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Asymptotic Expansion of Feynman Amplitudes

Part I – The Convergent Case*

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Abstract. Employing the technique of Mellin transforms to scalar convergent Feynman amplitude in the Schwinger integral representation, we determine its asymptotic expansion for large Euclidean momenta.

The determination of the coefficients of the expansion is effected via the use of generalized Taylor operators.

I. Introduction

Early in 1960, Weinberg [1] presented the famous power counting theorem to determine the asymptotic behaviour of a convergent scalar Feynman amplitude. This theorem was later extended by Fink [2] to further information on the logarithmic behaviour of the leading power. Although, this theorem is believed to be also valid for divergent graphs and is widely applied to obtain various properties of high energy physics, no successful attempt was ever made to establish it in this case. Moreover the studies of gauge fields have demonstrated that a generalization of this theorem is necessary to study those physical situations where some but not all masses are zero.

This communication is the first of a series of papers devoted to these generalizations. Applying the properties of Mellin transforms [3] on the Schwinger integral representation of Feynman amplitude, we will establish the technique of determining the asymptotic expansion of a scalar convergent graph and calculate its coefficients. The main tool to perform the analytic continuation of Mellin transforms is a generalization of the τ operators first introduced in Ref. [4] for the purpose of renormalization. In subsequent papers, this method will be generalized to divergent graphs and to the case where some masses and some momenta become large.

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The rest of this section is devoted to introduce to the reader the Schwinger representation of Feynman amplitudes, and to emphasize the pertinent properties required for subsequent discussion. In Section II, we shall adapt the technique of Mellin transforms for our purpose, while Section III will be divided into three parts – to discuss the meromorphic structure of Mellin transforms of Feynman graphs, to extract the poles corresponding to the dominant behaviour for large momenta, and to convert the information to the asymptotic expansion. Two Appendices are reserved for discussion of some properties of τ operators. A third Appendix is devoted to some technical details.

For a scalar Feynman graph G, which has *Euclidean* external momenta p, l lines, n vertices, L loops and incidence matrix ε_{ia} [1], the Feynman amplitude I_G is defined by

$$\delta^{(D)}\left(\sum_{i=1}^{n} p_i\right)\pi^{LD/2} I_G(p,m)=\int \prod_{a=1}^{l} d^D k_a \prod_{a=1}^{l} (k_a^2+m_a^2)^{-1} \prod_{i=1}^{n} \delta^{(D)}(p_i-\varepsilon_{ia}k_a)\,, \tag{I.1}$$

where D is the dimension of space with positive metric. The Schwinger integral representation for I_G is

$$I_G(p,m)=\int_0^\infty \prod_{a=1}^{l} d\alpha_a e^{-\sum_{a=1}^{l}\alpha_a m_a^2}\, e^{-\sum_{i,j=1}^{n-1} p_i [d_G^{-1}(\alpha)]_{ij} p_j}\, P_G(\alpha)^{-D/2}\,. \tag{I.2}$$

Here $P_G(\alpha)$ is a homogeneous polynomial of degree L in the α's and is defined as follows. Let $I=\{1,\ldots,l\}$ and define χ to be the set of all subsets $\mathscr{S}$ of I such that cutting the ith line for all i in $\mathscr{S}$ reduces the graph G to a *single* connected treegraph. Then $P_G(\alpha)$ is defined to be

$$P_G(\alpha)=\sum_{\mathscr{S}\in\chi}\prod_{i\in\mathscr{S}}\alpha_i\,. \tag{I.3}$$

The matrix $[d_G^{-1}(\alpha)]_{ij}$ is the ratio N_{ij}/P_G of two polynomials where N_{ij} is a homogeneous polynomial of degree $L+1$ in the α's and is defined as follows. Let $J=\{1,\ldots,n-1\}$; for any subset T of J we define Ψ_T to be the set of all subsets $\mathscr{S}$ of I, such that cutting the ith line for all i in $\mathscr{S}$ reduces G to two connected tree-graphs, one with external momenta p_j, $j\in T$, and the other with the rest of the external momenta. Then N_{ij} is defined by

$$p_i N_{ij}(\alpha)\, p_j=\sum_{T\subseteq J}\left(\sum_{j\in T} p_j\right)^2 \sum_{\mathscr{S}\in\Psi_T}\prod_{i\in\mathscr{S}}\alpha_i\,. \tag{I.4}$$

To any subset $\mathscr{S}$ of integers in I, we associate a subdiagram, that is a set of lines *and vertices*. It is well-known that $P_G(\alpha)$ and $d_G^{-1}(\alpha)$ have

[1] $\varepsilon_{ia}=+1$ if the line a points away from the vertex i, $\varepsilon_{ia}=-1$ if the line a points into the vertex i, and 0 otherwise.

the following properties:

1. $p_i N_{ij}(\alpha)\, p_j \geqq 0 \quad \forall \alpha \geqq 0$. (I.5)
2. $P_G(\alpha) \geqq 0 \quad \forall \alpha \geqq 0$, (I.6)
3. given a subdiagram $\mathscr{S} \subsetneqq I$,

$$P_G(\alpha)\Big|_{\substack{\alpha_i = \varrho\ \forall i \in \mathscr{S} \\ \alpha_i \neq 0\ \forall i \notin \mathscr{S}}} = \mathcal{O}(\varrho^{L(\mathscr{S})}) \tag{I.7}$$

$$p d_G^{-1}(\alpha)\, p\Big|_{\substack{\alpha_i = \varrho\ \forall i \in \mathscr{S} \\ \alpha_i \neq 0\ \forall i \notin \mathscr{S}}} = \begin{cases} \mathcal{O}(\varrho) & \text{if } \mathscr{S} \text{ is an essential subdiagram} \\ \mathcal{O}(\varrho^0) & \text{otherwise}. \end{cases} \tag{I.8}$$

In (I.7), $L(\mathscr{S})$ is the number of loops of the subdiagram $\mathscr{S}$, and in (I.8) an essential subdiagram is a subdiagram that alone conserves all the external momenta of G. From the above properties it can be shown that the integral in (I.2) is convergent if, for any $\mathscr{S} \subsetneqq I$, $\omega(\mathscr{S}) < 0$, where the superficial degree of divergence $\omega(\mathscr{S})$ is defined as

$$\omega(\mathscr{S}) = L(\mathscr{S})\, D - 2l(\mathscr{S}) \tag{I.9}$$

and $l(\mathscr{S})$ is the number of lines in $\mathscr{S}$.

The concepts of union, intersection and inclusion of subdiagrams will be understood as those of sets of lines and vertices. Two subdiagrams $\mathscr{S}_1$, and $\mathscr{S}_2$ are disjoint if their intersection is empty. If two subdiagrams are neither disjoint nor such that one is inside the other, they are said to overlap. A forest is a set of nonoverlapping subdiagrams. Given two subdiagrams $\mathscr{S}$ and $\mathscr{S}'(\subsetneqq \mathscr{S})$, the reduced diagram $\mathscr{S}/\mathscr{S}'$ is the diagram obtained from $\mathscr{S}$ by shrinking $\mathscr{S}'$ to a point. The functions $P_G(\alpha)$ and $p d_G^{-1}(\alpha)\, p$, have a power expansion in the dilatation variables corresponding to the subdiagrams of any forest, after all common factors have been removed. Finally, let us close this section by quoting two well-known important properties [5]. Given a subdiagram $\mathscr{S}$, let us dilate all α_i's, $i \in \mathscr{S}$, by μ in the functions $P_G(\alpha)$ and $p d_G^{-1}(\alpha)\, p$ and obtain $P_G(\alpha, \mu)$ and $p d_G^{-1}(\alpha, \mu)\, p$ respectively. Defining $P'_G(\alpha, \mu)$ by

$$P_G(\alpha, \mu) = \mu^{L(\mathscr{S})} P'_G(\alpha, \mu), \tag{I.10}$$

we have

$$P'_G(\alpha, 0) = P_{\mathscr{S}}(\alpha)\, P_{G/\mathscr{S}}(\alpha). \tag{I.11}$$

If $\mathscr{S}$ is an essential subdiagram, we define $d'_G(\alpha, \mu)$ by

$$p d_G^{-1}(\alpha, \mu)\, p = \mu p d'_G(\alpha, \mu)\, p, \tag{I.12}$$

and we find

$$p d'_G(\alpha, 0)\, p = p d_{\mathscr{S}}^{-1}(\alpha)\, p. \tag{I.13}$$

II. Mellin Transform

To determine the asymptotic expansion of a function $\varphi(\lambda)$ as λ tends to infinity we use the technique of Mellin transforms [6]. Given a function $\varphi(\lambda)$, which is finite in the region $0 \leqq \lambda < \infty$ and which possesses the asymptotic expansion

$$\varphi(\lambda) \sim \sum_{p=p_{\max}}^{-\infty} \sum_{q=0}^{q_{\max}(p)} a_{pq} \lambda^p (\ln \lambda)^q , \tag{II.1}$$

we define the Mellin transform $M(x)$ in the interval $A < \operatorname{Re} x < A+1$, where A is an integer $\geqq p_{\max}$, to be

$$M(x) = \int_0^\infty d\lambda \lambda^{-x-1} (1 - T_\lambda^A) \varphi(\lambda) . \tag{II.2}$$

In this equation T_λ^A is the Taylor operator defined by

$$T_\lambda^A \varphi(\lambda) = \sum_{n=0}^{A} \frac{1}{n!} \varphi^{(n)}(0) \lambda^n . \tag{II.3}$$

To continue $M(x)$ to the entire complex x-plane, we first split $M(x)$ into two pieces

$$M(x) = M_1(x) + M_2(x) \tag{II.4}$$

with

$$M_1(x) = \int_0^1 d\lambda \lambda^{-x-1} (1 - T_\lambda^A) \varphi(\lambda) \tag{II.5a}$$

and

$$M_2(x) = \int_1^\infty d\lambda \lambda^{-x-1} (1 - T_\lambda^A) \varphi(\lambda) . \tag{II.5b}$$

The functions $M_1(x)$ and $M_2(x)$ are analytic respectively in the region $\operatorname{Re} x < A+1$ and $\operatorname{Re} x > A$. Now, we continue $M_1(x)$ to the region $\operatorname{Re} x < A+n+1$, where n is an integer $\geqq 1$, by separating the integrand into two parts

$$M_1(x) = \int_0^1 d\lambda \lambda^{-x-1} (1 - T_\lambda^{A+n}) \varphi(\lambda) - \sum_{i=A+1}^{A+n} \frac{\varphi^{(i)}(0)}{i!} \frac{1}{x-i} . \tag{II.6}$$

Since n can be chosen arbitrarily large, equation (II.6) shows that the analytic continuation $\bar{M}_1(x)$ of $M_1(x)$ is analytic everywhere except for *simple* poles at $x = A+1, A+2, \ldots$. Next, we continue $M_2(x)$. For any integer $m \geqq 1$, we define for the function $\varphi(\lambda)$ given in (II.1) the analog W^m of the "Taylor" operator at ∞ by

$$W^m \varphi(\lambda) = \sum_{p=m}^{p_{\max}} \sum_{q=0}^{q_{\max}(p)} a_{pq} \lambda^p (\ln \lambda)^q . \tag{II.7}$$

For $\operatorname{Re} x > A$,

$$M_2(x) = \int_1^\infty d\lambda \lambda^{-x-1} (1 - W^{A-m+1}) (1 - T_\lambda^A) \varphi(\lambda) + \sum_{p=A-m+1}^{A} \sum_{q=0}^{q'_{\max}(p)} a'_{pq} q! \frac{1}{(x-p)^{q+1}}, \tag{II.8}$$

where

$$q'_{\max}(p) = \begin{cases} q_{\max}(p) & \text{for} \quad p \leqq p_{\max} \\ 0 & \text{for} \quad p > p_{\max}, \end{cases} \tag{II.9}$$

and

$$a'_{pq} = \begin{cases} a_{pq} & \text{for} \quad q \neq 0 \quad \text{or} \quad p < 0 \\ a_{p0} - \varphi^{(p)}(0) & \text{for} \quad q = 0 \quad \text{and} \quad 0 \leqq p \leqq p_{\max} \\ -\varphi^{(p)}(0) & \text{for} \quad q = 0 \quad \text{and} \quad p \geqq \operatorname{Sup}\{0, p_{\max}+1\}. \end{cases} \tag{II.10}$$

The integral in (II.8) exists for $\operatorname{Re} x > A - m$. Hence the analytic continuation $\bar{M}_2(x)$ of $M_2(x)$ is analytic everywhere except for *simple* poles at $x = A$, $A-1, \ldots, \operatorname{Sup}\{p_{\max}+1, 0\}$, and *multiple* poles at $x = p_{\max}$, $p_{\max} - 1, \ldots$. Thus, the analytic continuation $\bar{M}(x)$ of $M(x)$ is

$$\bar{M}(x) = \bar{M}_1(x) + \bar{M}_2(x). \tag{II.11}$$

The function $\bar{M}(x)$ is analytic everywhere in the complex x-plane except for *multiple* poles at $x = p_{\max}, p_{\max} - 1, \ldots$, and *simple* poles at $x = \operatorname{Sup}\{p_{\max}+1, 0\}, \operatorname{Sup}\{p_{\max}+1, 0\}+1, \ldots$.

Let us define

$$F(x) = \frac{\bar{M}(x)}{\Gamma(-x)}, \tag{II.12}$$

which is analytic everywhere except for multiple poles at

$$x = p_{\max}, \quad p_{\max} - 1, \ldots.$$

Since, from (II.6) and (II.8) we obtain

$$\bar{M}(x) = \sum_{p=p_{\max}}^{-\infty} \sum_{q=0}^{q_{\max}(p)} \frac{a_{pq} q!}{(x-p)^{q+1}} - \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \frac{1}{x-n} + \text{continuous part}, \tag{II.13}$$

therefore, we find

$$F(n) = (-)^n \varphi^{(n)}(0) \tag{II.14}$$

for integer $n \geqq p_{\max} + 1$.

From the properties we have found for Mellin transform of functions $\varphi(\lambda)$ satisfying (II.1), we now pose and solve the inverse problem to determine when $\varphi(\lambda)$ possesses an asymptotic expansion of the form (II.1). First let us remind ourselves of the following theorem on inverse Mellin transform.

Theorem. [6]. *Let $f(\lambda)$ be piecewise smooth for $\lambda > 0$, and let $\int_0^\infty d\lambda \lambda^{-x-1} f(\lambda)$ be absolutely convergent for $\alpha < \operatorname{Re} x < \beta$. Then, if*

$$g(x) = \int_0^\infty d\lambda \lambda^{-x-1} f(\lambda), \quad f(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \lambda^x g(x)\, dx$$

with $\alpha < \sigma < \beta$.

Then, if there exists an integer $A \geqq -1$ such that $(1 - T_\lambda^A)\,\varphi(\lambda)$ fulfills the conditions of the theorem for $A < \operatorname{Re} x < A + 1$,

$$(1 - T_\lambda^A)\,\varphi(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dx \lambda^x M(x), \tag{II.15}$$

where $M(x)$ is defined in (II.2) and $A < \sigma < A + 1$.

If moreover the analytic continuation $\bar{M}(x)$ of $M(x)$ is found to be analytic for $\operatorname{Re} x < A + 1$ except for multiple poles at integer values of $x \leqq A$, and, if for $\operatorname{Re} x$ in any interval $a < \operatorname{Re} x < A + \varepsilon$, $|M(x)| \to 0$ uniformly in $\operatorname{Re} x$ faster than $|\operatorname{Im} x|^{-1-\delta}$ when $|\operatorname{Im} x| \to \infty$ (ε and $\delta > 0$), then the contour of integration can be shifted towards the left to a value of σ such that $M(\sigma)$ exists. The residues of the poles between $A + 1$ and σ then generate an asymptotic expansion for $\varphi(\lambda)$ of the form (II.1).

III. Feynman Amplitudes at Large Momenta

a) The Mellin Transform

Let us scale all external momenta p by λ in the integral (I.2) for convergent scalar Feynman amplitudes in Euclidean space and obtain the function $\varphi(\lambda)$,

$$\varphi(\lambda) = \int_0^\infty d\alpha\, e^{-\alpha m^2} P(\alpha)^{-D/2}\, e^{-\lambda^2 p d^{-1}(\alpha) p}. \tag{III.1}$$

In Appendix C, we shall derive the following two estimates for the positive function $\varphi(\lambda)$

$$\begin{aligned} &\varphi(\lambda) \leqq \text{const} \\ &\varphi(\lambda) \leqq \text{const} \times |\lambda|^{\omega + \varepsilon} \quad \text{for} \quad |\lambda| > N, \end{aligned} \tag{III.2}$$

where $\varepsilon > 0$ and

$$\omega = \operatorname{Sup}[\omega(\mathscr{S})]. \tag{III.3}$$

In (III.3), Sup runs over the superficial degree of divergence of all essential subdiagrams. These two estimates allow us to define the Mellin transform of $\varphi(\lambda)$ as in (II.2) with $A = -1$. Indeed, splitting the integral into two parts

$$M(x) = \int_0^N d\lambda \lambda^{-x-1} \varphi(\lambda) + \int_N^\infty d\lambda \lambda^{-x-1} \varphi(\lambda), \tag{III.4}$$

and using (III.2), we see that the λ integral is absolutely convergent for $\omega < \mathrm{Re}\, x < 0$. Then, the theorem on Mellin transform in Section II can be applied and the inverse Mellin transform is valid for the function $\varphi(\lambda)$ and for $\omega < \sigma < 0$. Replacing $\varphi(\lambda)$ in (II.2) by its value in (III.1) and interchanging the order of integration by Fubini-Tonelli theorem, we obtain for the Mellin transform the integral representation

$$M(x) = \frac{1}{2}\Gamma\left(-\frac{x}{2}\right)\int_0^\infty d\alpha e^{-\alpha m^2} P(\alpha)^{-D/2}[pd^{-1}(\alpha)p]^{x/2}. \qquad \text{(III.5)}$$

Since $pd^{-1}(\alpha)p$ vanishes linearly with those α's belonging to an essential subdiagram (I.8), every essential subdiagram $\mathscr{S}$ will cause the divergence of the α-integral in (III.5) whenever $\mathrm{Re}\, x \leqq \omega(\mathscr{S})$, the superficial degree of divergence of $\mathscr{S}$. Then, ω is the largest negative integer where $\bar{M}(x)$ is singular. It will be shown in this paper that $\bar{M}(x)$ has a multiple pole at $x = \omega$ and that shifting the integral contour across the pole at $x = \omega$ in (II.15) will yield a leading behaviour for $\varphi(\lambda)$ of the form

$$\varphi(\lambda) \sim \lambda^{\omega}(\ln \lambda)^q a_{\omega q} + \cdots. \qquad \text{(III.6)}$$

This is the famous power counting theorem due to Weinberg [1]. In (III.6), q is a nonnegative integer and ω has been defined in (III.3). Let us remind ourselves of the rule to obtain essential subdiagrams; they are the subdiagrams which alone conserve the external energy-momentum flow.

Let us now show the meromorphic structure of $\bar{M}(x)$. For this purpose, we decompose the domain of integration in (III.5) into sectors [7] g as defined in Appendix C. Then,

$$M(x) = \sum_g M_g(x). \qquad \text{(III.7)}$$

In each sector

$$\mathscr{I}_g = \{\alpha \mid 0 \leqq \alpha_{i_1} \leqq \alpha_{i_2} \ldots \leqq \alpha_{i_l}\}. \qquad \text{(III.8)}$$

We define a sequence of nested subdiagrams

$$R^j = \{i_1, \ldots, i_j\} \qquad \text{(III.9)}$$

for $j = 1, \ldots, l$. The set of all essential subdiagrams is denoted by $\mathscr{E}$. If $R^j \in \mathscr{E}$ all $R^r \in \mathscr{E}$ for $r \geqq j$; let R^k be the smallest essential subdiagram among the R^j's. Performing the change of variables defined in (C.6) and integrating over β_l, we obtain

$$M_g(x) = 2^{l-2}\Gamma\left(-\frac{x}{2}\right)\Gamma\left(\frac{x-\omega(R^l)}{2}\right)\int_0^1 \prod_{i=1}^{l-1} \cdot [d\beta_i \beta_i^{-\omega(R^i)-1}] \prod_{i=k}^{l-1} \beta_i^x Q^g(m^2, p, \beta, x) \qquad \text{(III.10)}$$

with

$$Q^g(m^2, p, \beta, x) = \left[m_{i_l}^2 + \sum_{j=1}^{l-1} \beta_j^2 \ldots \beta_{l-1}^2 m_{i_j}^2\right]^{-[x-\omega(R^l)]/2} P_g(\beta)^{-D/2} [p\Delta_g(\beta)p]^{x/2}. \tag{III.11}$$

The functions $P_g(\beta)$ and $p\Delta_g(\beta)p$, defined in (C.8), are such that $Q^g(m^2, p, \beta, x)$ has a Taylor expansion in any subset of β's, convergent in a neighborhood of these β's $= 0$.

We introduce in (III.10) the identity

$$Q^g(m^2, p, \beta, x) = \sum_{S \subseteq \{k,\ldots,l-1\}} \prod_{t\in S} T_{\beta_t}^{q_t} \prod_{\substack{t\notin S \\ t\in\{k,\ldots,l-1\}}} (1 - T_{\beta_t}^{q_t}) Q^g(m^2, p, \beta, x),$$

where the q_s' are any set of nonnegative integers. Let us denote by

$$Q_{\{n\}S}^g(m^2, p, \{\beta_t : t\notin S\}, x) = \prod_{t\in S} \left\{\frac{1}{n_t!}\left(\frac{\partial}{\partial\beta_t}\right)^{n_t}\right\} Q^g(m^2, p, \beta, x)\Big|_{\beta_t=0, t\in S}. \tag{III.13}$$

Using (III.13) and (III.12) and integrating in (III.10) over β_t for $t\in S$ in the region $\mathrm{Re}\, x > \omega$, we obtain

$$M_g(x) = 2^{l-2}\Gamma\left(-\frac{x}{2}\right)\Gamma\left(\frac{x-\omega(R^l)}{2}\right) \sum_{S\subseteq\{k,\ldots,l-1\}} \cdot \left\{\sum_{\substack{n_t=0 \\ t\in S}}^{q_t} \prod_{t\in S} \frac{1}{x-\omega(R^t)+n_t} I_{\{n\},S}^g(m^2, p, x)\right\}, \tag{III.14}$$

where

$$I_{\{n\},S}^g(m^2, p, x) = \int_0^1 \prod_{t\notin S} [d\beta_t \beta_t^{-\omega(R^t)-1}] \prod_{\substack{t\notin S \\ R^t \text{ essential}}} \beta_t^x \prod_{\substack{t\notin S \\ t\in\{k,\ldots,l-1\}}} (1 - T_{\beta_t}^{q_t}) \cdot Q_{\{n\},S}^g(m^2, p, \{\beta_t : t\notin S\}, x). \tag{III.15}$$

The integral (III.15) is shown in Appendix C to be analytic in x for $\mathrm{Re}\, x > \sup_{\substack{t\notin S \\ R^t \text{ essential}, \neq R^l}} (\omega(R^t) - q_t - 1)$. Summing over all possible S in (III.14) shows that the only singularities of $\bar{M}_g(x)$ in the region $\mathrm{Re}\, x > \sup_{\substack{R^t \text{ essential}, \\ R^t \neq R^l}} (\omega(R^t) - q_t - 1)$ is a set of multiple poles at integer x; each essential subdiagram R^i of the sector develops single poles at $x = \omega(R^i), \omega(R^i) - 2, \ldots$. Since the q_t's are arbitrary, $\bar{M}_g(x)$ is a meromorphic function and so is $\bar{M}(x)$ by (III.7). Then, $\bar{M}(x)$ is analytic in the entire complex plane except for simple poles at $x = 0, 2, 4, \ldots$, and multiple poles at $x = \omega, \omega - 2, \omega - 4, \ldots$ for even dimension of space-time and at $x=\omega,\ \omega-1,\ \omega-2, \ldots$ for odd dimension of space-time. It will be shown by extracting the residue of the pole at $x=\omega$, that it is not zero.

It is also shown in Appendix C that the integral in (III.15) is polynomially bounded when $|\mathrm{Im}\,x| \to \infty$, and that this boundedness is uniform in $\mathrm{Re}\,x$ in any fixed interval. Then the presence of the Γ functions[2] in $\bar{M}_g(x)$ insures that we may shift the integration contour towards the left for the inverse Mellin transform (II.15). The asymptotic expansion for $\varphi(\lambda)$ is then of the form (II.1).

It is certainly possible to extract the coefficients of all pole terms from (III.14) but the result is not obviously sector independent. However, we will present in Section IIIb) an alternate determination of the coefficients which has the advantage that it is not dependent of the sector decomposition.

b) Leading Poles of $F(x)=M(x)/\Gamma(-x)$

In Section IIIa), we showed the pole structure of $\bar{M}(x)$ from the property that the integrand in (III.5), apart from a common factor, has a power series in the sector-dependent variables β. Here, we shall use the fact that, for any forest of subdiagrams, the same property holds with respect to the dilatation variables corresponding to the subdiagrams of the forest. In the following, we shall present a method to analytically continue $F(x)$ to the band $\omega-N<\mathrm{Re}\,x<\omega-N+1$ where N is any positive integer, and we shall explicitly determine the coefficients of the leading poles, that is, at $x=\omega$. The main tool used to perform the analytic continuation is the generalized Taylor operator as defined in Appendix A. This is a generalization of the operators introduced in Ref. [4] for the purpose of renormalization. We shall here merely restate the definition. Given a function $f(x)$ such that $x^{-\nu}f(x)$ (where ν may be complex) is infinitely differentiable at $x=0$, then we define τ^n as

$$\tau_x^n f(x) = x^{-\lambda-\varepsilon} T^{n+\lambda}\{x^{\lambda+\varepsilon} f(x)\}, \tag{III.16}$$

where $\lambda \geqq -E'(\nu)$ is an integer, $E'(\nu)$ is the smallest integer $\geqq \mathrm{Re}\,\nu$, and $\varepsilon = E'(\nu)-\nu$. As an application of (A.11), let us define for any subdiagram $\mathscr{S}$

$$\tau_{\mathscr{S}}^n f(\alpha) = [\tau_\varrho^n f(\alpha)|_{\alpha_i=\varrho^2\alpha_i,\ \forall i\in\mathscr{S}}]_{\varrho=1}\,. \tag{III.17}$$

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[2] See Eq. (6) on p. 47 of Bateman, Higher transcendental functions, Vol. I, McGraw-Hill (1953):

$$\lim_{|y|\to\infty} |\Gamma(x+iy)|\, e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}$$

for x, y real.

Let us now consider the quantity

$$\prod_{\mathscr{S}\subseteq G}(1-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\{[pd^{-1}(\alpha)p]^{x/2}P(\alpha)^{-D/2}\}, \qquad \text{(III.18)}$$

where the product runs over all subdiagrams $\mathscr{S}$ of the graph G, including G itself. Although the τ operators do not commute, it can be shown that the complete product $\prod_{\mathscr{S}\subseteq G}(1-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})$ is independent of the order of application upon the function between the brackets { }. For real integer x the proof of this statement can be obtained via the formula for the remainder of Taylor series, analogous to Refs. [4] and [8]. Such a proof does not hold for other x. Fortunately enough it is proved in Part b) of Appendix B that whatever is the order of application of the τ operators in (III.18) we obtain the forest formula

$$\left[1+\sum_{\mathscr{F}}\prod_{\mathscr{S}\in\mathscr{F}}(-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\right]\{[pd^{-1}(\alpha)p]^{x/2}P(\alpha)^{-D/2}\}, \qquad \text{(III.19)}$$

where the sum runs over all nonempty forests of subdiagrams. From property (A.4) it is easy to convince oneself that all τ operators give zero for $\mathrm{Re}\,x>\omega$. For $\mathrm{Re}\,x\leqq\omega$, only τ operators corresponding to essential subgraphs will give non-zero results. If no α is equal to zero the quantity (III.18) or (III.19) is band-wise analytic in the bands $\omega-N<\mathrm{Re}\,x<\omega-N+1$ with possible discontinuities at the boundaries.

For a given positive integer N, and for $\alpha\neq 0$ we define the functions $g^{(N)}(x,\alpha)$ and $h^{(N)}(x,\alpha)$, analytic in the entire x-plane, by the respective conditions

$$g^{(N)}(x,\alpha)=e^{-\alpha m^2}\prod_{\mathscr{S}\subseteq G}(1-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\{[pd^{-1}(\alpha)p]^{x/2}P(\alpha)^{-D/2}\}, \qquad \text{(III.20)}$$

$$h^{(N)}(x,\alpha)=e^{-\alpha m^2}\sum_{\mathscr{F}}\prod_{\mathscr{S}\in\mathscr{F}}(-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\{[pd^{-1}(\alpha)p]^{x/2}P(\alpha)^{-D/2}\}, \qquad \text{(III.21)}$$

for x in the band $\omega-N<\mathrm{Re}\,x\leqq\omega-N+1$. From (III.18) and (III.19) we show that in this band, and hence everywhere by analyticity,

$$g^{(N)}(x,\alpha)-h^{(N)}(x,\alpha)=e^{-\alpha m^2}[pd^{-1}(\alpha)p]^{x/2}P(\alpha)^{-D/2}. \qquad \text{(III.22)}$$

Integrating over the $d\alpha$'s for $\mathrm{Re}\,x>\omega$, we obtain

$$F(x)=\frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)}\int_0^\infty d\alpha g^{(N)}(x,\alpha)-\frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)}\int_0^\infty d\alpha h^{(N)}(x,\alpha). \qquad \text{(III.23)}$$

In Appendix C, it is shown in full detail that the integral $\int_0^\infty d\alpha g^{(N)}(x,\alpha)$ exists for $\mathrm{Re}\,x>\omega-N$ and is analytic in the same region.

The integral $\frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)}\int_0^\infty d\alpha h^{(N)}(x,\alpha)$ in the region $\mathrm{Re}\,x>\omega$ will give the singularity structure of $F(x)$ when analytically continued to the domain $\mathrm{Re}\,x>\omega-N$.

From the definition (III.21), we see that any forest containing a nonessential subdiagram contributes nothing to $h^{(N)}(x, \alpha)$. Replacing the τ operators by Taylor operators T, we obtain

$$h^{(N)}(x,\alpha) = e^{-\alpha m^2} \sum_{\mathscr{F}'} \prod_{\mathscr{S} \in \mathscr{F}'} (-T_{\xi_{\mathscr{S}}}^{\omega(\mathscr{S})-\omega+N-1}) \cdot \{[p\Delta_{\mathscr{F}'}(\alpha,\xi)\,p]^{x/2} P_{\mathscr{F}'}(\alpha,\xi)^{-D/2}\}|_{\xi=1}\,, \tag{III.24}$$

where the sum runs over the forests of essential subdiagrams with superficial degree of divergence $\omega(\mathscr{S}) \geqq \omega - N + 1$, and where $\Delta_{\mathscr{F}'}$, and $P_{\mathscr{F}'}$, defined by

$$\begin{cases} pd^{-1}(\alpha,\xi)\,p = \prod\limits_{\mathscr{S} \in \mathscr{F}'} \xi_{\mathscr{S}}^{-2} p\Delta_{\mathscr{F}'}(\alpha,\xi)\,p & \text{(III.25a)} \\ P(\alpha,\xi) = \prod\limits_{\mathscr{S} \in \mathscr{F}'} \xi_{\mathscr{S}}^{-2L(\mathscr{S})} P_{\mathscr{F}'}(\alpha,\xi)\,, & \text{(III.25b)} \end{cases}$$

have power expansions in the ξ corresponding to subdiagrams of $\mathscr{F}'$. Since a forest of essential subdiagrams is necessarily a forest of nested subdiagrams, in each of the forests of the sum (III.24) there exist a minimal essential subdiagram contained in all others of the same forest. We now regroup the forests $\mathscr{F}'$ into the classes of forests which have the same minimal elements:

$$h^{(N)}(x,\alpha) = e^{-\alpha m^2} \sum_{\mathscr{S}} \sum_{\mathscr{F}_{\mathscr{S}}} \prod_{\substack{\mathscr{S}' \in \mathscr{F}_{\mathscr{S}} \\ \mathscr{S}' \neq \mathscr{S}}} (-T_{\xi_{\mathscr{S}'}}^{\omega(\mathscr{S}')-\omega+N-1}) \cdot (-T_{\xi_{\mathscr{S}}}^{\omega(\mathscr{S})-\omega+N-1}) \{[p\Delta_{\mathscr{F}_{\mathscr{S}}}(\alpha,\xi)\,p]^{x/2} P_{\mathscr{F}_{\mathscr{S}}}(\alpha,\xi)^{-D/2}\}|_{\xi=1}\,, \tag{III.26}$$

where the first sum runs over all essential subdiagrams $\mathscr{S}$ and the second sum over all forest $\mathscr{F}_{\mathscr{S}}$ whose minimal element is $\mathscr{S}$. To obtain the pole structure of $h^{(N)}(x, \alpha)$ we operate $T_{\xi_{\mathscr{S}}}$ on $\{\ \}$, and each term so produced can be factorized into two parts: an x-independent part which is a function of the α's belonging to the reduced diagram $(G/\mathscr{S})$, and an x-dependent part, which is a function of the α's belonging to $\mathscr{S}$ and which can be integrated in the region $\mathrm{Re}\,x > \omega$ to yield the poles of $F(x)$. It is beyond the scope of this paper to determine the coefficients of all the poles of $F(x)$ and we shall contend ourselves with the determination of only those of the poles at $x = \omega$.

The poles of $F(x)$ at $x = \omega$ will be obtained from the function $h^{(1)}(x, \alpha)$. The essential subdiagrams with superficial degree of divergence equal to ω are said to be *leading* subdiagrams. Of course, only the leading subdiagrams contribute to the forests in $h^{(1)}(x, \alpha)$:

$$h^{(1)}(x,\alpha) = e^{-\alpha m^2} \sum_{\mathscr{S}} \sum_{\mathscr{F}_{\mathscr{S}}} \prod_{\substack{\mathscr{S}' \in \mathscr{F}_{\mathscr{S}} \\ \mathscr{S}' \neq \mathscr{S}}} (-T_{\xi_{\mathscr{S}'}}^{0}) \cdot (-T_{\xi_{\mathscr{S}}}^{0}) \{[p\Delta_{\mathscr{F}_{\mathscr{S}}}(\alpha,\xi)\,p]^{x/2} P_{\mathscr{F}_{\mathscr{S}}}(\alpha,\xi)^{-D/2}\}|_{\xi=1}\,. \tag{III.27}$$

On applying the Taylor operator $T^0_{\xi_{\mathscr{S}}}$ for the leading subdiagram $\mathscr{S}$, we obtain by virtue of (I.11) and (I.13)

$$[p d_{\mathscr{S}}^{-1}(\alpha) p]^{x/2} P_{\mathscr{S}}(\alpha)^{-D/2} P_{G/\mathscr{S}}(\alpha, \xi)^{-D/2} . \tag{III.28}$$

This factorisation allows us to separate every term in $h^{(1)}(x, \alpha)$ into two factors: one depending on $\mathscr{S}$ and the other one depending on the reduced diagram $G/\mathscr{S}$. Thus,

$$\begin{aligned} h^{(1)}(x, \alpha) = -e^{-\alpha m^2} \sum_{\mathscr{S}} \sum_{\mathscr{F}_{\mathscr{S}}} \prod_{\substack{\mathscr{S}' \in \mathscr{F}_{\mathscr{S}} \\ \mathscr{S}' \neq \mathscr{S}}} (-\tau_{\mathscr{S}'/\mathscr{S}}^{-2l(\mathscr{S}'/\mathscr{S})}) \{P_{G/\mathscr{S}}(\alpha)^{-D/2}\} \\ \cdot [p d_{\mathscr{S}}^{-1}(x) p]^{x/2} P_{\mathscr{S}}(\alpha)^{-D/2} , \end{aligned} \tag{III.29}$$

where we have reintroduced the τ operators for the reduced subdiagrams $\mathscr{S}'/\mathscr{S}$. Since, for a subdiagram $\mathscr{S}' \in \mathscr{F}_{\mathscr{S}}$, $\mathscr{S}'/\mathscr{S}$ is a subdiagram in $G/\mathscr{S}$ with superficial degree of divergence $\omega(\mathscr{S}'/\mathscr{S}) = 0$, we may express the sum over $\mathscr{F}_{\mathscr{S}}$ as a sum over forests of logarithmically divergent subdiagrams of $G/\mathscr{S}$:

$$\begin{aligned} h^{(1)}(x, \alpha) = -\sum_{\mathscr{S}} e^{-\sum_{i \in G/\mathscr{S}} \alpha_i m_i^2} \sum_{\mathscr{F}} \prod_{\mathscr{S}'/\mathscr{S} \in \mathscr{F}} (-\tau_{\mathscr{S}'/\mathscr{S}}^{-2l(\mathscr{S}'/\mathscr{S})}) P_{G/\mathscr{S}}(\alpha)^{-D/2} \\ \cdot e^{-\sum_{i \in \mathscr{S}} \alpha_i m_i^2} [p d_{\mathscr{S}}^{-1}(\alpha) p]^{x/2} P_{\mathscr{S}}(\alpha)^{-D/2} . \end{aligned} \tag{III.30}$$

We may enlarge the sum over $\mathscr{F}$ in (III.30) to cover all forests of $G/\mathscr{S}$, since the contribution from any convergent diagram is zero. Then, by applying the forest formula (part b of Appendix B) we obtain:

$$\begin{aligned} h^{(1)}(x, \alpha) = -\sum_{\mathscr{S}} e^{-\sum_{i \in G/\mathscr{S}} \alpha_i m_i^2} \prod_{R \subset G/\mathscr{S}} (1 - \tau_R^{-2l(R)}) P_{G/\mathscr{S}}(\alpha)^{-D/2} \\ \cdot e^{-\sum_{i \in \mathscr{S}} \alpha_i m_i^2} [p d_{\mathscr{S}}^{-1}(\alpha) p]^{x/2} P_{\mathscr{S}}(\alpha)^{-D/2} . \end{aligned} \tag{III.31}$$

Let us define

$$\gamma_G(x) = \frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)} \int_0^\infty d\alpha g_G^{(1)}(x, \alpha) , \tag{III.32}$$

and

$$\varrho_G = \int_0^\infty d\alpha e^{-\alpha m^2} \prod_{\mathscr{S} \subseteqq G} (1 - \tau_{\mathscr{S}}^{-2l(\mathscr{S})}) P_G(\alpha)^{-D/2} , \tag{III.33}$$

where we have displayed explicitly the dependence of $g^{(1)}$ on the diagram G. The function ϱ_G is nothing but the renormalized amplitude for the Feynman graph G at external momenta equal to zero [4]. Then,

$$F_G(x) = \gamma_G(x) + \sum_{\text{leading } \mathscr{S}} \varrho_{G/\mathscr{S}} F_{\mathscr{S}}(x) . \tag{III.34}$$

Since we are going to use the function $F(x)$ recurrently, we have indicated its dependence on the diagram G by writing $F_G(x)$. In Eq. (III.34), $\varrho_{G/G}$

is meant to be one. This equation is trivial if the entire graph is leading, since in that case $\gamma_G(x) = \varrho_{G/\mathscr{S}} = 0$ for $\mathscr{S} \neq G$.

The singularities of $F_G(x)$ at $x \sim \omega$ are now present in the functions $F_\mathscr{S}(x)$. To extract the pole from $F_\mathscr{S}(x)$ at $x = \omega$, we change all integration variables α in the integral representation of $F_\mathscr{S}(x)$, into $\mu\alpha$ ($\mu > 0$),

$$F_\mathscr{S}(x) = \frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)} \int_0^\infty d\alpha e^{-\mu\alpha m^2} \mu^{[x-\omega(\mathscr{S})]/2} [p d_\mathscr{S}^{-1}(\alpha) p]^{x/2} P_\mathscr{S}(\alpha)^{-D/2}, \quad \text{(III.35)}$$

and we differentiate by μ on both sides. On setting $\mu = 1$, we obtain

$$F_\mathscr{S}(x) = \frac{2}{x-\omega} \sum_{i \in \mathscr{S}} m_i^2 F_{\mathscr{S}_i}(x), \quad \text{(III.36)}$$

where $\mathscr{S}_i$ is the diagram obtained from $\mathscr{S}$ by inserting a 2-leg vertex (analogous to mass insertion) on the line i. Substituting (III.36) into (III.34), we obtain the following recurrent relation

$$F_G(x) = \gamma_G(x) + \frac{2}{x-\omega} \sum_{\text{leading}\,\mathscr{S}} \varrho_{G/\mathscr{S}} \sum_{i \in \mathscr{S}} m_i^2 F_{\mathscr{S}_i}(x). \quad \text{(III.37)}$$

Some of the functions $F_{\mathscr{S}_i}(x)$ are still singular at $x = \omega$. Such singularities are due to leading subdiagrams of $\mathscr{S}$ which do not contain the line i. We use exactly the same technique to extract the poles of $F_{\mathscr{S}_i}(x)$ and thus (III.37) can be applied recurrently. The recurrence stops when the functions $F_{m_i}(x)$ so obtained are not singular anymore at $x = \omega$, that is when m is a minimal leading subdiagram. In that case $F_{m_i}(x) = \gamma_{m_i}(x)$. The solution of the recurrence is

$$F_G(x) = \gamma_G(x) + \sum_{(\mathscr{S}_1, \ldots, \mathscr{S}_k)} \frac{2^k}{(x-\omega)^k} \varrho_{G/\mathscr{S}_1} \tilde{\varrho}_{\mathscr{S}_1/\mathscr{S}_2} \cdots \tilde{\varrho}_{\mathscr{S}_{k-1}/\mathscr{S}_k} \tilde{\gamma}_{\mathscr{S}_k}(x), \quad \text{(III.38)}$$

where the sum runs over all forests $(\mathscr{S}_1, \ldots, \mathscr{S}_k)$ of nested leading subdiagrams $\mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots \supset \mathscr{S}_k$, and where we define

$$\tilde{\gamma}_\mathscr{S}(x) = \sum_{i \in \mathscr{S}} m_i^2 \gamma_{\mathscr{S}_i}(x), \quad \text{(III.39)}$$

$$\tilde{\varrho}_\mathscr{S} = \sum_{i \in \mathscr{S}} m_i^2 \varrho_{\mathscr{S}_i}. \quad \text{(III.40)}$$

Note that in the case where G is leading, (III.38) simplifies into

$$F_G(x) = \sum_{(G=\mathscr{S}_1, \ldots, \mathscr{S}_k)} \frac{2^k}{(x-\omega)^k} \tilde{\varrho}_{G/\mathscr{S}_2} \cdots \tilde{\varrho}_{\mathscr{S}_{k-1}/\mathscr{S}_k} \tilde{\gamma}_{\mathscr{S}_k}(x), \quad \text{(III.41)}$$

where the sum is as above except that all forests must now contain $G = \mathscr{S}_1$. The functions $\gamma_G(x)$ and $\tilde{\gamma}_{\mathscr{S}_k}(x)$ are analytic in the region $\operatorname{Re} x > \omega - 1$. Hence, (III.38) and (III.41) display the poles of $F_G(x)$ at $x = \omega$.

c) The Asymptotic Expansion

The analytic continuation of the function $M(x)$ in (III.5) in the region $\mathrm{Re}\,x > \omega - 1$ is $\Gamma(-x)\,F(x)$, where the pole structure of $F(x)$ at $x = \omega$ is given by (III.38). To obtain the asymptotic expansion of $\varphi(\lambda)$, we first substitute (III.38) and $M(x) = \Gamma(-x)\,F(x)$ into the equation of inverse Mellin transform (II.15):

$$\begin{aligned}\varphi(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dx \lambda^x \Gamma(-x) \\ \cdot \left\{ \gamma_G(x) + \sum_{\mathscr{S}_1, \ldots, \mathscr{S}_k} \frac{2^k}{(x-\omega)^k} \varrho_{G/\mathscr{S}_1} \tilde{\varrho}_{\mathscr{S}_1/\mathscr{S}_2} \cdots \tilde{\varrho}_{\mathscr{S}_{k-1}/\mathscr{S}_k} \tilde{\gamma}_{\mathscr{S}_k}(x) \right\},\end{aligned} \qquad \text{(III.42)}$$

where $\omega < \sigma < 0$. It is shown in part c) of Appendix C that $\int_0^\infty dx g^{(1)}(x, \alpha)$ is analytic for $\mathrm{Re}\,x > \omega - 1$ and is polynomially bounded in $|\mathrm{Im}\,x|$ as $|\mathrm{Im}\,x| \to \infty$, and that this boundedness is uniform in $\mathrm{Re}\,x$ in any fixed interval of $\mathrm{Re}\,x > \omega - 1$. So, taking into account the property [2] of $\Gamma(-x/2)$ as $|\mathrm{Im}\,x| \to \infty$, we may integrate term by term in (III.42) and shift the contour of integration parallelly across the pole at $x = \omega$. Then we obtain

$$\varphi(\lambda) = \varphi_{as}(\lambda) + \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dx \lambda^x \Gamma(-x) \{ \ \}, \qquad \text{(III.43)}$$

where $\omega - 1 < \sigma < \omega$, $\{ \ \}$ is the same as that in (III.42), and where

$$\varphi_{as}(\lambda) = \lambda^\omega \sum_{q=0}^{q_{\max}(\omega)} a_{\omega q} (\ln \lambda)^q. \qquad \text{(III.44)}$$

In (III.44), ω, defined in (III.3), is the Weinberg power, $q_{\max}(\omega) =$ (number of elements in the largest set of nested leading subdiagrams) $- 1$, and

$$\begin{aligned}a_{\omega q} = \frac{1}{q!} \sum_{\substack{\mathscr{S}_1, \ldots \mathscr{S}_k \\ k > q}} \frac{2^k}{(k-q-1)!} \varrho_{G/\mathscr{S}_1} \tilde{\varrho}_{\mathscr{S}_1/\mathscr{S}_2} \cdots \tilde{\varrho}_{\mathscr{S}_{k-1}/\mathscr{S}_k} \\ \cdot \left[\frac{d^{k-q-1}}{dx^{k-q-1}} (\Gamma(-x)\, \tilde{\gamma}_{\mathscr{S}_k}(x)) \right]_{x=\omega},\end{aligned} \qquad \text{(III.45)}$$

where the sum runs over all forests $(\mathscr{S}_1, \ldots, \mathscr{S}_k)$ of $k\ (> q)$ nested leading subdiagrams $\mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots \supset \mathscr{S}_k$.

To justify that $\varphi_{as}(\lambda)$ are the leading terms of the asymptotic expansion, note that the absolute value of the integral in (III.43) is bounded from above by

$$\text{const} \times \lambda^\sigma \qquad (\sigma < \omega).$$

Let us remind the reader the following recipe to obtain the coefficients ϱ, $\tilde{\varrho}$, and the function $\tilde{\gamma}$:

1. $\varrho_{\mathscr{S}}$ is the renormalized Feynman amplitude for $\mathscr{S}$ at zero external momenta; $\mathscr{S}/\mathscr{S}'$ is the reduced diagram obtained from $\mathscr{S}$ by shrinking $\mathscr{S}'$ to a point; $\varrho_{G/G}$ is one by definition, while $\varrho_{G/\mathscr{S}}$ is zero if G is leading;

2. $\tilde{\varrho}_{\mathscr{S}}$ is the sum of renormalized Feynman amplitudes over graphs obtained from $\mathscr{S}$ by making a mass insertion;

3. $$\tilde{\gamma}_{\mathscr{S}}(x) = \frac{\Gamma(-\frac{x}{2})}{2\Gamma(-x)} \sum_{i \in \mathscr{S}} m_i^2 \int_0^\infty d\alpha\, e^{-\sum_{j \in \mathscr{S}_i} \alpha_j m_j^2} \cdot \prod_{R \subseteqq \mathscr{S}_i} (1 - \tau_R^{-2l(R)}) [p d_{\mathscr{S}_i}^{-1}(\alpha) p]^{x/2} P_{\mathscr{S}_i}(\alpha)^{-D/2}, \tag{III.46}$$

where the $\mathscr{S}_i$'s are obtained from $\mathscr{S}$ by making a mass insertion on the ith line.

IV. Conclusion

In Section IIIc), we have obtained an asymptotic expansion for any scalar convergent Feynman amplitude with all coefficients of the leading power determined for all logarithms. Weinberg's theorem [1] has been obtained at the very early stage of this paper, and later the leading logarithmic behaviour, as already given by Fink [2], has been recovered. It must be pointed out that our treatment does not distinguish the case where some partial sum of external momenta vanishes; in this case the essential subdiagrams may be disconnected, and this may result in a larger value of ω. Using the homogeneity equation, we also find, of course, that, in the limit all masses tending to zero, a convergent scalar Feynman amplitude exists only if the entire graph is the only leading subdiagram.

Our method using the Mellin transform [6] and the combinatorics of the τ operators [4] actually allows all coefficients of the asymptotic expansion to be determined. The main feature of the coefficients so obtained is a factorization property inside the forests of essential subdiagrams.

Although the result of Section IIIc) is not directly applicable to Lagrangian field theory (except in 2 dimensions where all scalar graphs without derivative couplings are convergent), this paper provides the basic ingredients which will be used in a following paper (Part II) to extend our results to divergent graphs. Later on, the results will be further generalized to the case where only some masses and some external momenta tend to infinity; application to physical situations will then be pointed out.

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Appendix A

Generalized Taylor Operators

The generalized Taylor operators have been defined and extensively employed in Ref. [4]. Here we want to further generalize the definition over a class of functions $f(x)$ which behaves like x^ν at $x=0$ (where ν is not necessarily an integer).

Definition. Given a function $f(x)$ such that $x^{-\nu}f(x)$ is C^∞ in $[0, a>0,)$ we define the generalized Taylor operator τ^n on $f(x)$ as

$$\tau^n f(x) = x^{-\lambda-\varepsilon} T^{n+\lambda}\{x^{\lambda+\varepsilon} f(x)\}\,, \tag{A.1}$$

where $\lambda \geqq -E'(\nu)$ is an integer, $E'(\nu)$ is the smallest integer $\geqq \mathrm{Re}\,\nu$, and $\varepsilon = E'(\nu) - \nu$. In (A.1), n is an integer and T is the usual Taylor operator.

The above definition is λ-independent. The purpose of introducing ε is to remove the cut of $f(x)$ at $x=0$.

Let us mention the following properties:

1. $$\tau^n f(x) \sim x^q \quad \text{at} \quad x \sim 0 \quad \text{with} \quad \mathrm{Re}\,q \leqq n\,, \tag{A.2}$$
2. $$(1-\tau^n) f(x) \sim x^q \quad \text{at} \quad x \sim 0 \quad \text{with} \quad \mathrm{Re}\,q > n\,, \tag{A.3}$$
3. $$\tau^n f(x) = 0 \quad \text{if} \quad n - E'(\nu) < 0\,. \tag{A.4}$$

By using the formula for the remainder of the Taylor expansion, we have

$$(1-\tau^n) f(x) = \int_0^1 d\xi\, \frac{(1-\xi)^{n+\lambda}}{(n+\lambda)!} \frac{\partial^{n+\lambda+1}}{\partial \xi^{n+\lambda+1}} \{\xi^{\lambda+\varepsilon} f(x\xi)\}\,. \tag{A.5}$$

The ε in (A.5) is essential to ensure the existence of the integral. In this equation, $\lambda \geqq \mathrm{Sup}(-E'(\nu), -n)$.

The generalization of the definition of τ to functions of several variables is straightforward but in general the τ operators do not commute. We observe the properties.

1. $$\tau_x^{n_x} \tau_y^{n_y} f(x,y) \sim \begin{cases} x^q & \text{for} \quad x \sim 0\,, \quad y \neq 0 \quad \text{and} \quad \mathrm{Re}\,q \leqq n_x \\ y^q & \text{for} \quad y \sim 0\,, \quad x \neq 0 \quad \text{and} \quad \mathrm{Re}\,q \leqq n_y \end{cases}. \tag{A.6}$$
2. $$(1-\tau_x^{n_x})(1-\tau_y^{n_y}) f(x,y) \sim x^q \quad \text{for} \quad x \sim 0\,, \quad y \neq 0 \quad \text{and} \quad \mathrm{Re}\,q > n_x\,, \tag{A.7}$$

but nothing can be said on the behaviour at $y \sim 0$, $x \neq 0$.

3. $$\ldots (1-\tau_{x_i}^{n_i}) \ldots \tau_{x_i}^{n_i'} \ldots f = 0 \qquad \text{if} \quad n_i' \leqq n_i\,. \tag{A.8}$$

Corollary

$$\ldots (1-\tau_{x_i}^{n_i}) \ldots f = \cdots (1-\tau_{x_i}^{n_i}) \ldots (1-\tau_{x_i}^{n_i'}) \ldots f \quad \text{if} \quad n_i' \leqq n_i \tag{A.9}$$

$$\ldots \tau_{x_i}^{n_i'} \ldots f = \cdots \tau_{x_i}^{n_i} \ldots \tau_{x_i}^{n_i'} \ldots f \qquad \text{if} \quad n_i' \leqq n_i, \tag{A.10}$$

where ... means a sequence of τ operators. The integral representation for the remainder of the Taylor expansion is not always generalizable to functions of several variables. Indeed in $(1-\tau_y)(1-\tau_x)f$, ε_y is generally different for each term in $(1-\tau_x)f$.

Finally let us define the τ operators relative to a family of variables. Given a function of several variables $f(\{x\},\{y\})$, where $\{x\}$ and $\{y\}$ are families of variables, we define

$$\tau^n_{\{x\}} f(\{x\},\{y\}) = [\tau^n_\varrho f(\{\varrho x\},\{y\}]|_{\varrho=1}\,. \tag{A.11}$$

Appendix B

1. Nested Forest Formula

A forest formula has been introduced in Ref. [4] for the purpose of renormalization. Here we present a similar formula for forests of nested elements only, and relaxing the requirement of a special type of ordering, as in Ref. [4], in the product $\Pi(1-\tau)$. We extend the proof to a larger class of functions, on which this product acts. These functions $Z(\alpha)$ have Taylor expansions in the dilatation variables corresponding to the subdiagrams of any forest of nested elements, after all common factors, which may be non integer and complex powers of these variables, have been removed. This nested forest formula can be precisely stated as follows. Given any ordering, $\mathscr{S}_1$, $\mathscr{S}_2$ ... of all the 2^l-1 subdiagrams of a diagram G (consisting of l lines), then

$$\prod_{i=1}^{2^l-1} (1-\tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)})\, Z(\alpha) = \left[1+\sum_{\mathscr{N}} \prod_{\mathscr{S}\in\mathscr{N}} (-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\right] Z(\alpha)\,, \tag{B.1}$$

where the sum runs over all non-empty forests $\mathscr{N}$ of nested subdiagrams of G.

The proof is by recurence. Consider

$$\begin{aligned}\prod_{i=1}^{2^l-1} &(1-\tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)})\, Z(\alpha)\\ &= \prod_{i=m}^{2^l-1} (1-\tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)}) \left[1+\sum_{\mathscr{N}\in\mathscr{E}_{m-1}} \prod_{\mathscr{S}\in\mathscr{N}} (-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\right] Z(\alpha)\,,\end{aligned} \tag{B.2}$$

where $\mathscr{E}_{m-1}$ is the set of all forests of nested elements built from the subdiagrams in $W_{m-1}=\{\mathscr{S}_1,\ldots,\mathscr{S}_{m-1}\}$. For $m=2$, it is trivially true, while, for $m=2^l$, it reduces to (B.1). If we assume it to be valid for $m=n-1$, then it is valid for $m=n$ provided that

$$\prod_{i=n+1}^{2^l-1} (1-\tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)})(-\tau_{\mathscr{S}_n}^{-2l(\mathscr{S}_n)}) \sum_{\mathscr{N}\in\mathscr{E}'_{n-1}} \prod_{\mathscr{S}\in\mathscr{N}} (-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\, Z(\alpha) = 0\,, \tag{B.3}$$

where $\mathscr{E}'_{n-1}$ is the set of all forests of nested elements built from the subdiagrams $\mathscr{S}_1, \ldots, \mathscr{S}_{n-1}$ with at least one element either disjoint or overlapping with $\mathscr{S}_n$. Hence, the nested forest formula is proved by establishing (B.3).

Given a set of elements $T_1 \subset \cdots \subset T_r$, we define an extended set of elements

$$T_0 = \emptyset \subset T_1 \subset \cdots \subset T_r \subset T_{r+1} = G' , \tag{B.4}$$

where $\emptyset$ is the empty subgraph and G' is any graph containing the entire graph G. These two elements are introduced for matter of convenience. We define for $T_j \subset T_i \neq \emptyset$

$$\omega(T_i, T_j) = T_j \cup (\mathscr{S}_n \cap T_i) = T_i \cap (\mathscr{S}_n \cup T_j) . \tag{B.5}$$

Then it follows

a) $$T_j \subseteqq \omega(T_i, T_j) \subseteqq T_i , \tag{B.6}$$

b) $$\omega(T_i, T_j) = T_i \Leftrightarrow T_i \cup \mathscr{S}_n = T_j \cup \mathscr{S}_n , \tag{B.7}$$

c) $$\omega(T_i, T_j) = T_j \Leftrightarrow T_i \cap \mathscr{S}_n = T_j \cap \mathscr{S}_n , \tag{B.8}$$

d) $$\omega(T_i, T_j) = R \Leftrightarrow \begin{cases} \omega(T_i, R) = R, \text{ and} \\ \omega(R, T_j) = R . \end{cases} \tag{B.9}$$

We define a maximal nest $\mathscr{G}$ with respect to $\mathscr{S}_n$ to be a nest $T_0 = \emptyset \subset T_1 \subset \cdots \subset T_r \subset T_{r+1} = G'$ such that for any $T_i \neq \emptyset$ in $\mathscr{G}$,

$$\omega(T_i, T_{i-1}) = T_i \quad \text{or} \quad T_{i-1} . \tag{B.10}$$

Let us map $\mathscr{G}$ into the oriented positive real line by mapping each T_i to the integer i. If $\omega(T_i, T_{i-1}) = T_i$, we asign to the interval between $i-1$ and i an arrow pointing into i, and if $\omega(T_i, T_{i-1}) = T_{i-1}$, an arrow pointing into $i-1$. Then we can partition the nest $\mathscr{G}$ into three nests respectively $\mathscr{H}, \mathscr{K}, \mathscr{B}$. $\mathscr{H}$ is the set of $T's$ represented by those integers such that two arrows point into it; $\mathscr{K}$ is the set of $T's$ represented by those integers such that no arrow point into it; and $\mathscr{B}$ is the rest of the elements having only one arrow pointing into it. Then, between two consecutive $T's \in \mathscr{H}$, there is one and only one element $T \in \mathscr{K}$, and between two consecutive $T's \in \mathscr{K}$, there is one and only one element $T \in \mathscr{H}$. $\mathscr{K}$ is never empty since $G' \in \mathscr{K}$, and $\mathscr{K}$ contains at least two elements. Indeed, if G' is the only element of $\mathscr{K}$, then $\omega(T_i, T_{i-1}) = T_{i-1}$ for all $i > 0$, which implies $T_i \cap \mathscr{S}_n = \emptyset$ for all i, in contradiction with the fact that $T_r \supseteqq \mathscr{S}_n$ for a maximal nest $\mathscr{G}$. Since $\mathscr{K}$ has one more element than $\mathscr{H}$, $\mathscr{H}$ is also never empty. If $\emptyset$ and G' are the only elements of $\mathscr{K}$, then $\mathscr{H} = \{\mathscr{S}_n\}$ and $\mathscr{G}$ is a nest containing $\mathscr{S}_n$; conversely, if $\mathscr{S}_n$ belongs to the nest $\mathscr{G}$, then $\mathscr{K} = \{\emptyset, G'\}$ and $\mathscr{H} = \{\mathscr{S}_n\}$.

Given two elements $T_j \subset T_i$ then $\omega(T_i, T_j) = T_i$ if all the arrows between j and i point towards i, and $\omega(T_i, T_j) = T_j$ if all the arrows between j and i point towards j. The converse is also true. Given two elements $T_j \subset T_i$ so that in between them, only one element R is in $\mathscr{H}$ and none in $\mathscr{K}$, then $\omega(T_i, T_j) = R$.

Given a nest $\mathscr{N}$, we define $\Omega(\mathscr{N})$ to be the set consisting of elements of the form $\omega(T_i, T_{i-1})$ for T_i and T_{i-1} in $\mathscr{N}$ (T_i nonminimal). If $\mathscr{G}$ is a maximal nest, then by definition

$$\mathscr{G} = \mathscr{G} \cup \Omega(\mathscr{G}) . \tag{B.11}$$

We define the corresponding minimal nest $\mathscr{N}^-$ corresponding to $\mathscr{G}$ to be

$$\mathscr{N}^- = \mathscr{K} \cup \mathscr{B} . \tag{B.12}$$

Then, it is clear that the Ω operation on $\mathscr{N}^-$ reconstruct $\mathscr{H}$ $(\Omega(\mathscr{N}^-) \supset \mathscr{H})$ and

$$\mathscr{G} = \mathscr{N}^- \cup \Omega(\mathscr{N}^-) . \tag{B.13}$$

It is straightforward to show that no subnest of $\mathscr{N}^-$ satisfy (B.13). Any nest $\mathscr{N}'$ such that $\mathscr{N}^- \subseteqq \mathscr{N}' \subseteqq \mathscr{G}$ also reconstruct $\mathscr{H}$ by the Ω operation on $\mathscr{N}'$ and

$$\mathscr{G} = \mathscr{N}' \cup \Omega(\mathscr{N}') . \tag{B.14}$$

Conversely, any nest $\mathscr{N}'$ satisfying (B.14) also satisfies $\mathscr{N}^- \subseteqq \mathscr{N}' \subseteqq \mathscr{G}$. From the properties mentioned between (B.10) and (B.11), it follows that

$$\Omega(\mathscr{K}) = \mathscr{H} . \tag{B.15}$$

For any set $\mathscr{N} \in \mathscr{E}'_{n-1}$ given in (B.3) we add the elements $\emptyset$ and G', and we determine by (B.14) its maximal nest $\mathscr{G}$ which we decompose into $\mathscr{K}$, $\mathscr{B}$, and $\mathscr{H}$. Since $\mathscr{S}_n \notin \mathscr{N}$, then $\mathscr{S}_n$ does not belong to $\mathscr{G}$, otherwise $\mathscr{N}^-$ and $\mathscr{N}$ should form a nest with $\mathscr{S}_n$ which contradicts (B.3). Let us partition the set of forest $\mathscr{N} \in \mathscr{E}'_{n-1}$ into groups which have the same $\mathscr{G}$, and consequently the same $\mathscr{K}$ and $\mathscr{H}$ satisfying (B.15). The sum on the left hand side of (B.3) restricted to each group can now be written

$$\begin{aligned} &\prod_{i=n+1}^{2^l-1} (1 - \tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)}) (-\tau_{\mathscr{S}_n}^{-2l(\mathscr{S}_n)}) \prod_{\mathscr{S}' \in \mathscr{N}^- - \{\emptyset, G'\}} \\ &\qquad (-\tau_{\mathscr{S}'}^{-2l(\mathscr{S}')}) \prod_{R' \in \mathscr{H} \cap W_{n-1}} (1 - \tau_{R'}^{-2l(R')}) Z(\alpha) , \end{aligned} \tag{B.16}$$

where W_{n-1} is the set $\{\mathscr{S}_1, \ldots, \mathscr{S}_{n-1}\}$. Let us remind the reader that, because of the property of the function $Z(\alpha)$, all operators τ to the right of $\tau_{\mathscr{S}_n}$ in (B.16) commute. By virtue of (A.9), we can complete the product over $\mathscr{H}$. We obtain

$$\prod_{i=n+1}^{2^l-1} (1-\tau_{\mathscr{S}_i}^{-2l(\mathscr{S}_i)})(-\tau_{\mathscr{S}_n}^{-2l(\mathscr{S}_n)}) \prod_{\mathscr{S}'\in\mathscr{N}^- -\{\emptyset,G'\}} (-\tau_{\mathscr{S}'}^{-2l(\mathscr{S}')}) \prod_{R'\in\mathscr{H}} (1-\tau_{R'}^{-2l(R')}) Z(\alpha). \tag{B.17}$$

We now prove that the expression (B.17) is zero for every group of forest as described above. Let us study the property of

$$\prod_{\mathscr{S}'\in\mathscr{K}-\{\emptyset,G'\}} (-\tau_{\mathscr{S}'}^{-2l(\mathscr{S}')}) \prod_{R'\in\mathscr{H}} (1-\tau_{R'}^{-2l(R')}) Z(\alpha). \tag{B.18}$$

The nest obtained by $\mathscr{S}'\in\mathscr{K}-G'$ and $R'\in\mathscr{H}$ can be written by (B.15)

$$\mathscr{S}_1'\subset R_1'\subset\mathscr{S}_2'\subset\cdots\subset\mathscr{S}_m'\subset\mathscr{R}_m'. \tag{B.19}$$

Note that in the above nest, if $\mathscr{S}_1'=\emptyset$, then $R_1'=\mathscr{S}_2'\cap\mathscr{S}_n$, and, if $\mathscr{S}_1'\neq\emptyset$, then $\mathscr{S}_1'\cap\mathscr{S}_n=\emptyset$; on the other hand $R_m'=\mathscr{S}_m'\cup\mathscr{S}_n$. The number of lines of the nested subgraphs in (B.19) satisfy the topological relation

$$\sum_{i=1}^{m} l(R_i') - \sum_{i=1}^{m} l(\mathscr{S}_i') = l(\mathscr{S}_n). \tag{B.20}$$

Let us dilate the α's in $Z(\alpha)$ in the following fashion: If the line $a\in\mathscr{S}_n$, then $\alpha_a\to\sigma^2\alpha_a$; if $a\in R_i'$, $\alpha_a\to\varrho_i^2\alpha_a$ and if $a\in\mathscr{S}_i'$, $\alpha_a\to\sigma_i^2\alpha_a$. If $\mathscr{S}_1'=\emptyset$ the dilated Z function does not depend on σ_1, but for sake of simplicity we keep the variable σ_1 in what follows. Now, if a does not belong to $\mathscr{S}_n$

$$\alpha_a\to\alpha_a \prod_{i,a\in\mathscr{S}_i'} \sigma_i^2\varrho_i^2. \tag{B.21}$$

If a belongs to $\mathscr{S}_n$, let $\mathscr{S}_i'$ be the minimal element of $\mathscr{K}$ containing a, then it is easy to see that

$$\alpha_a\to\alpha_a \prod_{j=1}^{m} \sigma_j^2\varrho_j^2 \cdot \sigma^2\varrho_{i-1}^2. \tag{B.22}$$

Hence, under this dilatation operation

$$Z\to Z'(\sigma\varrho_i, \sigma_j\varrho_j) \tag{B.23}$$

with $1\leqq i\leqq m$ and $1\leqq j\leqq m$. Since Z' can be expanded in a common Taylor expansion in ϱ_i and σ_i (but not a priori in σ) once common powers of ϱ_i and σ_i have been extracted, we find

$$Z' = \prod_{j=1}^{m} (\varrho_j\sigma_j)^{\mu_j} \prod_{i=1}^{m} (\sigma\varrho_i)^{\nu_i-\mu_i} \sum_{\substack{a_i+b_i\geqq 0\\ b_i\geqq 0}} (\sigma\varrho_i)^{a_i} (\sigma_j\varrho_j)^{b_j} C_{\{a,b\}}, \tag{B.24}$$

where μ_j is the power extracted for the variable σ_j and ν_i the power extracted for the variable ϱ_i; these powers may be complex numbers. Note that in the sum, the power a_i is not bounded from below and a priori preclude a common Taylor series with σ. Also $\mu_1 = b_1 = 0$ if $\mathcal{S}_1' = \emptyset$.

From the discriminating properties of (A.2) and (A.3) generalized to a set of commuting τ operators, and applying the τ operators of (B.18) on (B.24), we reduce the sum over only those a's and b's satisfying

$$\begin{aligned} \mathrm{Re}\,\mu_j + b_j &\leqq -2l(\mathcal{S}_j') \quad \text{for} \quad 1 \leqq j \leqq m, \\ \mathrm{Re}\,\nu_j + a_j + b_j &> -2l(R_j') \quad \text{for} \quad 1 \leqq j \leqq m. \end{aligned} \tag{B.25}$$

From (B.25) and (B.20), we find

$$\sum_{j=1}^{m} \mathrm{Re}\,\nu_j - \sum_{j=1}^{m} \mathrm{Re}\,\mu_j + \sum_{j=1}^{m} a_j > -2l(\mathcal{S}_n). \tag{B.26}$$

Consequently, $\sum_{j=1}^{m} a_j$ is now bounded from below and (B.18) has a Taylor series in σ apart from a possible complex power. The effect of the τ operators, relative to elements of $\mathcal{B} - \{\emptyset\}$ in (B.17) can only increase the minimum power of σ. It is clear now, that the application of $\tau_{\mathcal{S}_n}^{-2l(\mathcal{S}_n)}$ makes (B.17) vanish. This completes the proof of the nested forest formula (B.1). The left hand side of (B.1) is then independent of the ordering.

2. *The Forest Formula*

We shall generalize the nested forest formula for those functions $Z(\alpha)$ which have Taylor series in the dilatation variables corresponding to the subdiagrams of some forests in addition to the forests of nested elements, after all common factors have been removed.

Let us note that any forest which is not a nest has some disjoint elements. Given a forest, a set of disjoint elements of this forest is said to be maximal if any element of the forest that does not contain all of them, is contained in one of them. If no such set exists, then the forest is a forest of nested elements. We group all forests which are not a nest into pairs of the form

$$\{\mathcal{S}_1, \ldots, \mathcal{S}_n, \text{rest}\} \quad \text{and} \quad \{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n, \mathcal{S}_1, \ldots, \mathcal{S}_n, \text{rest}\}, \tag{B.27}$$

where $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ is the maximal disjoint set of elements and where of course $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n$ does not belong to the rest of the forest. Either $Z(\alpha)$ has a Taylor series property with respect to both forests of the pair or with respect to none. If $Z(\alpha)$ has a Taylor series property, the τ's relative to the elements of these forests commute and we can form the sum

$$\dots(1-\tau_{\cup\mathcal{S}_i}^{-2l(\cup\mathcal{S}_i)})(-\tau_{\mathcal{S}_1}^{-2l(\mathcal{S}_1)})\dots(-\tau_{\mathcal{S}_n}^{-2l(\mathcal{S}_n)})\,Z(\alpha)\,, \tag{B.28}$$

which vanishes by virtue of (A.2) and (A.3). Summing over all such pairs of forests and adding the result to the nested forest formula (B.1), we obtain

$$\prod_{i=1}^{2^l-1}(1-\tau_{\mathcal{S}_i}^{-2l(\mathcal{S}_i)})\,Z(\alpha)=\left[1+\sum_{\mathcal{F}}\prod_{\mathcal{S}\in\mathcal{F}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\right]Z(\alpha)\,, \tag{B.29}$$

where the sum runs over all forests which have the Taylor series property. Then, again, the left hand side of (B.29) is independent of the ordering.

If $Z(\alpha)=\dfrac{e^{-pd^{-1}(\alpha)p}}{P(\alpha)^{D/2}}$, the sum runs over all possible forests. If $Z(\alpha)=\dfrac{[pd^{-1}(\alpha)\,p]^{x/2}}{P(\alpha)^{D/2}}$, the sum also runs over all possible forests in the case of non-exceptional momenta. Otherwise, forests containing non-essential elements whose union is essential are not included in the sum. However, in the case of convergent Feynman graph, any τ operator relative to a nonessential element annihilates $Z(\alpha)=\dfrac{[pd^{-1}(\alpha)\,p]^{x/2}}{P(\alpha)^{D/2}}$. Consequently, in the case of convergent Feynman graph, the sum in (B.29) runs over all forests even for exceptional momenta.

3. *Theorem*

Theorem. *If $Z(\alpha)$ has Taylor expansions in the dilatation variables corresponding to the subdiagrams of any forest of nested elements, after all common factors have been removed, then*

a) $\prod_{\mathcal{S}\in\mathcal{N}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\,Z(\alpha)$ *has the same property for any nest $\mathcal{N}$, and*

b) $\left[1+\sum_{\mathcal{N}}\prod_{\mathcal{S}\in\mathcal{N}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\right]Z=\prod_{\mathcal{S}\subset G}(1-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\,Z$ *has a Taylor series with respect to any nest $R^1,\dots,R^m$ after common factors have been removed and the real part of the common power of the dilatation variable corresponding to R^i is $>-2l(R^i)$.*

Proof. This proof is a direct generalization of Part 1.

Let us consider a set of nested elements $R^1,\dots,R^m$. For any nest $T_0=\emptyset\subset T_1\subset\dots\subset T_r\subset T_{r+1}=G'$, where $\emptyset$ and G' are definined in (B.4), we define the ω^i operation relative to one R^i as

$$\omega^i(T_j,T_k)=T_k\cup(R^i\cap T_j)=T_j\cap(R^i\cup T_k)\,, \tag{B.30}$$

where j runs from 1 to $r+1$ and k from 0 to $j-1$. We note that $\omega^p(T_j,T_k)\subseteqq\omega^q(T_j,T_k)$ if $R^p\subset R^q$.

A nest $\mathscr{G}$ of elements $T_0 = \emptyset \subset T_1 \subset \cdots \subset T_r \subset T_{r+1} = G'$ is said to be maximal in regards to $R^1, \ldots, R^m$ if $\omega^j(T_i, T_{i-1})$ belongs to $\mathscr{G}$ for all j and all $T_i \neq \emptyset$, $T_{i-1} \in \mathscr{G}$. Then $\omega^j(T_i, T_{i-1})$ is either T_{i-1} or T_i. There exists for each $i = 1, \ldots, r+1$ a number $p(i)$ such that $0 \leqq p(i) \leqq m$ and

$$\begin{aligned} \omega^j(T_i, T_{i-1}) &= T_{i-1} \quad \text{for} \quad j \leqq p(i) \\ \omega^j(T_i, T_{i-1}) &= T_i \quad \text{for} \quad j > p(i) . \end{aligned} \tag{B.31}$$

Let us map $\mathscr{G}$ into the oriented positive real line by mapping each T_i to the integer i. To the interval $[i-1, i]$, we associate $p(i)$ arrows pointing towards T_{i-1} and $m - p(i)$ arrows pointing towards T_i. Each arrow corresponds to an ω^j operation and points towards the image of this operation. We define $\mathscr{K}$ to be the set consisting of G' $(= T_{r+1})$, of T_i if $p(i) > p(i+1)$, and of $\emptyset(=T_0)$ if at least one arrow points away from it.

Similarly, we define $\mathscr{H}$ as the set of $T's$ in $\mathscr{G}$ such that $p(i) < p(i+1)$, and $\mathscr{B}$ as the set of the remaining elements in $\mathscr{G}$. The elements $\emptyset$ and G' are never in $\mathscr{H}$. We define $\mathscr{N}^-$ as

$$\mathscr{N}^- = \mathscr{K} \cup \mathscr{B} = \mathscr{G} - \mathscr{H} . \tag{B.32}$$

If $\Omega^j(\mathscr{N}^-)$ is the set of elements obtained by performing the ω^j operation upon two consecutive elements of $\mathscr{N}^-$ and if $\Omega(\mathscr{N}^-) = \bigcup_{j=1}^{m} \Omega^j(\mathscr{N}^-)$ then $\mathscr{N}^-$ is the minimal nest such that

$$\Omega(\mathscr{N}^-) \cup \mathscr{N}^- = \mathscr{G} . \tag{B.33}$$

Every nest $\mathscr{N}'$ such that $\mathscr{N}^- \subset \mathscr{N}' \subset \mathscr{G}$ satisfies

$$\Omega(\mathscr{N}') \cup \mathscr{N}' = \mathscr{G} , \tag{B.34}$$

and is obtained by adding to $\mathscr{N}^-$ some elements of $\mathscr{H}$.

If $\mathscr{G}$ is maximal in regards to the Ω operation $(\Omega(\mathscr{G}) \subset \mathscr{G})$, it is clear that $\mathscr{G}$ is also maximal for all Ω^i operation separately. Thus, we can as in Part 1 define respectively $\mathscr{K}^i$ and $\mathscr{H}^i$ relative to the element R^i.

Then, by construction

$$\begin{aligned} \bigcup_{i=1}^{m} \mathscr{K}^i &= \mathscr{K} \\ \bigcup_{i=1}^{m} \mathscr{H}^i &= \mathscr{H} . \end{aligned} \tag{B.35}$$

Although $\mathscr{K}$ and $\mathscr{H}$ are disjoint, the $\mathscr{K}$'s may overlap among themselves for different i, and so do the $\mathscr{H}$'s.

In the nested forest formula, we group all forests of nested elements with the same $\mathscr{G}$. Then, they have the same $\mathscr{B}$, $\mathscr{K}$, $\mathscr{H}$, $\mathscr{K}^i$'s, and $\mathscr{H}^i$'s.

We denote the elements of $\mathscr{K}^i$ by $\{\mathscr{S}_1^i, \mathscr{S}_2^i, \dots \mathscr{S}_{r_i}^i = G'\}$ and the elements of $\mathscr{H}^i$ by $\{H_1^i, \dots, H_{r_i-1}^i\}$ where

$$H_j^i = \mathscr{S}_j^i \cup (\mathscr{S}_{j+1}^i \cap R^i) . \tag{B.36}$$

With this notation, several $\mathscr{S}_j^i$ may represent the same subgraph, and so do H_j^i, but of course no subgraph can be a $\mathscr{S}_j^i$ and a H_j^i. By hypothesis $Z(\alpha)$ has a simultaneous Taylor expansion property in regards to the elements of $\mathscr{G}$. This Taylor expansion property will be used later after performing dilatations in regards to the elements of $\mathscr{B}$, $\mathscr{K}$, and $\mathscr{H}$. Any subgraph which has several names ($\mathscr{S}_j^i$ or H_j^i) will have its variables dilated by several factors, one for each name. To be precise, let a be a line in the graph G; if a belongs to $B_s \in \mathscr{B}$, $\alpha_a \to \alpha_a(\lambda_s)^2$; for every pair $\{i,j\}$ such that a is in $\mathscr{S}_j^i$, $\alpha_a \to \alpha_a(\sigma_j^i)^2$; for every pair $\{i,j\}$ such that a is in H_j^i, $\alpha_a \to \alpha_a(\chi_j^i)^2$; if $a \in R^i$, $\alpha_a \to \alpha_a(\beta^i)^2$. We also dilate in that scheme the empty element $\emptyset$ because $\emptyset$ may be $\mathscr{S}_1^i$ for some i; however in that case, the dilated function Z remains σ_1^i independent. We do not need to dilate the element G' and we might set $\sigma_{r_i}^i = 1$.

If the line $a \notin R^i$ for all i but $a \in \mathscr{S}_j^i$ for $j < r_i$, then $a \in H_j^i$ and α_a is dilated by

$$\alpha_a \to \alpha_a \prod_{a \in \mathscr{S}_j^i} (\sigma_j^i \chi_j^i)^2 . \tag{B.37}$$

If $a \in R^i$ and $a \notin R^{i-1}$, let $\mathscr{S}$ be the minimal element in $\mathscr{K}$ which contains a, then $a \in \mathscr{S}_j^k$ and $a \in H_j^k$ for $\mathscr{S}_j^k \supseteqq \mathscr{S}$, $j < r_k$. For any k such that $i \leqq k \leqq m$, we can determine a number $J(\mathscr{S}, k)$, $1 \leqq J(\mathscr{S}, k) \leqq r_k - 1$, such that $\mathscr{S}_{J(\mathscr{S},k)}^k$ is the largest element in $\mathscr{K}^k$ but not equal to $\mathscr{S}$. Then a is also in $H_{J(\mathscr{S},k)}^k$. Obviously, a is also in R^j for $j \geqq i$. Then no other R, $\mathscr{S}$ and H elements contain a, except G'.

Hence

$$\alpha_a \to \alpha_a \prod_{\substack{k=1,\dots,m \\ j=1,\dots,r_k-1 \\ \mathscr{S}_j^k \supseteqq \mathscr{S}}} (\sigma_j^k \chi_j^k)^2 \prod_{k=i}^{m} (\beta^k \chi_{J(\mathscr{S},k)}^k)^2 , \tag{B.38}$$

so that

$$Z \to Z'(\sigma_j^k \chi_j^k, \beta^k \chi_j^k, \lambda_s) , \tag{B.39}$$

where k runs from 1 to m, and j from 1 to $r_k - 1$.

By hypothesis Z has a simultaneous Taylor expansion in the dilatation variables of the elements of $\mathscr{B}$, $\mathscr{H}$, and $\mathscr{K}$, after the common powers have been extracted, and, consequently, Z' has the same property in regards to the variables λ_s, σ_j^i and χ_j^i. Let $(\lambda_s)^{n_s}$, $(\sigma_j^i)^{\nu_j^i}$ and $(\chi_j^i)^{\mu_j^i}$ be the common powers which we extract from Z' to obtain a Taylor expansion.

Then,

$$Z' = \prod_s (\lambda_s)^{\eta_s} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\sigma_j^i \chi_j^i)^{\nu_j^i} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\beta^i \chi_j^i)^{\mu_j^i - \nu_j^i} \cdot \sum_{a_j=0}^{\infty} \sum_{d_s=0}^{\infty} (\lambda_s)^{d_s} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\sigma_j^i \chi_j^i)^{a_j^i} \Lambda'_{\{d,a\}}(\beta^i \chi_j^i), \tag{B.40}$$

where $\nu_1^i = a_1^i = 0$ if $\mathscr{S}_1^i = \emptyset$, and where we have used only the Taylor property in regards to the elements of $\mathscr{B}$ and $\mathscr{K}$. To use the Taylor property in regards to the elements of $\mathscr{H}$, we have to take into account the variable χ_j^i grouped in $(\sigma_j^i \chi_j^i)^{a_j^i}$ in the above equation. Let us define $\Lambda_{\{d,a\}}(\beta^i \chi_j^i)$ by

$$\Lambda_{\{d,a\}}(\beta^i \chi_j^i) = \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\beta^i \chi_j^i)^{a_j^i} \Lambda'_{\{d,a\}}(\beta^i \chi_j^i), \tag{B.41}$$

then $\Lambda_{\{d,a\}}(\beta^i \chi_j^i)$ has a Taylor expansion in $\beta^i \chi_j^i$. Equation (B.40) is now

$$Z' = \prod_s (\lambda_s)^{\eta_s} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\sigma_j^i \chi_j^i)^{\nu_j^i} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} (\beta^i \chi_j^i)^{\mu_j^i - \nu_j^i} \cdot \sum_{a_j^i=0}^{\infty} \sum_{d_s=0}^{\infty} (\lambda_s)^{d_s} \prod_{i=1}^{m} \prod_{j=1}^{r_i - 1} \left(\frac{\sigma_j^i}{\beta^i}\right)^{a_j^i} \Lambda_{\{d,a\}}(\beta^i \chi_j^i). \tag{B.42}$$

It is then clear that a priori we do not have in addition a common Taylor expansion with the variable β's.

Any forest of nested elements $\mathscr{N}$ belongs to a group characterized by a maximal nest $\mathscr{G}$ and it can be decomposed into $\mathscr{N}^{-}(=\mathscr{B} \cup \mathscr{K})$ and $\mathscr{H}'(\subseteqq \mathscr{H})$. It is certainly possible to duplicate the τ operators for the graphs with several names in $\mathscr{K}$ or for the graphs with several names in $\mathscr{H}'$ so that to each variable σ_j^i or χ_j^i corresponds a τ operator, and so

$$\begin{aligned} &\prod_{\mathscr{S} \in \mathscr{N}} (-\tau_{\mathscr{S}}^{-2l(\mathscr{S})}) Z(\beta^i) \\ &= \prod_s (-\tau_{\lambda_s}^{-2l(B_s)}) \prod_{\substack{\{i,j\} \\ \mathscr{S}_j^i \in \mathscr{K}}} (-\tau_{\sigma_j^i}^{-2l(\mathscr{S}_j^i)}) \prod_{\substack{\{i,j\} \\ H_j^i \in \mathscr{H}'}} (-\tau_{\chi_j^i}^{-2l(H_j^i)}) Z' \Big|_{\lambda=\sigma=\chi=1}. \end{aligned} \tag{B.43}$$

Substituting Z' from (B.42) into this equation, and applying the τ_σ's, we obtain zero if $-2l(\mathscr{S}_j^i) - E'(\nu_j^i) < 0$ for any (i,j); otherwise we obtain an expression similar to (B.42) except that the sum over a_j^i is bounded from above by

$$a_j^i \leqq -2l(\mathscr{S}_j^i) - E'(\nu_j^i). \tag{B.44}$$

Hence, the proof of part a) of the theorem is now completed.

If we sum over all forests of nested elements which belong to the same group characterized by a maximal nest $\mathscr{G}$, we obtain an expression of the form

$$\sum_{\mathcal{N} \in \text{Group}} \prod_{\mathcal{S} \in \mathcal{N}} (-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\, Z(\beta^i)$$

$$= \prod_{s} (-\tau_{\lambda_s}^{-2l(B_s)}) \prod_{\substack{\{i,j\} \\ \mathcal{S}^i_j \in \mathcal{K}}} (-\tau_{\sigma^i_j}^{-2l(\mathcal{S}^i_j)}) \prod_{\substack{\{i,j\} \\ H^i_j \in \mathcal{H}}} (1-\tau_{\chi^i_j}^{-2l(H^i_j)})\, Z' \Big|_{\lambda=\sigma=\chi=1} \tag{B.45}$$

$$= \prod_{i=1}^{m} (\beta^i)^{\sum_{j=1}^{r_i-1} (\mu^i_j - v^i_j)} \sum_{a^i_j=0}^{-2l(\mathcal{S}^i_j)-E'(v^i_j)} \sum_{d_s=0}^{-2l(B_s)-E'(\eta_s)} \prod_{i=1}^{m} (\beta^i)^{-\sum_{j=1}^{r_i-1} a^i_j}$$

$$\cdot \prod_{\substack{\{i,j\} \\ m^i_j \geqq 0}} (\beta^i)^{m^i_j+1} \int_0^1 \prod_{\substack{\{i,j\} \\ m^i_j \geqq 0}} \left\{ d\xi^i_j \frac{(1-\xi^i_j)^{m^i_j}}{m^i_j!} \left(\frac{\partial}{\partial \beta^i \xi^i_j} \right)^{m^i_j+1} \right\} \Lambda_{\{d,a\}}(\beta^i \xi^i_j) \Big|_{\xi^i_j=1,\, m^i_j<0} \tag{B.46}$$

where

$$m^i_j = -2l(H^i_j) - E'(\mu^i_j)\,, \tag{B.47}$$

and where $-2l(\mathcal{S}^i_j) - E'(v^i_j) \geqq 0$ for all (i,j), otherwise (B.45) is zero for that group. The integral in (B.46) exists in a neighbourhood of the β's $=0$ and has a common Taylor expansion in the β's. The real part of the common power of β^i so obtained is larger or equal to

$$\sum_{j=1}^{r_i-1} (\operatorname{Re}\mu^i_j - \operatorname{Re} v^i_j - a^i_j + m^i_j + 1)\,. \tag{B.48}$$

Using (B.44), (B.47) and the topological relation

$$\sum_{j=1}^{r_i-1} [l(\mathcal{S}^i_j) - l(\mathcal{H}^i_j)] = -l(R^i) \quad \text{for} \quad i=1,\dots,m\,, \tag{B.49}$$

the real part of the common power of β^i is found to be strictly larger than $-2l(R^i)$. The second part of the theorem is then proved.

Appendix C

a) Two Estimates about the Function $\varphi(\lambda)$

Given the convergent scalar Feynman amplitude $\varphi(\lambda)$ where all external momentum are scaled by λ,

$$\varphi(\lambda) = \int_0^\infty d\alpha\, e^{-\alpha m^2} e^{-\lambda^2 p d^{-1}(\alpha) p}\, P(\alpha)^{-D/2}\,, \tag{C.1}$$

a trivial bound is obtained since

$$\varphi(\lambda) \leqq \varphi(0)\,. \tag{C.2}$$

On the other hand, let us decompose the domain of integration in (C.1) into sectors [7]. Given a permutation $g=(i_1,\dots,i_l)$ of the integers

$1, \ldots, l$, we define a sector dependent on g to be

$$\mathscr{I}_g = \{\alpha \mid 0 \leqq \alpha_{i_1} \leqq \alpha_{i_2} \leqq \cdots \leqq \alpha_{i_l}\}. \tag{C.3}$$

Then,

$$\varphi(\lambda) = \sum_g \varphi_g(\lambda). \tag{C.4}$$

In each sector $\mathscr{I}_g$, we define a sequence of nested subdiagrams

$$R^j = \{i_1, \ldots, i_j\}, \tag{C.5}$$

for $j = 1, \ldots, l$.

Let us introduce the change of variables

$$\alpha_{i_j} = \beta_j^2 \beta_{j+1}{}^2 \ldots \beta_l^2 \tag{C.6}$$

for $j = 1, \ldots, l$. The domain of integration in β is now

$$\begin{aligned} &0 \leqq \beta_l < \infty \\ &0 \leqq \beta_j \leqq 1, \quad j < l, \end{aligned} \tag{C.7}$$

and the Jacobian of the transformation is $2^l \prod_{i=1}^{l} \beta_i{}^{2i-1}$. In this change of variables, the Symanzik functions become

$$\begin{aligned} p d^{-1}(\alpha) p &\to \prod_{i=k}^{l} \beta_i{}^2 p \Delta_g(\beta) p \\ P(\alpha) &\to \prod_{i=1}^{l} \beta_i{}^{2L(R^i)} P_g(\beta), \end{aligned} \tag{C.8}$$

where $R^k, \ldots, R^l$ are essential subdiagrams. The functions $P_g(\beta)$ and $p\Delta_g(\beta)p$ are strictly positive in the domain of integration and are β_l independent.

Then,

$$\begin{aligned} \varphi_g(\lambda) = 2^l \int_0^\infty d\beta_l \int_0^1 \prod_{i=1}^{l-1} d\beta_i \prod_{i=1}^{l} \beta_i^{-\omega(R^i)-1} e^{-\beta_l^2\left[m_{i_l}^2 + \sum_{j=1}^{l-1} \beta_j^2 \ldots \beta_{l-1}^2 m_{i_j}^2\right]} \\ \cdot e^{-\lambda^2 \prod_{i=k}^{l} \beta_i^2 p \Delta_g(\beta) p} P_g(\beta)^{-D/2}. \end{aligned} \tag{C.9}$$

After the β_l integration, we have

$$\begin{aligned} \varphi_g(\lambda) = 2^{l-1} \Gamma(-\omega(R^l)/2) \int_0^1 \prod_{i=1}^{l-1} [d\beta_i \beta_i^{-\omega(R^i)-1}] P_g(\beta)^{-D/2} \\ \cdot \left[m_{i_l}{}^2 + \sum_{j=1}^{l-1} \beta_j^2 \ldots \beta_{l-1}{}^2 m_{i_j}{}^2 + \lambda^2 \prod_{i=k}^{l-1} \beta_i{}^2 p \Delta_g(\beta) p\right]^{\omega(R^l)/2}. \end{aligned} \tag{C.10}$$

In (C.10), the bracket [ ] is larger or equal to $m_{i_l}{}^2\left[1+\frac{\lambda^2}{m_{i_l}{}^2}\prod_{i=k}^{l-1}\beta_i^2 p\Delta_g(\beta)p\right]$, and we also have

$$\left[1+\frac{\lambda^2}{m_{i_l}{}^2}\prod_{i=k}^{l-1}\beta_i^2 p\Delta_g(\beta)p\right]^{\omega(R^l)/2}<\left(\frac{\lambda}{m_{i_l}}\right)^{\omega_g+\varepsilon}\prod_{i=k}^{l-1}\beta_i^{\omega_g+\varepsilon}[p\Delta_g(\beta)p]^{(\omega_g+\varepsilon)/2}, \tag{C.11}$$

where

$$\omega_g=\sup_{i\in\{k,\ldots,l\}}\omega(R^i) \quad \text{and} \quad 0<\varepsilon<1.$$

Consequently

$$\varphi_g(\lambda)<\lambda^{\omega_g+\varepsilon}\frac{2^{l-1}\Gamma(-\omega(R^l)/2)}{m_{i_l}^{\omega_g-\omega(R^l)+\varepsilon}}\int_0^1\prod_{i=1}^{l-1}[d\beta_i\beta_i^{-\omega(R^i)-1}]\prod_{i=k}^{l-1}\beta_i^{\omega_g+\varepsilon} \tag{C.12}$$
$$P_g(\beta)^{-D/2}[p\Delta_g(\beta)p]^{(\omega_g+\varepsilon)/2}.$$

The integral in (C.12) is convergent.

Using (C.4), we obtain the estimate

$$\varphi(\lambda)<\text{const}\times\lambda^{\omega+\varepsilon} \quad \text{for } \lambda \text{ large enough}, \tag{C.13}$$

where $\omega=\sup_g\omega_g=\sup_{\mathscr{S}\text{ essential}}\omega(\mathscr{S})$.

b) Convergence and Estimate of the Integral (III.15) *when* $|\mathrm{Im}\,x|\to\infty$

We consider the integral given in (III.15)

$$I^g_{\{n\},S}(m^2,p,x)$$
$$=\int_0^1\prod_{t\notin S}[d\beta_t\beta_t^{-\omega(R^t)-1}]\prod_{\substack{t\in S\\ R^t\text{ essential}}}\beta_t^x\prod_{\substack{t\notin S\\ t\in\{k,\ldots l-1\}}}(1-T_{\beta_t}^{q_t}) \tag{C.14}$$
$$\cdot Q^g_{\{n\},S}(m^2,p,\{\beta_i:t\notin S\},x).$$

Then, using the remainder formula for the Taylor expansion, we have

$$\prod_{\substack{t\notin S\\ t\in\{k,\ldots,l-1\}}}(1-T_{\beta_t}^{q_t})\,Q^g_{\{n\},S}(m^2,p,\{\beta_t:t\notin S\},x)$$
$$=\prod_{\substack{t\notin S\\ R^t\text{ essential}}}\beta_t^{q_t+1}\int_0^1\prod_{\substack{t\notin S\\ R^t\text{ essential}}}\left[d\xi_t\frac{(1-\xi_t)^{q_t}}{q_t!}\left(\frac{\partial}{\partial\beta_t\xi_t}\right)^{q_t+1}\right]Q'^g_{\{n\},S}(\beta_t\xi_t,\beta_t) \tag{C.15}$$

where the $Q'^g_{\{n\},S}$ function is the $Q^g_{\{n\},S}$ function with the β_t's for $t\notin S$, but R_t essential, dilated by ξ_t.

Now from (III.11) and (III.13) we see that any number of derivatives of $Q^g(m^2, p, \beta, x)$ in regards to any subset of β's is a finite sum of terms of the form

$$\text{Polyn}(x)\ \text{Polyn}(\beta\text{'s})\ C(\beta, x) \tag{C.16}$$

where Polyn means a polynomial,

$$\begin{aligned} C(\beta, x) = &\left[m_{i_l}{}^2 + \sum_{j=1}^{l-1} \beta_j^2 \dots \beta_{l-1}{}^2 m_{i_j}{}^2\right]^{-[x-\omega(R^l)]/2 - v_1} \\ &\cdot P_g(\beta)^{-D/2 - v_2} [p\Delta_g(\beta)\, p]^{x/2 - v_3}, \end{aligned} \tag{C.17}$$

and v_1, v_2, v_3 are non negative integers depending on the number of derivatives which has been performed. The terms $C(\beta, x)$ are continous in the β's for $\beta \geqq 0$. Consequently, the ξ integral in (C.15) is analytic for all x and continous for the β's in the domain of integration.

$$\begin{aligned} |I^g_{\{n\},S}(m^2, p, x)| \leqq &\sum_{\text{finite}} |\text{Polyn}(x)| \int_0^1 \prod_{t \notin S} [d\beta_t \beta_t^{-\omega(R^t)-1}] \\ &\cdot \prod_{\substack{t \notin S \\ R^t \text{ essential}}} \beta_t^{\text{Re}\,x + q_t + 1} \int_0^1 \prod_{\substack{t \notin S \\ R^t \text{ essential}}} \left[d\xi_t \frac{(1-\xi_t)^{q_t}}{q_t!}\right] \\ &\cdot |\text{Polyn}(\xi_t \beta_t, \beta_t)|\ C(\xi_t \beta_t, \beta_t, \text{Re}\,x)\,. \end{aligned} \tag{C.18}$$

$I^g_{\{n\},S}(m^2, p, x)$ is analytic in x for $\text{Re}\,x > \sup_{\substack{t \notin S \\ R^t \text{ essential}}} [\omega(R^t) - q_t - 1]$ and is polynomially bounded in $|\text{Im}\,x|$ when $|\text{Im}\,x| \to \infty$. For any finite interval in $\text{Re}\,x$ inside the region of analyticity of $I^g_{\{n\},S}(m^2, p, x)$, the boundedness is uniform in $\text{Re}\,x$.

c) Properties of the Integral $\int_0^\infty d\alpha g^{(N)}(x, \alpha)$

In the region $\omega - N < \text{Re}\,x \leqq \omega - N + 1$, the function $g^{(N)}(x, \alpha)$ of (III.20) may be expressed as

$$g^{(N)}(x, \alpha) = e^{-\alpha m^2}\left[1 + \sum_{\mathcal{N}} \prod_{\mathcal{S} \in \mathcal{N}} (-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})\right] \{[pd^{-1}(\alpha)\, p]^{x/2} P(\alpha)^{-D/2}\}\,. \tag{C.19}$$

For a given sector (C.3), and its sequence of nested subdiagrams R's in (C.5), we gather all forests of nested elements into groups according to the R's as in Appendix B.

Let

$$Z(\alpha) = [pd^{-1}(\alpha)\, p]^{x/2} P(\alpha)^{-D/2}\,. \tag{C.20}$$

Then, adopting the same dilatation scheme of Appendix B, we find that the common powers η_s, μ^i_j, and ν^i_j in (B.42) are now given by

$$\eta_s = \begin{cases} x - DL(B_s) & \text{if} \quad B_s \text{ is essential} \\ -DL(B_s) & \text{if} \quad B_s \text{ is non-essential} \end{cases} \tag{C.21}$$

$$\nu^i_j = \begin{cases} x - DL(\mathscr{S}^i_j) & \text{if} \quad \mathscr{S}^i_j \text{ is essential} \\ -DL(\mathscr{S}^i_j) & \text{if} \quad \mathscr{S}^i_j \text{ is non-essential and } \neq \emptyset \\ 0 & \text{if} \quad \mathscr{S}^i_j = \emptyset \end{cases} \tag{C.22}$$

$$\mu^i_j = \begin{cases} x - DL(H^i_j) & \text{if} \quad H^i_j \text{ is essential} \\ -DL(H^i_j) & \text{if} \quad H^i_j \text{ is non-essential}. \end{cases} \tag{C.23}$$

Because $g^{(N)}(x, \alpha)$ is defined in the band $\omega - N < \mathrm{Re}\, x \leqq \omega - N + 1$ we have

$$E'(x) = \omega - N + 1\,. \tag{C.24}$$

Then in a way analogous to (B.46), the partial contribution to the right hand side of (C.19) from such a group of forests can be written in a given sector

$$e^{-\beta_l^2\left[m_{l l}^2 + \sum\limits_{j=1}^{l-1} \beta_j^2 \ldots \beta_{l-1}^2 m_{lj}^2\right]} \prod_{i=1}^{l} \beta_i^{\sum\limits_{j=1}^{r_i - 1} (\mu^i_j - \nu^i_j)} \sum_{\substack{d_s = 0 \\ B_s \text{ essential}}}^{\omega(B_s) - \omega + N - 1} \sum_{\substack{d_s = 0 \\ B_s \text{ nonessential}}}^{\omega(B_s)}$$

$$\sum_{\substack{a^i_j = 0 \\ \mathscr{S}^i_j \text{ essential}}}^{\omega(\mathscr{S}^i_j) - \omega + N - 1} \sum_{\substack{a^i_j = 0 \\ \mathscr{S}^i_j \text{ nonessential}}}^{\omega(\mathscr{S}^i_j)} \prod_{\substack{\{i,j\} \\ \mathscr{S}^i_j \neq \emptyset}} \beta_i^{-a^i_j} \prod_{\substack{\{i,j\} \\ m^i_j \geqq 0}} \beta_i^{m^i_j + 1} \tag{C.25}$$

$$\cdot \int_0^1 \prod_{\substack{\{i,j\} \\ m^i_j \geqq 0}} \left[d\xi^i_j \frac{(1 - \xi^i_j)^{m^i_j}}{m^i_j!} \left(\frac{\partial}{\partial \beta^i \xi^i_j} \right)^{m^i_j + 1} \right] A_{\{d,a\}}(\beta^i \xi^i_j, \beta^i)\,,$$

where

$$m^i_j = \begin{cases} \omega(H^i_j) - \omega + N - 1 & \text{if } H^i_j \text{ is essential} \\ \omega(H^i_j) & \text{if } H^i_j \text{ is not essential}, \end{cases} \tag{C.26}$$

and where we have performed the change of variables (C.6). If there exists a nonempty $\mathscr{S}^i_j$ such that $\omega(\mathscr{S}^i_j) - \omega + N - 1 < 0$ for $\mathscr{S}^i_j$ essential or $\omega(\mathscr{S}^i_j) < 0$ for $\mathscr{S}^i_j$ nonessential, then the corresponding group of forests contributes nothing to (C.19).

The same remark holds for the elements of $\mathscr{B}$. The expression (C.25) may now be analytically continued to the region

$$\mathrm{Re}\, x > \omega - N \tag{C.27}$$

(the variable x in (C.25) appears in some μ^i_j, v^i_j, and in $A_{\{d,a\}}$). Indeed from the definition of A and (C.20) the function $A_{\{d,a\}}(\beta^i\xi^i_j)$ is nothing but

$$A_{\{d,a\}}(\beta^i\xi^i_j,\beta^i)=\prod_s\frac{1}{d_s!}\left(\frac{\partial}{d\lambda_s}\right)^{d_s}\prod_{i,j}\frac{\beta_i^{a^i_j}}{a^i_j!}\left(\frac{\partial}{\partial\sigma^i_j}\right)^{a^i_j} \cdot\{[p\Delta(\lambda_s,\sigma^i_j/\beta^i,\beta^i\xi^i_j,\beta^i)\,p]^{x/2}\,P'(\lambda_s,\sigma^i_j/\beta^i,\beta^i\xi^i_j,\beta^i)^{-D/2}\}_{\lambda=\sigma=0}, \tag{C.28}$$

where the functions $p\Delta p$ and P' are similar to the functions in (C.8), and are strictly positive and β_l independent.

Any derivatives of $A_{\{d,a\}}(\beta^i\xi^i_j,\beta^i)$ in regards to $\beta^i\xi^i_j$ is of the same form as $A_{\{d,a\}}(\beta^i\xi^i_j,\beta^i)$, that is a finite sum of terms of the form

$$\mathrm{Polyn}(x)\cdot\mathrm{Polyn}(\beta^i\xi^i_j,\beta^i)\,C(\beta^i\xi^i_j,\beta^i,x), \tag{C.29}$$

where

$$C(\beta^i\xi^i_j,\beta^i,x)=[p\Delta(\beta^i\xi^i_j,\beta^i)\,p]^{x/2-v_1}\,P'(\beta^i\xi^i_j,\beta^i)^{-D/2-v_2}, \tag{C.30}$$

Polyn means a polynomial, and where v_1 and v_2 are nonnegative integers. Note that the β^i's coming from σ^i_j/β^i are cancelled by $(\beta^i)^{a^i_j}$ and $\sigma^i_j=0$. The function C is analytic for all x and continuous in β^i and ξ^i_j in the region of integration.

Substituting (C.28) into (C.25), using (C.29) and integrating over the sector $\mathscr{I}_g$, we have for each sector and for each group of forests

$$\begin{aligned}&\left|\int_{\mathscr{I}_g} d\alpha e^{-\alpha m^2}\sum_{\mathscr{N}\in\mathrm{Group}}\prod_{\mathscr{S}\in\mathscr{N}}(-\tau_{\mathscr{S}}^{-2l(\mathscr{S})})\,[pd^{-1}(\alpha)\,p]^{x/2}\,P(\alpha)^{-D/2}\right|\\ &\leqq \sum_{\substack{d_s=0\\ B_s\ \text{essential}}}^{\omega(B_s)-\omega+N-1}\ \sum_{\substack{d_s=0\\ B_s\ \text{nonessential}}}^{\omega(B_s)}\ \sum_{\substack{a^i_j=0\\ \mathscr{S}^i_j\ \text{essential}}}^{\omega(\mathscr{S}^i_j)-\omega+N-1}\ \sum_{\substack{a^i_j=0\\ \mathscr{S}^i_j\ \text{nonessential}}}^{\omega(\mathscr{S}^i_j)}\ \sum_{\text{finite}}|\mathrm{Polyn}(x)|\\ &\quad\cdot\int_0^1\prod_{i=1}^{l-1}d\beta_i\beta_i^{p^i}\int_0^\infty d\beta_l\beta_l^{p_l}\,e^{-\beta_l^2\left[m_{i_l^2}+\sum_{j=1}^{l-1}\beta_j^2\ldots\beta_{l-1}^2 m_{i_j^2}\right]}\\ &\quad\cdot\int_0^1\prod_{\substack{\{i,j\}\\ m^i_j\geqq 0}}\left[d\xi^i_j\frac{(1-\xi^i_j)^{m^i_j}}{m^i_j!}\right]|\mathrm{Polyn}(\beta^i\xi^i_j,\beta^i)|\,C(\beta^i\xi^i_j,\beta,\mathrm{Re}\,x)\,.\end{aligned} \tag{C.31}$$

In (C.31) the integer p^i's are

$$p^i=2i-1+\sum_{j=1}^{r_i-1}\mathrm{Re}(\mu^i_j-v^i_j)-\sum_{j=1}^{r_i-1}a^i_j+\sum_{\substack{j=1\\ m^i_j\geqq 0}}^{r_i-1}(m^i_j+1)\,. \tag{C.32}$$

From (C.22), (C.23), and (C.26), we have

$$\begin{aligned}p^i\geqq 2l(R^i)-1+&\sum_{\substack{j\\ H^i_j\supseteqq T^i}}\{\mathrm{Re}\,x-\omega+N-1-2l(H^i_j)\}+\sum_{H^i_j\subset T^i}[-2l(H^i_j)]\\ &-\sum_{\substack{j\\ \mathscr{S}^i_j\supseteqq T^i}}\{\mathrm{Re}\,x-\omega+N-1-2l(\mathscr{S}^i_j)\}-\sum_{\substack{j\\ \mathscr{S}^i_j\subset T_i}}[-2l(\mathscr{S}^i_j)]+r_i-1\,,\end{aligned} \tag{C.33}$$

where T^i is the smallest element of $\mathscr{H}^i \cup \mathscr{K}^i$ defined in Appendix B, to be essential. Using the topological identity (B.47)

$$p^i \geqq r_i - 2 + \sum_{\substack{j \\ H^i_j \supseteqq T^i}} \{\mathrm{Re}\,x - \omega + N - 1\} - \sum_{\substack{j \\ \mathscr{S}^i_j \supseteqq T^i}} \{\mathrm{Re}\,x - \omega + N - 1\}. \quad (C.34)$$

From (B.36), $\mathscr{S}^i_j \subset H^i_j$. Then if T^i belongs to $\mathscr{K}^i$, the brackets { } cancel each other and since $r_i \geqq 2$, $p_i \geqq 0$. On the other hand, if T^i belongs to $\mathscr{H}^i$, there is one more essential H^i_j than there are essential $\mathscr{S}^i_j$ and

$$p^i \geqq \mathrm{Re}\,x - \omega + N - 1\,. \quad (C.35)$$

For $\mathrm{Re}\,x > \omega - N, p^i > -1$.

In view of the continuity property of $C(\beta^i \xi^i_j, \beta^i, \mathrm{Re}\,x)$ the integrals over ξ^i_j in (C.31) exist and are bounded for all β^i in the domain of the remaining integrations, whose convergence is then insured by the bound $p^i > -1$ for $\mathrm{Re}\,x > \omega - N$.

Summing (C.31) over all groups of forests of nested elements and then over all sectors, we conclude that the integral $\int_0^\infty d\alpha g^{(N)}(x, \alpha)$ exists for $\mathrm{Re}\,x > \omega - N$ and is polynomially bounded in $|\mathrm{Im}\,x|$ as $|\mathrm{Im}\,x| \to \infty$. Furthermore this boundedness is uniform in $\mathrm{Re}\,x$ in the interval $\omega - N + \varepsilon \leqq \mathrm{Re}\,x \leqq \delta$, where ε is any positive number and δ any number.

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