v' are the two vertices of an edge E of T, then $\hat{v} \cap \hat{v}' = \hat{E}$, and $v' - v$ is an outer normal vector to $\hat{v}$ at its facet $\hat{E}$. Thus $\mathscr{T}^*$ is pegged, with peg-set $V := \operatorname{vert} \mathscr{T}$.

Finally, the construction makes it clear that $\mathscr{T}^{**} (:= (\mathscr{T}^*)^*) = \mathscr{T}$, and this completes the proof of the theorem. □

Two tilings which are dual in the way described above (that is, the vertices of one form the pegs of the other) shall be called *strongly dual* (a term also occasionally used is *othogonally dual*).

4. Examples

Before proceeding further, we give several examples of tilings which can or cannot be pegged.

In the first example, we merely make an injudicious choice of a peg-set. Let $\mathscr{T}$ be the usual tiling of $\mathbb{E}^2$ by unit squares, whose vertices form the integer lattice $\mathbb{Z}^2$, with $\mathbb{Z}$ as usual denoting the integers. Let $T(m, n)$ be the tile whose bottom left corner is $(m, n) \in \mathbb{Z}^2$. Define

$$k' := \begin{cases} 2 - \dfrac{1}{k} & \text{if} \quad k > 0, \\ k & \text{if} \quad k \leqslant 0, \end{cases}$$

and let the peg $p(m, n)$ of $T(m, n)$ be

$$p(m, n) := (m', n'),$$

for all $(m, n) \in \mathbb{Z}^2$. It is easy to see that this indeed forms a peg-set of $\mathscr{T}$. However, the dual $\mathscr{T}^*$ is only a partial tiling of $\mathbb{E}^2$, since it only covers $\{(\xi, \eta) \in \mathbb{E}^2 | \xi < 2, \eta < 2\}$.

A modification of the tiling $\mathscr{T}$ forces such bad behaviour. We now define

$$k'' := \begin{cases} 2k - 1 + \dfrac{1}{k} & \text{if} \quad k > 0, \\ 2k & \text{if} \quad k \leqslant 0, \end{cases}$$

and let $\mathscr{T}$ be the tiling whose tiles $T(m, n)$ have vertices

$$\begin{aligned} \operatorname{vert} T(m, n) := \{&(m'', n''), ((m+1)'', n''), (m'', (n+1)''), \\ &((m+1)'', (n+1)'')\}, \end{aligned}$$

with $(m, n) \in \mathbb{Z}^2$. This distorts the square tiles with vertex set $2\mathbb{Z}^2$ in the positive quadrant.

We can choose a peg-set as follows. Define

$$p(m, n) := \begin{cases} (m, n) & \text{if} \quad m \leqslant 0, n \leqslant 0, \\ \left(2 - \dfrac{1}{m}, n\right) & \text{if} \quad m > 0, n \leqslant 0, \\ \left(m, 2 - \dfrac{1}{n}\right) & \text{if} \quad m \leqslant 0, n > 0, \\ \left(2 - \dfrac{2n[2m(n+1)+1]}{4mn(m+1)(n+1)-1}, 2 - \dfrac{2m[2n(m+1)+1]}{4mn(m+1)(n+1)-1}\right) & \text{if} \quad m > 0, n > 0. \end{cases}$$

(The pegs $p(m, n)$ for fixed $m > 0$ and variable $n \geqslant 0$ lie on the line through $(2 - 1/m, 0)$ with slope $2m(m + 1)$.) Again, the dual tiling only covers $\{(\xi, \eta) \in \mathbb{E}^2 \mid \xi < 2, \eta < 2\}$.

In this case, no peg-set for $\mathscr{T}$ can be discrete. For fixed $m > 0$, the tiles $T(m, n)$ with $n \geqslant 0$ meet on parallel sides, and so their pegs must lie on a line with slope $2m(m + 1)$. Any two such lines for positive m must meet, and so the corresponding pegs have a cluster point.

As a final example, consider the part of the tiling of $\mathbb{E}^2$ by regular hexagons consisting of one hexagon, and the two successive rings of six and twelve hexagons which surround it. Now split the central hexagon into three rhombi. Then this patch of tiles cannot belong to any pegged tiling. In trying to peg the patch, we are forced to have a ring of equilateral triangles corresponding to the two rings of hexagons. But we cannot split the peg corresponding to the central hexagon into three separate pegs, since these pegs would still have to lie on the same three lines as before.

Similar examples which cannot be pegged for local reasons can be constructed from non-stretchable arrangements of pseudo-lines in the plane.

Let us end the section with a positive result.

THEOREM 4.1. *Let $\mathscr{T}$ be a face-to-face tiling in $\mathbb{E}^n$, such that each tile P in $\mathscr{T}$ has a circumsphere whose centre lies in* int P. *Then $\mathscr{T}$ is a pegged tiling.*

A *circumsphere* of a convex polytope is a sphere which contains all its vertices. The result is obvious; the pegs are the circumcentres. □

5. Sections

Let $\mathscr{T}$ be a pegged tiling in $\mathbb{E}^n$, let L be a d-dimensional linear subspace of $\mathbb{E}^n$, and let $M := L + p$ be a translate of L which meets dom $\mathscr{T}$. We say that M is *general* (with respect to $\mathscr{T}$) if M meets no face of $\mathscr{T}$ of dimension less than $n - d$. Of course, 'most' (in an obvious sense) such translates M are general, since M only has to avoid countably many $(n - d - 1)$-faces of $\mathscr{T}$. The following result is intuitively obvious, although not straightforward to prove.

LEMMA 5.1. *Let $\mathscr{T}$ be a tiling in $\mathbb{E}^n$, and let M be a general d-dimensional affine subspace of $\mathbb{E}^n$ which meets* dom $\mathscr{T}$. *If F is a face of $\mathscr{T}$ such that $M \cap F \neq \emptyset$, then* $\dim(M \cap F) = \dim F - (n - d)$.

Suppose first that $M \cap \operatorname{relint} F \neq \emptyset$. Counting dimensions (actually of affine hulls), we have

$$\begin{aligned}\dim F - \dim(M \cap F) &= \dim(M + F) - \dim M \\ &\leqslant n - d.\end{aligned}$$

Now suppose that $\dim(M \cap F) \geqslant 1$. We have $M \cap \operatorname{relbd} F = \operatorname{relbd}(M \cap F)$, where relbd denotes the relative boundary. Thus we can find a face G of F with $\dim G \leqslant \dim F - 1$, such that $M \cap \operatorname{relint} G \neq \varnothing$, and $\dim(M \cap G) = \dim(M \cap F) - 1$, so that

$$\dim G - \dim(M \cap G) \leqslant \dim F - \dim(M \cap F).$$

Proceeding in this way, we eventually find a face J of F (and hence of $\mathscr{T}$) for which $M \cap \operatorname{relint} J \neq \varnothing$ and $\dim(M \cap J) = 0$. Since M is a general subspace, we must have $\dim(M \cap J) \geqslant n - d$, and from

$$\begin{aligned} n - d &\leqslant \dim J - \dim(M \cap J) \\ &\leqslant \dim F - \dim(M \cap F) \\ &\leqslant n - d, \end{aligned}$$

we have equality throughout.

Finally, suppose, if possible, that $M \cap F \neq \varnothing$, but that $M \cap \operatorname{relint} F = \varnothing$. There is then a maximal face G of F such that $M \cap \operatorname{relint} G \neq \varnothing$. By a standard separation argument, we can find a hyperplane H in $\mathbb{E}^n$, such that $M \subseteq H$ and $H \cap \operatorname{relint} F = \varnothing$. Then necessarily $G \subseteq H$. We now carry out the same calculations as above, with H replacing $\mathbb{E}^n$, to conclude that

$$\dim G - \dim(M \cap G) = (n - 1) - d.$$

But this contradicts the previous calculation. Thus, in fact, $M \cap \operatorname{relint} F \neq \varnothing$, as before. This completes the proof of the lemma. □

If M is a general affine subspace with respect to the tiling $\mathscr{T}$, we define the tiling $M \cap \mathscr{T}$ by

$$M \cap \mathscr{T} := \{M \cap P \mid P \text{ is a tile of } \mathscr{T} \text{ with } M \cap P \neq \varnothing\}.$$

Such a tiling is called a *general section* of $\mathscr{T}$. It seems to be well known that a general section of a Voronoĭ tiling is a generalized Voronoĭ or Laguerre tiling (compare [20], which provides an alternative proof to what follows for Laguerre tilings). The definition of the latter which we adopt is the following. Let V^* be an infinite set of points in $\mathbb{E}^n$ which has no cluster points in $\operatorname{conv} V^*$. With each $v_i^* \in V^*$ is associated a *weight* ω_i, and a tile

$$P_i := \{x \in \mathbb{E}^n \mid \|x - v_i^*\|^2 - \omega_i \leqslant \|x - v_j^*\|^2 - \omega_j \text{ for each } j \neq i\}.$$

Eliminating all points v_i^* for which $\dim P_i < n$, the result is a *Laguerre tiling.* As originally defined, the weights ω_i were non-negative, but this has little effect on the theory. The name 'Laguerre' itself comes from his theorem, which states that the locus of points whose tangents to two circles in $\mathbb{E}^2$ have equal lengths is a line (the obvious analogue holds in $\mathbb{E}^n$).

We may note that Laguerre tilings are pegged. In a pegged tiling, adjacent tiles P_i and P_j, with corresponding pegs v_i^* and v_j^*, are separated by a hyperplane with equation

$$\langle x, v_j^* - v_i^* \rangle = \alpha_{ij},$$

for some number α_{ij}. For a Laguerre tiling as above, the numbers α_{ij} are given by

$$\alpha_{ij} = \tfrac{1}{2}(\|v_j^*\|^2 - \omega_j - \|v_i^*\|^2 + \omega_i).$$

The property of sections above generalizes to pegged tilings.

THEOREM 5.2. *A general section of a pegged tiling is itself a pegged tiling.*

We first translate our tiling $\mathscr{T}$ so that the section is by a linear subspace L. If $P_i \in \mathscr{T}$ is such that $Q_i := L \cap P_i \neq \emptyset$, we define the peg of Q_i to be the image w_i^* of v_i^* under orthogonal projection on L. If P_i and P_j are adjacent tiles of $\mathscr{T}$ such that L meets $P_i \cap P_j$, then, as required, the intersection $Q_i \cap Q_j = L \cap P_i \cap P_j$ lies in

$$\{x \in L \mid \langle x, v_j^* - v_i^* \rangle = \alpha_{ij}\} = \{x \in L \mid \langle x, w_j^* - w_i^* \rangle = \alpha_{ij}\}.$$

Since $L \cap \mathscr{T}$ is automatically locally finite, this proves the theorem. □

6. The projection method

The projection method is now easily described in our more general context. Starting with a pegged tiling $\mathscr{T}$ in $\mathbb{E}^n$ (and we recall once again that $\mathscr{T}$ is locally finite, but need not tile all of $\mathbb{E}^n$), we form its dual $\mathscr{T}^*$ (based on its peg set), take a general section $M \cap \mathscr{T}^*$, and then dualize again, to give

$$\mathscr{T}_M := (M \cap \mathscr{T}^*)^*.$$

We say that $\mathscr{T}_M$ is obtained from $\mathscr{T}$ by the *projection method.*

Let us give an explicit description of the tiles of $\mathscr{T}_M$. With our previous convention $\dim M = d$, we deduce from the sections above

THEOREM 6.1. *The tiles of $\mathscr{T}_M$ are the images under orthogonal projection on M of the d-faces F of $\mathscr{T}$ such that $M \cap \hat{F} \neq \emptyset$.*

Since duality reverses the direction of mappings, it seems to be more appropriate to regard this set of d-faces F, which form the *broken surface,* as the inverse image of $\mathscr{T}_M$ under the orthogonal projection.

We make an easy observation. Since a general section of a general section of a tiling $\mathscr{T}$ is itself a general section of $\mathscr{T}$, it is clear that tilings obtained by the projection method display a similar hereditary behaviour.

7. Convex functions

What we have done so far gives a somewhat unsurprising generalization of the known theory, even though the results are rigorously established in this more general context for the first time. (It is, perhaps, useful to note that our description of the projection method does not involve any restrictions on positioning of a space orthogonal to the subspace M, as seems to be needed in the block (Klotz) construction of [12].)

In two different contexts, it has been observed that certain tilings lift to graphs of convex functions. First, Edelsbrunner and Seidel [6] showed that a Voronoĭ tiling with *sites* (that is, pegs) V^* lifts to the function

$$f := \sup\{\langle \cdot, v^* \rangle - \tfrac{1}{2}\|v^*\|^2 \mid v^* \in V^*\}.$$

Aurenhammer [1] later generalized this result to Laguerre tilings (we shall comment on his later paper [2] below). Further, Bohne *et al.* [4] have described the grid method used by de Bruijn [5] to construct Penrose-type tilings in terms of the graph of a certain convex function. It is our purpose here to show that all pegged tilings have an analogous lifting property; a consequence will be that pegged tilings are actually Laguerre tilings (in the general sense of this paper).

Let f be a convex function on $\mathbb{E}^n$. We shall follow the conventions of Rockafellar [19], and regard f as a function taking values in $\mathbb{R} \cup \{\pm\infty\}$, so that f is convex if its *epigraph*

$$\operatorname{epi} f := \{(x, \eta) \in \mathbb{E}^{n+1} \mid \eta \geqslant f(x)\}$$

is convex. This easily allows for the possibility that the *domain*

$$\operatorname{dom} f := \{x \in \mathbb{E}^n \mid f(x) < +\infty\}$$

is not the whole of $\mathbb{E}^n$; observe that $\operatorname{dom} f$ is always convex. Our convex functions will be *proper*, in that $\operatorname{dom} f \neq \emptyset$, and $f(x) \neq -\infty$ for any $x \in \mathbb{E}^n$.

A convex function f is called *locally polyhedral* if the restriction $f|_Q$ of f to any convex polytope $Q \subseteq \operatorname{dom} f$ is *polyhedral*, so that $\operatorname{epi} f|_Q = \operatorname{epi} f \cap (Q \times [0, \infty))$ is a polyhedral set (see [8]). We call f *strongly locally polyhedral* if, in addition, the regions on which f restricts to a (finite) affine function are compact. Then we have the following fundamental result.

THEOREM 7.1. *To each pegged tiling $\mathscr{T}$ in $\mathbb{E}^n$ corresponds a strongly locally* *polyhedral convex function f; the tiles of $\mathscr{T}$ are the maximal regions of $\mathbb{E}^n$ on which f is affine.*

Once again, we remind the reader of our use of the term tiling. We remark that the converse of Theorem 7.1 is obvious.

Our proof of Theorem 7.1 proceeds in several steps. We pick any fixed tile in $\mathscr{T}$, and call it P_0; we then define $\varphi_0^* := 0$. For any other tile P_j, there is a chain $P_0 = P_{k_0}, P_{k_1}, \ldots, P_{k_r} = P_j$, such that $P_{k_{s-1}}$ and P_{k_s} are adjacent for $s = 1, \ldots, r$. With the α_{ij} as before, we now define

$$\varphi_j^* := \sum_{s=1}^{r} \alpha_{k_{s-1}k_s}.$$

LEMMA 7.2. *The numbers φ_j^* are well defined.*

This follows from the fact that, if (with changed notation) $P_0, P_1, \ldots, P_j = P_0$ is a closed chain of tiles containing a fixed $(n-2)$-face G, then for any $x \in G$, we have

$$\langle x, v_i^* - v_{i-1}^* \rangle = \alpha_{i-1,i},$$

so that $\Sigma_{i=1}^{j} \alpha_{i-1,i} = 0$. In the general case (of a closed chain of tiles with empty intersection), we argue as in the proof of Theorem 3.1. Such a chain gives rise to a loop which can be contracted to a point avoiding $(n-3)$-faces of $\mathscr{T}$, and in contracting over a $(n-2)$-face we can appeal to the previous case. □

At this point, it is appropriate to comment on [2]. The definition adopted there is a local Laguerre condition: it only applies to adjacent tiles. However, the condition is equivalent to ours. The proof, though, effectively proceeds only as far as establishing (the analogue of) the first part of Lemma 7.2 (where the tiles have a common intersection); the general case is not discussed. What follows below is also lacking; in fact, the local convexity property is clear in the alternative formulation, but is not specifically mentioned.

We now define our function f by $f(x) := \langle x, v_j^* \rangle - \varphi_j^*$ if $x \in P_j$. Certainly, f is continuous, and further has a restricted local convexity property, in that, if P_i and P_j are adjacent tiles, then for $x \in P_i$ we have

$$\langle x, v_j^* - v_i^* \rangle \leqslant \alpha_{ij} = \varphi_j^* - \varphi_i^*,$$

with strict inequality if $x \in \operatorname{int} P_i$, so that

$$f(x) = \langle x, v_i^* \rangle - \varphi_i^* \geqslant \langle x, v_j^* \rangle - \varphi_j^*.$$

However, this restricted property, together with local finiteness of $\mathscr{T}$, implies that f is convex in the usual sense, since a line segment lying in $\operatorname{dom} \mathscr{T}$ is the limit of such line segments which do not meet $(n-2)$-faces of $\mathscr{T}$, and it is clear that f is convex on such a segment.

The remaining properties of a strongly polyhedral function are obvious.

□

The relationship between tiling and function given above is described by saying that f is *lifted* from $\mathscr{T}$, or $\mathscr{T}$ is *dropped* from f, and that f and $\mathscr{T}$ are *associated*. Note that f drops to a unique tiling $\mathscr{T}$, whereas $\mathscr{T}$ can lift into many different functions f. We shall consider the latter point in more detail below.

As we remarked above, Aurenhammer ([1]) showed that Laguerre tilings admit such liftings to convex functions. If we allow Laguerre tilings to have domains other than $\mathbb{E}^n$ itself, what we have done actually proves that this result is the most general possible.

THEOREM 7.3. *A tiling $\mathscr{T}$ in $\mathbb{E}^n$ can be lifted to a convex function is and only if it is a Laguerre tiling. In particular, pegged tilings are Laguerre tilings.*

We have just seen that pegged tilings do lift to convex functions, and Laguerre tilings are pegged. But conversely, if the tiling $\mathscr{T}$ does lift to a convex function f, then corresponding to each tile $P_i \in \mathscr{T}$ are a point $v_i^* \in \mathbb{E}^n$ and a number $\varphi_i^* \in \mathbb{R}$, such that $f(x) = \langle x, v_i^* \rangle - \varphi_i^*$ for $x \in P_i$. If we now define the weight ω_i by

$$\omega_i := \|v_i^*\|^2 - 2\varphi_i^*,$$

then this expresses $\mathscr{T}$ as a Laguerre tiling with sites v_i^* and these weights.

□

We finally remark here that, although we have demanded that our tiles be compact, this is not necessary for the validity of the proof above (when we change the language appropriately).

8. Conjugacy and Duality

Let f be a closed convex function on $\mathbb{E}^n$. The *conjugate* f^* of f is defined by

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) \mid x \in \mathbb{E}^n\}$$

(see Rockafellar [19]). Then f^* is also a closed convex function, and $f^{**}(:= (f^*)^*) = f$. Actually, the conjugate is defined for any function f on $\mathbb{E}^n$; f^* is still closed and convex, and

$$f^{**} = \operatorname{cl}\operatorname{conv} f := \sup\{g \mid g \text{ is closed and convex and } g \leqslant f\}.$$

Intimately connected with conjugacy are the notions of subgradient and subdifferential. We call a^* a *subgradient* of the convex function f at the point a at which f is finite if

$$f(x) \geqslant f(a) + \langle x - a, a^* \rangle$$

for all $x \in \mathbb{E}^n$. The family of subgradients of f at a is called the *subdifferential* of f at a, and is denoted $\partial f(a)$. Note that $a^* \in \partial f(a)$ precisely when $(a^*, -1)$ is an outer normal vector to a support hyperplane of epi f at the point $(a, f(a))$, and hence it is easy to see that $\partial f(a)$ is a convex set, which is closed and non-empty if $a \in \operatorname{relint} \operatorname{dom} f$. A crucial relationship to which we need to refer is (see [19]):

LEMMA 8.1. *Let f be a closed convex function on $\mathbb{E}^n$, and suppose that $f(a)$ is finite. Then the following are equivalent:*

(a) $a^* \in \partial f(a)$;
(b) $a \in \partial f^*(a^*)$;
(c) $f(a) + f^*(a^*) = \langle a, a^* \rangle$.

Our first result using conjugacy is

THEOREM 8.2. *Let f be a strongly locally polyhedral convex function on $\mathbb{E}^n$, and let $\mathscr{T}$ be the tiling associated with f. Then the conjugate f^* of f is also strongly locally polyhedral, and the tiling $\mathscr{T}^*$ associated with f^* is strongly dual to $\mathscr{T}$.*

In fact, f^* has the following explicit description, which shows why it is strongly locally polyhedral. As usual, if V^* is the peg-set of $\mathscr{T}$, and φ_j^* is defined as before for each $v_j^* \in V^*$, then

$$f(x) = \sup\{\langle x, v_j^* \rangle - \varphi_j^* \mid v_j^* \in V^*\}$$

shows that $f^* = \operatorname{cl} \operatorname{conv} g^*$, where

$$g^*(x^*) = \begin{cases} \varphi_j^* & \text{if } x^* = v_j^* \text{ for some } j, \\ +\infty & \text{otherwise.} \end{cases}$$

Again as before, we let $V := \operatorname{vert} \mathscr{T}$, and for $v_i \in V$, define $\varphi_i := f(v_i)$. If $P_j \in \mathscr{T}$ is any tile containing v_i, then

$$\varphi_i = f(v_i) = \langle v_i, v_j^* \rangle - \varphi_j^*,$$

so that

$$f^*(v_j^*) = \langle v_j^*, v_i \rangle - \varphi_i$$

for each such peg v_j^*. If $P_i^* := \hat{v}_i$ is the tile of $\mathscr{T}^*$ corresponding to v_i, we thus see that

$$f^*(x^*) = \langle x^*, v_i \rangle - \varphi_i$$

for all $x^* \in P_j^*$. It follows that $\mathcal{T}^*$ lifts to f^*, and since $\mathcal{T}^*$ is strongly dual to $\mathcal{T}$, this completes the proof. □

The subdifferential relationships of the lemma can be used to show what is, perhaps, now fairly intuitive, that if F is a face of $\mathcal{T}$, and if $a \in \operatorname{relint} F$, then $\partial f(a) = \hat{F}$, the corresponding face of $\mathcal{T}^*$. In particular, $\partial f(v_i) = P_i^*$ for $v_i \in \operatorname{vert} \mathcal{T}$, and similarly, $P_j = \partial f^*(v_j^*)$ for a peg $v_j^* \in V^*$.

We have already remarked that a pegged tiling $\mathcal{T}$ does not lift to a unique convex function f; indeed, the lifting is determined by the peg-set, and this can vary, for example, by replacing it by a homothetic copy of itself. We now consider the corresponding effects on convex functions.

We begin by introducing some operations on convex functions; we shall make further use of these operations later. Let f, g be convex functions on $\mathbb{E}^d$, and let $\lambda > 0$. The *sum* $f + g$ and *left scalar multiple* λf are defined in the usual way; the *infimal convolution* $f \,\Box\, g$ is defined by

$$(f \,\Box\, g)(x) := \inf\{f(y) + g(z) \mid y + z = x\},$$

and the *right scalar multiple* $f\lambda$ is defined by

$$(f\lambda)(x) := \lambda f(\lambda^{-1}x).$$

In terms of epigraphs, the last two are given by

$$\operatorname{epi}(f \,\Box\, g) := \operatorname{cl}(\operatorname{epi} f + \operatorname{epi} g); \quad \operatorname{epi}(f\lambda) := \lambda(\operatorname{epi} f).$$

Conjugacy interacts with these operations in the following way.

LEMMA 8.3. *Let f, g be closed convex functions on $\mathbb{E}^n$, and let $\lambda > 0$. Then*

$$(f + g)^* = f^* \,\Box\, g^*, \quad (f \,\Box\, g)^* = \operatorname{cl}(f^* + g^*),$$

$$(\lambda f)^* = f^*\lambda, \quad (f\lambda)^* = \lambda f^*.$$

The sum of any two convex functions which occur here will actually be closed, so the closure condition for $(f \,\Box\, g)^*$ can always be dropped.

For the moment, we only need special cases of such operations. The *convex indicator* function $\delta(K, \cdot)$ of a convex set K in $\mathbb{E}^n$ is given by

$$\delta(K, x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

We shall write $\delta(a, \cdot)$ rather than $\delta(\{a\}, \cdot)$ if $K = \{a\}$ consists of a point. Then we have

LEMMA 8.4. (a) *If $a^* \in \mathbb{E}^n$ and $\alpha \in \mathbb{R}$, then*

$$(\langle \cdot, a^* \rangle + \alpha)^* = \delta(a^*, \cdot) - \alpha.$$

(b) *If L is a linear subspace of $\mathbb{E}^n$, then*

$$\delta(L, \cdot)^* = \delta(L^\perp, \cdot),$$

where $L^\perp$ is the orthogonal complement of L.

Now, if f is one lift of the pegged tiling $\mathscr{T}$, then so is $\lambda f + \langle \cdot, a^* \rangle + \alpha$, for any $\lambda > 0$, $a^* \in \mathbb{E}^d$ and $\alpha \in \mathbb{R}$. Its conjugate is then $f^* \lambda \,\Box\, (\delta(a^*, \cdot) - \alpha) = g^*$, say, where

$$g^*(x^*) = \lambda f^*(\lambda^{-1}(x^* - a^*)) - \alpha,$$

which we recognize as a lift of the tiling $\lambda\mathscr{T}^* + a^*$ homothetic to $\mathscr{T}^*$. (Since adding a constant to a function results in subtracting the same constant from its conjugate, we shall henceforth ignore such changes in lift.)

9. Tilings of the Whole Space

It is convenient here to change a definition of Rockafellar [19] somewhat, and call a vector $y \in \mathbb{E}^n$ a *direction of recession* of the convex function f if f is bounded above by an affine function on $\{x + \lambda y \mid \lambda \geqslant 0\}$ for some (and hence for all) $x \in \operatorname{relint} \operatorname{dom} f$. This means that (y, α) is a recession vector of $\operatorname{epi} f$ for some $\alpha \in \mathbb{R}$. In this context, we recall that z is a *recession vector* of the convex set K if $x + \lambda z \in K$ for some $x \in \operatorname{relint} K$ and all $\lambda \geqslant 0$; the set of recession vectors of K is denoted $\operatorname{rec} K$, and is called the *recession cone* of K. The *recession function* $\operatorname{rec} f$ of f is then the convex function defined by

$$\operatorname{epi}(\operatorname{rec} f) := \operatorname{rec}(\operatorname{epi} f).$$

The function $\operatorname{rec} f$ is denoted $f0^+$ in [19]. Note that the zero vector o is always a (trivial) direction of recession of f.

Now (see [19]) $\operatorname{rec} f$ is the support function of $\operatorname{dom} f^*$, and so, interchanging the roles of f and f^*, we have immediately

THEOREM 9.1. *If $\mathscr{T}$ is the tiling associated with the strongly locally polyhedral convex function f, then $\mathscr{T}$ tiles the whole of space if and only if the conjugate function f^* has no non-zero directions of recession.*

It is helpful to have an alternative description of this situation. We keep to the notation introduced in the previous section.

THEOREM 9.2. *Let $\mathscr{T}$ be a pegged tiling, with peg-set V^*. Then* dom $\mathscr{T} = \mathbb{E}^n$ *if and only if, for every sequence* $(v_j^* \mid j \in \mathbb{N})$ *in* V^* *with* $\|v_j^*\| \to \infty$,

$$\frac{\varphi_j^*}{\|v_j^*\|} \to \infty.$$

First, suppose that dom $\mathscr{T} = \mathbb{E}^n$, that is, that f^* has no non-zero directions of recession. We show that the condition of the theorem must hold.

Suppose, if possible, that it does not. Then we have some sequence $(v_j^* \mid j \in \mathbb{N})$ in V^* and some constant μ, such that $\|v_j\| \to \infty$, but $\varphi_j^*/\|v_j^*\| \leqslant \mu$. By compactness of the unit sphere, the sequence $(\|v_j^*\|^{-1} v_j^* \mid j \in \mathbb{N})$ has a convergent subsequence, which we can identify with the original sequence, and so assume that

$$\lim_{j \to \infty} \|v_j^*\|^{-1} v_j^* =: y^*$$

exists. Let $x^* \in$ relint dom f^* be arbitrary. For any j and any $\lambda > 0$, we have

$$x^* + \lambda y^* = \lim_{j \to \infty} \left(\left(1 - \frac{\lambda}{\|v_j^*\|} \right) x^* + \frac{\lambda}{\|v_j^*\|} v_j^* \right),$$

so that, using the continuity of a convex function at a relatively interior point of its domain, we have

$$\begin{aligned} f^*(x^* + \lambda y^*) &= \lim_{j \to \infty} f^* \left(\left(1 - \frac{\lambda}{\|v_j^*\|} \right) x^* + \frac{\lambda}{\|v_j^*\|} v_j^* \right) \\ &\leqslant \lim_{j \to \infty} \left(\left(1 - \frac{\lambda}{\|v_j^*\|} \right) f^*(x^*) + \frac{\lambda}{\|v_j^*\|} f^*(v_j^*) \right) \\ &\leqslant \lim_{j \to \infty} \left(\left(1 - \frac{\lambda}{\|v_j^*\|} \right) f^*(x^*) + \lambda \mu \right) \\ &= f^*(x^*) + \lambda \mu \end{aligned}$$

since $f^*(v_j^*) = \varphi_j^*$. It follows that y^* is a direction of recession of f^*, contrary to assumption.

For the converse, suppose that the condition of the theorem holds. Thus, for every $\mu > 0$, there exists a $\rho > 0$, such that whenever $v_j^* \in V^*$ satisfies $\|v_j^*\| > \rho$, then $\varphi_j^* > \mu \|v_j^*\|$. It follows that $\inf f^* =: \alpha^*$, say, is finite, and hence that epi $f^* \subseteq K_\rho + C_\mu$, where

$$K_\rho := \{(x^*, \alpha^*) \in \mathbb{E}^{d+1} \mid \|x^*\| \leqslant \rho\}$$

$$C_\mu := \{(x^*, \eta^*) \in \mathbb{E}^{d+1} \mid \eta^* \geqslant \mu \|x^*\|\}.$$

Thus the recession cone of epi f^* lies in C_μ; since μ is arbitrary, this shows that f^* has no non-zero directions of recession, and hence that dom $\mathscr{T} = \mathbb{E}^n$. This proves the theorem. □

We remark that Schlottmann [20] has a different criterion (expressed in terms of Laguerre tilings) which ensures that both $\mathscr{T}$ and $\mathscr{T}^*$ tile the whole of $\mathbb{E}^n$.

10. Operations on tilings

We now consider various operations on tilings, which correspond to the operations on convex functions which we described in Section 8. Since left scalar multiplication (by a positive number) of a convex function associated with a tiling gives another function associated with the same tiling, and right scalar multiplication merely leads to a homothetic tiling, we can confine our attention to the results of taking sums and infimal convolutions of such convex functions.

We begin with the sum. If the tilings $\mathscr{T}_1$ and $\mathscr{T}_2$ are such that dom $\mathscr{T}_1 \cap$ dom $\mathscr{T}_2 \neq \varnothing$, and if $\mathscr{T}_i$ lifts to the convex function f_i for $i = 1, 2$, then $f_1 + f_2$ gives rise to a tiling which we denote $\mathscr{T}_1 \wedge \mathscr{T}_2$, and call the *meet* of $\mathscr{T}_1$ and $\mathscr{T}_2$. The reason for the name is clear from its description–the tiles of $\mathscr{T}_1 \wedge \mathscr{T}_2$ are just the intersections of tiles of $\mathscr{T}_1$ and $\mathscr{T}_2$ which have non-empty interiors, since these are the maximal regions on which $f_1 + f_2$ is affine. (We use a slightly different name and symbol from the ordinary intersection, to avoid confusion with this.) We observe that the tiling $\mathscr{T}_1 \wedge \mathscr{T}_2$ does not depend upon the particular functions to which $\mathscr{T}_1$ and $\mathscr{T}_2$ are lifted.

The situation is quite different for the infimal convolution; we shall make use of this fact in the next section as well as in this. Let the tiling $\mathscr{T}_i$ lift to f_i $(i = 1, 2)$ as before. Then the tiling corresponding to $f_1 \,\Box\, f_2$ is denoted $\mathscr{T}_1 + \mathscr{T}_2$, and is called a *sum* of $\mathscr{T}_1$ and $\mathscr{T}_2$. Again, the reason for the terminology is clear. Since the epigraph of $f_1 \,\Box\, f_2$ is epi f_1 + epi f_2, we see that the tiles of $\mathscr{T}_1 + \mathscr{T}_2$ are sums $F_1 + F_2$ of faces F_i of $\mathscr{T}_i$ for $i = 1, 2$. In the most interesting case, when the tilings $\mathscr{T}_1$ and $\mathscr{T}_2$ are in general position relative to each other (or, rather, when the epigraphs of the associated convex functions are), the dimensions of the faces F_1 and F_2 are complementary.

That the sum does depend on the choice of liftings is obvious. Indeed, we have

THEOREM 10.1. *For $i = 1, 2$, let the tiling $\mathscr{T}_i$ lift to the convex function f_i, let F_i be a face of $\mathscr{T}_i$, and let $a_i \in \operatorname{relint} F_i$. Then $F_1 + F_2$ is a tile of the corresponding sum $\mathscr{T}_1 + \mathscr{T}_2$ if and only if $\dim(F_1 + F_2) = d$, and $\partial f_1(a_1) \cap \partial f_2(a_2) \neq \varnothing$.*

The theorem just expresses the condition for the epigraphs of f_i to have parallel support hyperplanes at $(a_i, f_i(a_i))$ for $i = 1, 2$. This condition is not usually preserved when f_i is replaced by a different lifting.

11. Sections and projections

We now set the projection method in the context of convex functions; to do this, we just translate the material of Sections 5 and 6 into the appropriate language.

We can clearly relate a tiling of an affine subspace of $\mathbb{E}^n$ to a strongly locally polyhedral convex function whose domain is not full dimensional. However, the conjugate of such a function is not strongly locally polyhedral; instead, its epigraph is the sum of that of a strongly locally polyhedral convex function whose domain has the same dimension, and a certain affine subspace. More precisely, suppose that g is a convex function, whose domain spans a linear subspace $L := \operatorname{aff} \operatorname{dom} g$. If $a \in L^\perp$, the orthogonal complement of L (this is general enough for our purposes), then the domain of $g \,\square\, \delta(a, \cdot)$ spans the translate $M := L + a$ of L. The conjugate of $g \,\square\, \delta(a, \cdot)$ is then

$$(g \,\square\, \delta(a, \cdot))^* = g^* + \langle a, \cdot \rangle,$$

whose epigraph is the sum of that of $g^*|_L$ (which is the conjugate of g calculated in L), and the affine subspace

$$\tilde{M} := \{(x^*, \langle x^*, a \rangle) \mid x^* \in L^\perp\}.$$

More generally, let f be a strongly locally polyhedral convex function, associated with the tiling $\mathscr{T}$ of $\mathbb{E}^n$, let L be a linear subspace of $\mathbb{E}^n$, and let $a \in L^\perp$ be such that $M \cap \operatorname{dom} \mathscr{T} \neq \varnothing$, where $M := L + a$. Then $f + \delta(M, \cdot)$ is a convex function associated with $M \cap \mathscr{T}$. We can work out its conjugate as above, obtaining

$$\begin{aligned}(f + \delta(M, \cdot))^* &= f^* \,\square\, \delta(M, \cdot)^* \\ &= f^* \,\square\, (\delta(L, \cdot) \,\square\, \delta(a, \cdot))^* \\ &= f^* \,\square\, (\delta(L^\perp, \cdot) + \langle a, \cdot \rangle),\end{aligned}$$

whose epigraph is $\operatorname{epi} f^* + \tilde{M}$, where $\tilde{M}$ is defined as above. We can now associate with this conjugate a tiling of L itself; its lifting is just the sum of this function with $\delta(L, \cdot)$, which restricts it to L. This, then, gives the projection method in terms of associated convex functions.

There is an appealing picture of what is happening here. The epigraph of the final function

$$(f^* \,\square\, (\delta(L^\perp, \cdot) + \langle a, \cdot \rangle)) + \delta(L, \cdot)$$

is just the projection of epi f^* on $L \times \mathbb{R}$ in direction $\tilde{M}$, or, more picturesquely, what we see when we look at epi f^* along $\tilde{M}$. As we vary a (keeping L fixed), we tilt $\tilde{M}$, and so change the viewpoint.

As a final remark in this section, if the hereditary nature of the projection method were not already apparent from Section 6, it should now be obvious.

12. Grid methods

We shall now discuss several kinds of grid methods in the context of convex functions. Bohne *et al.* [4] first made the observation that related the original grid construction of de Bruijn [5] to the epigraph of a certain convex function. We shall exhibit this and other grid constructions as particular cases of sums of tilings.

The general idea of a grid method is the following. We overlay (that is, take the meet of) several copies in different orientations of a fixed tiling (which may degenerate in some way, for example, to a set of parallel strips), and then take the strong dual. (Previous descriptions have usually been somewhat less succinct.) This dual is then, as we have seen above, a sum of the duals of the component tilings; in degenerate cases, these tilings may be lower dimensional.

We begin with planar tilings. The original grid method of [5] took five families of tilings by equal strips, rotated relative to each other by multiples of $2\pi/5$. If these tilings are in general position (relative to each other), then the resulting dual is a tiling by rhombi of angles $\pi/5$ and $2\pi/5$. In the present context, we think of this tiling as a sum of tilings of lines, in these five directions, by equal line segments. Similar sums of other numbers of linear tilings rotated by angles such as $\pi/4$ (see also immediately below), $\pi/6$, and so on, have also appeared in the literature.

There are three regular tessellations of the plane–by squares, equilateral triangles, and regular hexagons. If we take two equal copies of the square tessellation, rotate one with respect to the other by $\pi/4$, and take a sum, we obtain a quasi-periodic tiling of squares and rhombi of angle $\pi/4$. Such tilings were discovered (independently) by Ammann and Beenker; they can also be obtained as sums of four linear tilings. Similarly, we may take two equal tessellations of triangles (or hexagons), rotate one against the other by $\pi/6$ (or $\pi/2$), and take a general sum. The resultings tilings consist of triangles (or hexagons), together with squares and rhombi of angle $\pi/6$. Tilings of this kind have been described by Stampfli ([21]), and with different rotation angles by Niizeki ([15]–[17]).

There are two analogous kinds of four-dimensional tilings. For the first, we begin with the regular honeycomb $\{3, 3, 4, 3\}$, whose tiles are regular cross-polytopes (analogues of the octahedron). We can rotate a second copy of this, so that the new vertex-figure is in reciprocal position to the first.

A general sum of these tilings has tiles which are regular cross-polytopes, two kinds of prisms on regular tetrahedra (right and oblique), and two kinds of direct sum of equilateral triangles (again, right and oblique). These tilings are obtained in a different way in McMullen ([14]), where it is shown that they are quasi-periodic. The second kind of tiling in $\mathbb{E}^4$ is obtained in a similar way from the dual honeycomb $\{3, 4, 3, 3\}$; this time, the tiles are 24-cells $\{3, 4, 3\}$, two kinds of prism on octahedra, and two kinds of direct sum of triangles.

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