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Symmetric and Nonsymmetric Macdonald Polynomials

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Abstract. The symmetric Macdonald polynomials may be constructed from the nonsymmetric Macdonald polynomials. This allows us to develop the theory of the symmetric Macdonald polynomials by first developing the theory of their nonsymmetric counterparts. In taking this approach we are able to obtain new results as well as simpler and more accessible derivations of a number of the known fundamental properties of both kinds of polynomials.

Keywords: Macdonald polynomials, q-series, Hecke algebras

1. Introduction

The symmetric Macdonald polynomial $P_\kappa := P_\kappa(x; q, t)$ is a polynomial of n variables $x = (x_1, \ldots, x_n)$ having coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in indeterminants q and t. The symmetric Macdonald polynomial $P_\kappa(x; q, t)$ is labeled by a partition of length $\leq n$ and can be defined as the unique eigenfunction of the operator

$$D_n^1(q,t) = \sum_{i=1}^{n} \sum_{i \neq j} \frac{tx_i - x_j}{x_i - x_j} \tau_i, \tag{1.1}$$

which is of the form

$$P_\kappa(x; q, t) = m_\kappa(x) + \sum_{\mu < \kappa} u_{\kappa\mu} m_\mu(x). \tag{1.2}$$

In (1.2), $m_\kappa(x)$ is the monomial symmetric function in variables $x_1, \ldots, x_n$ and the sum is over the partitions μ which have the same modulus as κ, but are smaller in dominance ordering. The q-shift operator τ_i in (1.1) acts on functions so that

$$(\tau_i f)(x_1 \ldots x_n) = f(x_1, \ldots, qx_i, \ldots x_n).$$

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The symmetric Macdonald polynomials have been the subject of much recent study, both for their mathematical properties [5,20,28] and their applications to the the trigonometric Ruijsenaars–Schneider quantum many body model [22]. They can be viewed as a t-generalization of the symmetric Jack polynomials, the latter being obtained from the former in the limit $t \to 1$ with $q = t^{\alpha}$ and α fixed. In this paper we will develop the theory of the Macdonald polynomials by generalizing the approach taken by Baker and Forrester [6] towards the Jack polynomials.

The strategy is to first develop the theory of nonsymmetric Macdonald polynomials. These polynomials were first introduced [12,25] some time after the seminal work of Macdonald [24] on the symmetric Macdonald polynomials. The symmetric polynomials can be constructed from their nonsymmetric counterparts. This opens the way to using the theory of the nonsymmetric Macdonald polynomials to develop the theory of the symmetric Macdonald polynomials. In taking this approach we will obtain new results as well as new and simpler derivations of known results. In the later case, references will be provided to the original contributors.

2. Preliminaries

In this section we will revise the basic definitions and results of the nonsymmetric, symmetric, and t-antisymmetric polynomials. Following Macdonald [25], the symmetric and t-antisymmetric polynomials will be constructed in terms of their nonsymmetric counterparts, rather than as an independent entity as would stem from making (1.1) and (1.2) the starting point. In addition, dual nonsymmetric Macdonald polynomials will be defined and related to the symmetric and t-antisymmetric Macdonald polynomials. The results presented on this topic are for the most part new.

The nonsymmetric Macdonald polynomials are defined in terms of operators which generate a realization of an extended affine Hecke algebra (see, e.g., [19]). Let s_{ij} be the operator which acts on functions of $x := (x_1, \dots, x_n)$ by interchanging the variables x_i and x_j. The Demazure–Lusztig operators are defined by

$$T_i := t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(s_i - 1), \quad i = 1, \dots, n-1, \tag{2.1}$$

and

$$T_0 := t + \frac{qtx_n - x_1}{qx_n - x_1}(s_0 - 1), \tag{2.2}$$

where $s_i := s_{i\,i+1}$ and $s_0 := s_{1n}\tau_1\tau_n^{-1}$. T_i have the following action on the monomial $x_i^a x_{i+1}^b$ for $1 \le i \le n-1$ (see, e.g, [19]):

$$T_i x_i^a x_{i+1}^b = \begin{cases} (1-t)x_i^{a-1}x_{i+1}^{b+1} + \cdots + (1-t)x_i^{b+1}x_{i+1}^{a-1} + x_i^b x_{i+1}^a, & a > b, \\ tx_i^a x_{i+1}^a, & a = b, \\ (t-1)x_i^a x_{i+1}^b + \cdots + (t-1)x_i^{b-1}x_{i+1}^{a+1} + tx_i^b x_{i+1}^a, & a < b. \end{cases} \tag{2.3}$$

The operator ω is defined by

$$\omega := s_{n-1}\cdots s_2 s_1 \tau_1 = s_{n-1}\cdots s_i \tau_i s_{i-1}\cdots s_1.$$

A realization $\tilde{\mathcal{H}}_n(q,t)$ of an extended affine Hecke algebra is then generated by elements T_i, $0 \le i \le n-1$ and ω, satisfying the relations

$$(T_i - t)(T_i + 1) = 0, \tag{2.4}$$

$$T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1}, \tag{2.5}$$

$$T_i\, T_j = T_j\, T_i, \qquad |i-j| \ge 2, \tag{2.6}$$

$$\omega\, T_i = T_{i-1}\, \omega, \tag{2.7}$$

where the indices $0, 1, \ldots, n-1$ are understood as elements of $\mathbb{Z}_n$. From the quadratic relation (2.4), we have the identity

$$T_i^{-1} = t^{-1} - 1 + t^{-1} T_i. \tag{2.8}$$

Given a permutation σ with reduced word decomposition $\sigma := s_{i_1}\cdots s_{i_p}$, we define

$$T_\sigma := T_{i_1}\cdots T_{i_p}.$$

A composition of n-components is an n-tuple $\eta := (\eta_1, \ldots, \eta_n)$ of non-negative integers. Each η_i is called a component of η. The Cherednik operators [10, 11] are defined by

$$Y_i = t^{-n+i}\, T_i \cdots T_{n-1}\, \omega\, T_1^{-1}\cdots T_{i-1}^{-1}, \qquad 1 \le i \le n.$$

The fact that the Cherednik operators commute with each other, along with the triangularity of their action on $x^\eta := x^{\eta_1}\cdots x^{\eta_n}$, implies that they possess a set of simultaneous eigenfunctions. These are the nonsymmetric Macdonald polynomials E_η which can be defined by the conditions

$$E_\eta(x;q,t) = x^\eta + \sum_{\nu \prec \eta} b_{\eta\nu} x^\nu, \tag{2.9}$$

$$Y_i E_\eta(x;q,t) = q^{\eta_i} t^{-l'_\eta(i)} E_\eta(x;q,t), \quad 1 \le i \le n, \tag{2.10}$$

where

$$l'_\eta(i) := \#\{k \,|\, k < i, \eta_k \ge \eta_i\} + \#\{k \,|\, k > i, \eta_k > \eta_i\}. \tag{2.11}$$

Let η^+ be the unique partition obtained by permuting η. The partial order $\prec$ is defined on compositions having the same modulus so that

$$\nu \prec \eta \quad \text{if either } \nu^+ < \eta^+, \text{ or } \nu^+ = \eta^+ \text{ and } \nu < \eta.$$

where $<$ is the usual dominance ordering for n-tuples, that is, $\nu < \eta$ if and only if $\sum_{i=1}^p (\eta_i - \nu_i) \ge 0$, for all $1 \le p \le n$.

Following [29], $l'_\eta(s) := l'_\eta(i)$ is called the leg co-length of the node $s = (i, j)$ in the composition η. The arm length $a_\eta(s)$, arm co-length $a'_\eta(s)$, and leg length $l_\eta(s)$ are defined by

$$\begin{aligned} a_\eta(s) &:= \eta_i - j, \\ l_\eta(s) &:= \#\{k \mid k < i, j \le \eta_k + 1 \le \eta_i\} + \#\{k \mid k > i\, j \le \eta_k \le \eta_i\}, \\ a'_\eta(s) &:= j - 1. \end{aligned} \tag{2.12}$$

The following associated quantities occur frequently in the theory of the Macdonald polynomials.

$$\begin{aligned} d_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a_\eta(s)+1}t^{l_\eta(s)+1}\right), \quad d'_\eta(q,t) := \prod_{s\in\eta}\left(1 - q^{a_\eta(s)+1}t^{l_\eta(s)}\right), \\ e_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a'_\eta(s)+1}t^{n-l'_\eta(s)}\right), \quad e'_\eta(q,t) := \prod_{s\in\eta}\left(1 - q^{a'_\eta(s)+1}t^{n-1-l'_\eta(s)}\right), \\ b_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a'_\eta(s)}t^{n-l'_\eta(s)}\right). \end{aligned} \tag{2.13}$$

All these constants are equal to unity if $\eta = 0$. We also have the constants

$$l(\eta) := \sum_{s\in\eta} l(s), \qquad l'(\eta) := \sum_{s\in\eta} l'(s). \tag{2.14}$$

For future reference, some properties of these quantities, easily derivable from [29], are listed.

Lemma 2.1. *Let* $\Phi\eta := (\eta_2, \eta_3, \ldots, \eta_n, \eta_1 + 1)$ *and* $\delta_{i\eta}(q,t) := q^{(\eta_i - \eta_{i+1})}t^{(l'_\eta(i+1) - l'_\eta(i))}$. *We have*

$$\begin{aligned} \frac{d_{\Phi\eta}(q,t)}{d_\eta(q,t)} &= \frac{e_{\Phi\eta}(q,t)}{e_\eta(q,t)} = 1 - q^{\eta_1+1}t^{n-l'_\eta(1)}, \\ \frac{d'_{\Phi\eta}(q,t)}{d'_\eta(q,t)} &= \frac{e'_{\Phi\eta}(q,t)}{e'_\eta(q,t)} = 1 - q^{\eta_1+1}t^{n-1-l'_\eta(1)}, \\ l(\Phi\eta) &= l(\eta) + l'_\eta(1), \quad l'(\Phi\eta) = l'(\eta) + n - 1 - l_\eta(1), \end{aligned}$$

where $l'_\eta(1) = \#\{k | k > 1, \eta_k \le \eta_1\}$. *If* $\eta_i > \eta_{i+1}$, *we have*

$$\begin{aligned} \frac{d_{s_i\eta}(q,t)}{d_\eta(q,t)} &= \frac{1 - t\delta_{i,\eta}(q,t)}{1 - \delta_{i,\eta}(q,t)}, \quad \frac{d'_{s_i\eta}(q,t)}{d'_\eta(q,t)} = \frac{1 - \delta_{i,\eta}(q,t)}{1 - t^{-1}\delta_{i,\eta}(q,t)}, \\ e_{s_i\eta}(q,t) &= e_\eta(q,t), \qquad e'_{s_i\eta}(q,t) = e'_\eta(q,t), \qquad b_{s_i\eta}(q,t) = b_\eta(q,t), \\ l(s_i\eta) &= l(\eta) + 1, \qquad l'(s_i\eta) = l'(\eta). \end{aligned}$$

Define a q-shifted factorial by

$$f(a,b) := \frac{(a;q)_\infty}{(ab;q)_\infty}, \quad \text{where} \quad (c,q)_\infty := \prod_{i=0}^{\infty}(1-cq^i).$$

The q-shifted factorial $f(a,b)$ is an analytic function provided $|q|, |a|, |b| < 1$. It can then be identified with the formal series associated with its series expansion about $a = b = q = 0$. Given polynomials h and h' of indeterminants $a_1, \ldots a_m$, the formal Laurent series $f(h(a_1,\ldots,a_m), h'(a_1,\ldots,a_m))$ is fixed by substituting $a \mapsto h(a_1,\ldots,a_m)$, $b \mapsto h'(a_1,\ldots,a_m)$ into the formal series identified with $f(a,b)$. If $b = q^k$, we can write

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

The q-gamma function is defined by

$$\Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}, \quad 0 < q < 1.$$

We remark that with the generalized factorial defined by

$$\begin{aligned} [q^x]_\eta^{(q,t)} &:= \prod_{s\in\eta^+}\left(t^{l'_\eta(s)} - q^{a'_\eta(s)+x}\right) \\ &= t^{l'(\eta^+)}(1-q)^{|\eta|}\prod_{i=1}^{n}\frac{\Gamma_q(x-\frac{1}{\alpha}(i-1)+\eta_i^+)}{\Gamma_q(x-\frac{1}{\alpha}(i-1))}, \end{aligned} \tag{2.15}$$

we have

$$\begin{aligned} e_\eta(q,t) &= t^{-l'(\eta^+)}[q^{1+\frac{n}{\alpha}}]_{\eta^+}^{q,t}, \\ e'_\eta(q,t) &= t^{-l'(\eta^+)}[q^{1+\frac{1}{\alpha}(n-1)}]_{\eta^+}^{q,t}, \\ b_\eta(q,t) &= t^{-l'(\eta^+)}[q^{\frac{n}{\alpha}}]_{\eta^+}^{q,t}. \end{aligned} \tag{2.16}$$

For any Laurent formal series f, let $\mathrm{CT}(f)$ denote the constant term of that series with respect to x. We can define a scalar product on $\mathbb{Q}(q,t)[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ by

$$\langle f,g\rangle_{q,t} := \mathrm{CT}\left(f(x;q,t)g(\frac{1}{x};\frac{1}{q},\frac{1}{t})W(x)\right),$$

where

$$W(x) := W(x;q,t) := \prod_{1\le i<j\le n}\frac{(\frac{x_i}{x_j};q)_\infty}{(t\frac{x_i}{x_j};q)_\infty}\frac{(q\frac{x_j}{x_i};q)_\infty}{(qt\frac{x_j}{x_i};q)_\infty}.$$

The nonsymmetric Macdonald polynomials have the following orthogonality property, which can be deduced from (2.10).

Proposition 2.2. [24] *The polynomials $E_\eta(x;q,t)$ form an orthogonal set with respect to $\langle\cdot,\cdot\rangle_{q,t}$.*

A consequence of this is that the nonsymmetric Macdonald polynomials are able to be constructed by means of a Gram–Schmidt procedure. Let $\eta^{(1)} \prec \cdots \prec \eta^{(p)}$ be a chain of compositions satisfying:

$$\text{If} \quad \eta^{(i)} \prec \mu \prec \eta^{(i+1)}, \quad \text{then } \mu = \eta^{(i)} \text{ or } \mu = \eta^{(i+1)}. \tag{2.17}$$

The nonsymmetric Macdonald polynomial $E_{\eta^{(p)}}$ can be determined as the unique polynomial satisfying (2.9) which is orthogonal to all $E_{\eta^{(i)}}$ with $i < p$.

The nonsymmetric Macdonald polynomials are elements of the ring of n variable polynomials with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions in q and t. Let the hat symbol $\hat{}$ denote the involution on the ring $\mathbb{Q}(q,t)[x_1^{\pm 1},\ldots,x_n^{\pm}]$ which sends $x_i \mapsto x_{n-i+1}$, $q \mapsto q^{-1}$ and $t \mapsto t^{-1}$. Extend this operator to act on operators so that for any operator T and polynomial f, $\hat{T}\hat{f} = \widehat{(Tf)}$. This operator is an involution on the extended affine Hecke algebra $\tilde{\mathcal{H}}_n(q,t)$ since $\hat{\omega} = \omega^{-1}$ and $\hat{T}_i = T_{n-i}^{-1}$ for all $i \in \mathbb{Z}_n$. These relations follow from the respective definitions and (2.8). We define the dual nonsymmetric Macdonald polynomial by $\hat{E}_\eta(x;q,t) := E_\eta(\underline{x};q^{-1},t^{-1})$, where $\underline{x} := (x_n,\ldots,x_1)$. These polynomials are uniquely determined by the conditions

$$\hat{E}_\eta(x;q,t) = x^{\underline{\eta}} + \sum_{\nu \hat{\prec} \eta} c_{\eta\nu} x^\nu, \tag{2.18}$$

$$\hat{Y}_i \hat{E}_\eta(x;q,t) = q^{-\eta_i} t^{l'_\eta(i)} \hat{E}_\eta(x;q,t), \quad 1 \le i \le n, \tag{2.19}$$

where $\hat{Y}_i = t^{n-i} T_{n-i}^{-1} \cdots T_1^{-1} \omega^{-1} T_{n-1} \cdots T_{n-i+1}$ and the partial order $\hat{\prec}$ is defined on compositions so that

$$\nu \hat{\prec} \eta \ \text{ if either } \ \nu^+ < \eta^+ \ \text{or} \ \nu^+ = \eta^+ \ \text{ and } \ \nu > \eta.$$

Equivalently, $\nu \hat{\prec} \eta$ if and only if $\underline{\nu} \prec \underline{\eta}$.

The dual nonsymmetric Macdonald polynomials are simply related to the nonsymmetric Macdonald polynomials by means of the Demazure–Lusztig operators. We require the following lemma which provides the action of T_σ on the monomial x^η.

Lemma 2.3. *For any composition η and permutation σ, we have*

$$T_\sigma x^\eta = c_{\sigma\eta} x^{\sigma\eta} + \sum_{\upsilon \prec \sigma\eta} c_\upsilon x^\upsilon, \tag{2.20}$$

for scalars $c_\upsilon \in \mathbb{Q}(q,t)$.

Proof. We can write σ as a reduced decomposition $\sigma := t_n \cdots t_1 := s_{k_l} \cdots s_{k_1}$, where

$$t_j := \begin{cases} s_j \cdots s_{\sigma^{-1}(j)+K_j(\sigma^{-1})-1}, & \sigma^{-1}(j) + K_j(\sigma^{-1}) - 1 \ge j, \\ 1, & \text{else}, \end{cases}$$

with $K_j(\sigma) := \#\{i | i < j, \sigma(i) > \sigma(j)\}$. Application of each t_i moves i to the required position, that is, for each $p \in \mathbb{N}_n$, $(t_p \cdots t_1)^{-1}(i) = \sigma^{-1}(i)$ for all $i \leq p$. Since $\sigma\eta := (\eta_{\sigma^{-1}(1)}, \dots, \eta_{\sigma^{-1}(n)})$, we have

$$t_p \cdots t_1 \eta = (\eta_{\sigma^{-1}(1)}, \dots, \eta_{\sigma^{-1}(p)}, \varepsilon_p^{(p)}, \dots, \varepsilon_n^{(p)})$$

for some $\varepsilon_j^{(p)} \geq 0$. It follows from (2.3) that

$$T_\sigma x^\eta = \sum_{\omega \in A(\sigma,\eta)} c_{\omega\eta} x^{\omega\eta} + \sum_{\upsilon | \upsilon^+ < \eta^+} c_\upsilon x^\upsilon,$$

where $\omega \in A(\sigma, \eta)$ if and only if ω has a reduced decomposition.

(1) If $(\omega_{k_{i-1}} \cdots \omega_{k_1}\eta)_{k_i} < (\omega_{k_{i-1}} \cdots \omega_{k_1}\eta)_{k_i+1}$, then $\omega_{k_i} = s_{k_i}$ or $\omega_{k_i} = 1$.
(2) If $(\omega_{k_{i-1}} \cdots \omega_{k_1}\eta)_{k_i} \geq (\omega_{k_{i-1}} \cdots \omega_{k_1}\eta)_{k_i+1}$, then $\omega_{k_i} = s_{k_i}$.

To prove (2.20) it suffices to show that, if $\omega \in A(\sigma, \eta)$, then $\omega\eta \preceq \sigma\eta$. Let $\omega \in A(\sigma, \eta)$ and $\omega_{k_l} \cdots \omega_{k_1}$ be a reduced decomposition of ω satisfying (1) and (2). If t_i is the word $s_{k_r} \cdots s_{k_s}$, let $\omega^{(i)}$ be the word $\omega_{k_r} \cdots \omega_{k_s}$ so that $\omega = \omega^{(n)} \cdots \omega^{(1)}$. For $p \in \mathbb{N}_n$, let $\mu^{(p)} := t_p \dots t_1\eta - \omega^{(p)} \cdots \omega^{(1)}\eta$. Suppose $\mu^{(p)} \succeq 0$ and $\mu_j^{(p)} \leq 0$ for all $j > p$. Let $\mu' := t_{p+1} \cdots t_1\eta - t_{p+1}\omega^{(p)} \cdots \omega^{(1)}\eta$. Then $\mu'_j = \mu_j^{(p)}$ for $j \leq p$ while $(\mu'_{p+1}, \dots, \mu'_n)$ is an arrangement of $(\mu_{p+1}^{(p)}, \dots, \mu_n^{(p)})$. Hence, $\mu' \succeq 0$ and $\mu'_j \leq 0$ for $j \geq p+1$. Let $\mu'' := t_{p+1}\omega^{(p)} \cdots \omega^{(1)}\eta - \omega^{(p+1)}\omega^{(p)} \cdots \omega^{(1)}\eta$. Then by inspection, $\mu''_j = 0$ for $j \leq p$ while $\mu''_{p+1} \geq 0$ and $\mu''_j \leq 0$ for all $j > p+1$. Let $\mu^{(p+1)} := t_{p+1} \dots t_1\eta - \omega^{(p+1)} \cdots \omega^{(1)}\eta$. Since $\mu^{(p+1)} = \mu' + \mu''$, it follows that $\mu^{(p+1)} \succeq 0$ and $\mu_j^{(p+1)} \leq 0$ for all $j < p+1$. By induction, it follows that $\omega\eta \preceq \sigma\eta$. ∎

Lemma 2.4.

$$T_{(n,\dots,1)} \hat{E}_\eta(x; q, t) = t^{\#\{(i,j) | i<j, \eta_i \geq \eta_j\}} E_\eta(x; q, t). \tag{2.21}$$

Proof. Using Lemma (2.3) and noting that $(n, \dots, 2, 1) = t_n \cdots t_1$ with $t_i = s_i \cdots s_{n-1}$, it is simple to verify that

$$T_{(n,\dots,2,1)} x^\eta = t^{\#\{(i,j)|i<j,\eta_i \geq \eta_j\}} x^{\underline{\eta}} + \sum_{\mu \prec \underline{\eta}} a_{\eta\mu} x^\mu,$$

for some $a_{\eta\mu} \in \mathbb{Q}(t)$. It follows that

$$T_{(n,\dots,1)} \hat{E}_\eta(x; q, t) = t^{\#\{(i,j)|i<j,\eta_i \geq \eta_j\}} \left(x^\eta + \sum_{\mu \prec \eta} a'_{\eta\mu} x^\mu \right). \tag{2.22}$$

It suffices then to show that, given a chain $\eta^{(1)} \prec \cdots \prec \eta^{(p)} = \eta$ satisfying (2.17), $T_{(n,\dots,1)} \hat{E}_{\eta^{(p)}}$ is orthogonal to $E_{\eta^{(i)}}$ for all $i < p$. This will be done by induction. If μ is a minimal composition under the partial ordering $\prec$, then $E_\mu(x; q, t) = x^\mu$. It follows from (2.22) that (2.21) is true for the composition $\eta^{(1)}$. Suppose (2.21) is true for $\eta^{(1)},\dots,\eta^{(r-1)}$. Then for any $k < r$,

$$\langle E_{\eta^{(k)}}, T_{(n,\dots,1)}\hat{E}_{\eta^{(r)}}\rangle_{q,t} = t^{-\#\{(i,j)|i<j,\eta_i^{(k)}\geq\eta_j^{(k)}\}}\langle T_{(n,\dots,1)}\hat{E}_{\eta^{(k)}}, T_{(n,\dots,1)}\hat{E}_{\eta^{(r)}}\rangle_{q,t}.$$

Since T_i^{-1} is the adjoint operator of T_i with respect to $\langle\cdot,\cdot\rangle_{q,t}$ and $\langle\hat{f},\hat{g}\rangle_{q,t} = \langle f,g\rangle_{q,t}$, we have

$$\begin{aligned}\langle E_{\eta^{(k)}}, T_{(n,\dots,1)}\hat{E}_{\eta^{(r)}}\rangle_{q,t} &= t^{-\#\{(i,j)|i<j,\eta_i^{(k)}\geq\eta_j^{(k)}\}}\langle \hat{E}_{\eta^{(k)}}, \hat{E}_{\eta^{(r)}}\rangle_{q,t}\\ &= t^{-\#\{(i,j)|i<j,\eta_i^{(k)}\geq\eta_j^{(k)}\}}\langle E_{\eta^{(k)}}, E_{\eta^{(r)}}\rangle_{q,t}\\ &= 0.\end{aligned}$$

∎

Next, we revise the construction of the symmetric and t-antisymmetric Macdonald polynomials from the nonsymmetric Macdonald polynomials. This requires t-analogs of the symmetrization and antisymmetrization operators defined by

$$U^+ := \sum_{\sigma\in S_n} T_\sigma, \quad U^- := \sum_{\sigma\in S_n}(-t)^{-l(\sigma)}T_\sigma,$$

where $l(\sigma) := \#\{(i,j)\,|\,i<j,\,\sigma_i>\sigma_j\}$ is the length of the permutation σ. These operators have the following properties:

$$T_i^{\pm}U^+ = U^+T_i^{\pm} = t^{\pm}U^+, \tag{2.23}$$

$$T_i^{\pm}U^- = U^-T_i^{\pm} = -U^-. \tag{2.24}$$

The operators U^+ and U^- are able to be factorized in terms of the antisymmetrization operator $\mathcal{A}$ [13].

(a)
$$U^+f(x) = \frac{\mathcal{A}(\Delta_{t^{-1}}(x)f(x))}{\Delta(x)} = \mathrm{Sym}\left(\frac{\Delta_{t^{-1}}(x)}{\Delta(x)}f(x)\right), \tag{2.25}$$

(b)
$$U^-f(x) = \frac{\Delta_t(x)}{\Delta(x)}\mathcal{A}f(x), \tag{2.26}$$

where $\mathcal{A} := \sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma$ and $\Delta_t(x) := \prod_{i<j}(x_i - t^{-1}x_j)$. One consequence of (2.25) is that the operator U^+ sends the symmetric monomials onto the Hall–Littlewood polynomials. We have

$$P_\kappa(x;t) = U^+m_\kappa(x),$$

where $P_\kappa(x;t)$ is the Hall–Littlewood polynomial associated with the partition κ (see, e.g., [24]). A further consequence is that

$$U^-s_i = -U^-. \tag{2.27}$$

Now, when acting on symmetric functions, the Macdonald operator $D_n^1(q,t)$ can be decomposed in terms of the Cherednik operators according to [19]

$$D_n^1(q,t) = t^{n-1}\sum_{i=1}^{n} Y_i.$$

Since the operator U^+ commutes with $\sum_{i=1}^{n} Y_i$, it follows from (2.9) and (2.10) that there exist unique symmetric polynomials indexed by partitions which satisfy

$$\prod_{i=1}^{n}(1+uY_i)\, P_\kappa(x;q,t) = \prod_{i=1}^{n}(1+uq^{\kappa_i}t^{-l'_\kappa(i)})\, P_\kappa(x;q,t), \tag{2.28}$$

$$P_\kappa(x;q,t) = m_\kappa(x) + \sum_{\mu<\kappa} u_{\kappa\mu} m_\mu(x). \tag{2.29}$$

From Section 1, these are the symmetric Macdonald polynomials. The symmetric Macdonald polynomials can also be determined as eigenfunctions of a generating function of symmetric functions in Y_i. This implies that P_κ is an eigenfunction of any $f(Y_1,\ldots,Y_n)$ for which f is symmetric. One has the relation

$$P_{\eta^+}(x;q,t) = \frac{1}{\gamma_\eta^+(q,t)} U^+ E_\eta(x;q,t), \tag{2.30}$$

for some scalar $\gamma_\eta^+(q,t)$. We can also define the t-antisymmetric Macdonald polynomials [25]. The t-antisymmetric monomial m'_κ, indexed by the partition κ with non-repeating components, is

$$m'_\kappa := U^- x^\kappa.$$

There are no t-antisymmetric monomials associated with partitions with repeating components as it follows from (2.27) that for such partitions $U^- x^\kappa = 0$. It follows from (2.25) that

$$m'_\kappa = \Delta_t(x)\, s_{\kappa-\delta}(x),$$

where $s_{\kappa-\delta}$ is the Schur polynomial associated with the partition $\kappa-\delta$. When $\kappa=\delta$, we then have

$$m'_\delta = \Delta_t(x). \tag{2.31}$$

A function f is t-antisymmetric if $T_i f = -f$ for all $i = 1,\ldots,n-1$. The t-antisymmetric monomials are a basis for the analytic t-antisymmetric functions. The t-antisymmetric Macdonald polynomials $S_\kappa(x;q,t)$ are indexed by partitions with non-repeating components and can be defined by the following conditions:

$$\prod_{i=1}^{n}(1+uY_i)\, S_\kappa(x;q,t) = \prod_{i=1}^{n}(1+uq^{\kappa_i}t^{-l'_\kappa(i)})\, S_\kappa(x;q,t), \tag{2.32}$$

$$S_\kappa(x;q,t) = m'_\kappa(x) + \sum_{\mu<\kappa} v_{\kappa\mu} m'_\mu(x). \tag{2.33}$$

Analogous to the derivation of (2.30), we have

$$S_{\eta^+}(x;q,t) = \frac{1}{\gamma_\eta^-(q,t)} U^- E_\eta(x;q,t). \tag{2.34}$$

The symmetric and t-antisymmetric Macdonald polynomials can also be expressed as linear combinations of the nonsymmetric Macdonald polynomials.

Lemma 2.5. [25]

(a)
$$P_\kappa(x;q,t) = \sum_{\eta:\eta^+=\kappa} \frac{d'_{\eta^+}(q,t)}{d'_\eta(q,t)} E_\eta(x;q,t) \tag{2.35}$$

(b)
$$S_\kappa(x;q,t) = \sum_{\sigma\in S_n} (-t)^{-l(\sigma)} \frac{d_{\sigma(\kappa)}(q,t)}{d_\kappa(q,t)} E_{\sigma(\kappa)}(x;q,t). \tag{2.36}$$

Proof. A simple generalization of the derivation of the analog results in the case of the Jack polynomials [8]. ∎

It immediately follows from the orthogonality of the nonsymmetric Macdonald polynomials and Lemma 2.5 that

Proposition 2.6. [25] *Both the symmetric Macdonald polynomials* $\{P_\kappa(x;q,t)\}$ *and the* t*-antisymmetric Macdonald polynomials* $\{S_\kappa(x;q,t)\}$ *form orthogonal sets with respect to* $\langle\cdot,\cdot\rangle_{q,t}$.

It follows that both the symmetric and t-antisymmetric Macdonald polynomials are able to be constructed by means of a Gram–Schmidt procedure similar to that in the case of the nonsymmetric polynomials.

The dual nonsymmetric Macdonald polynomials share many properties with the nonsymmetric Macdonald polynomials. In particular, they are equally able to serve as building blocks for the symmetric and t-antisymmetric Macdonald polynomials. This is explained by the following results, the first of which follows from (2.26).

Lemma 2.7.

(a)
$$\hat{U}^+ = t^{-\frac{n(n-1)}{2}} U^+, \tag{2.37}$$

(b)
$$\hat{U}^- = t^{\frac{n(n-1)}{2}} U^-. \tag{2.38}$$

Lemma 2.8.

(a)
$$P_\kappa(x;q,t) = \hat{P}_\kappa(x;q,t) = P_\kappa(x;q^{-1},t^{-1}), \tag{2.39}$$

(b)
$$S_\kappa(x;q,t) = (-t)^{-\frac{n(n-1)}{2}} \hat{S}_\kappa(x;q,t). \tag{2.40}$$

Proof. We shall consider only the second identity as (a) is well known and is proven in a similar way as (b). It follows from Lemma 2.7 that $\hat{m'}_\lambda(x) = (-t)^{n(n-1)/2} m'_\lambda(x)$. Using the defining property (2.33), we then have

$$\hat{S}_\kappa(x;q,t) = (-t)^{\frac{n(n-1)}{2}} \left(m'_\kappa(x) + \sum_{\mu<\kappa} \hat{v}_{\kappa\mu} m'_\mu(x) \right).$$

Since $\{(-t)^{-n(n-1)/2}\hat{S}_\kappa(x;q,t)\}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_{q,t}$ and posseses the triangular structure (2.33), (b) must be true. ∎

Using the above two lemmas in conjunction with (2.30), (2.34), and Lemma 2.5, we obtain the following two lemmas.

Lemma 2.9.

(a) $$P_{\eta^+}(x;q,t) = \frac{t^{-\frac{n(n-1)}{2}}}{\gamma_\eta^+(q^{-1},t^{-1})} U^+ \hat{E}_\eta(x;q,t),$$

(b) $$S_{\eta^+}(x;q,t) = \frac{(-1)^{n(n-1)/2}}{\gamma_\eta^-(q^{-1},t^{-1})} U^- \hat{E}_\eta(x;q,t).$$

Lemma 2.10.

(a) $$P_\kappa(x;q,t) = \sum_{\eta:\eta^+=\kappa} \frac{d'_{\eta^+}(q^{-1},t^{-1})}{d'_\eta(q^{-1},t^{-1})} \hat{E}_\eta(x;q,t),$$

(b) $$S_\kappa(x;q,t) = \sum_{\sigma\in S_n} (-t)^{l(\underline{\sigma})} \frac{d_{\sigma(\kappa)}(q^{-1},t^{-1})}{d_\kappa(q^{-1},t^{-1})} \hat{E}_{\sigma(\kappa)}(x;q,t).$$

where for any permutation σ, $\underline{\sigma} := (\sigma_n,\ldots,\sigma_1)$.

3. Nonsymmetric Macdonald Polynomial Theory

In this section we will derive some of the basic properties of the nonsymmetric Macdonald polynomials independently of the theory of the symmetric Macdonald polynomials. A required preliminary result is the Cauchy-type formula for the nonsymmetric Macdonald polynomials.

Proposition 3.1. [26] *Define*

$$\Omega(x,y;q,t) := \prod_{i=1}^{n} \frac{(qtx_iy_i;q)_\infty}{(x_iy_i;q)_\infty} \prod_{1\le i<j\le n} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} \frac{(qtx_jy_i;q)_\infty}{(qx_jy_i;q)_\infty}, \tag{3.1}$$

then

$$\Omega(x,y;q,t) = \sum_\eta \frac{1}{u_\eta(q,t)} E_\eta(x;q,t) E_\eta(y;q^{-1},t^{-1}), \quad u_\eta(q,t)\in\mathbb{Q}(q,t).$$

The nonsymmetric Macdonald polynomials have the following stability property:

$$E_\eta(x_1,\ldots,x_{n-1},0;q,t) = \begin{cases} E_{(\eta_1,\ldots,\eta_{n-1})}(x_1,\ldots,x_{n-1};q,t), & \eta_n = 0, \\ 0, & \eta_n > 0. \end{cases} \tag{3.2}$$

Applying this property to (3.1) shows that the scalars $u_\eta(q,t)$ are independent of n.

Dunkl has introduced a family of multivariable polynomials which allow a workable treatment of some important constructions and has a close relationship to the theory of the nonsymmetric Macdonald polynomials [14]. The t-analog of these polynomials are the polynomials $q_\eta(x;q,t)$ defined by

$$\Omega(x,y;q,t) := \sum_\eta q_\eta(x;q,t)y^\eta.$$

Corollary 3.2. *Let $\delta_{\nu\eta}$ be 1 if $\nu=\eta$ and 0 otherwise. Define a scalar product by $\langle E_\nu(x;q,t), E_\nu(x;q^{-1},t^{-1})\rangle_q := u_\eta(q,t)\delta_{\nu\eta}$. We have*

$$\langle q_\nu(x;q,t), x^\eta\rangle_q = \delta_{\nu\eta}.$$

Hence, $q_\eta(x;q,t)$ are a basis for the multivariable polynomials with coefficients in $\mathbb{Q}(q,t)$.

Proof. From the triangular structure of the nonsymmetric Macdonald polynomials, $\{\frac{1}{u_\eta(q,t)}E_\eta(x;q,t)\}$ and $\{E_\eta(x;q^{-1},t^{-1})\}$ are a basis for the multivariable polynomials. The scalar product $\langle\cdot,\cdot\rangle_q$ is then well defined. An argument similar to Macdonald's [24, p. 310, 311] can now be used to show that (3.2) is equivalent to (3.1). ∎

The nonsymmetric Macdonald polynomials can be computed recursively by just two kinds of operators. The first are the Demazure–Lusztig operators T_i, $1\le i\le n-1$. The second is the raising-type operator [5, 21]:

$$\Phi_q := x_n T_{n-1}^{-1}\cdots T_2^{-1}T_1^{-1} = t^{i-n}T_{n-1}\cdots T_i x_i T_{i-1}^{-1}\cdots T_1^{-1}.$$

These operators have the following action on the nonsymmetric Macdonald polynomials [5, 26]:

$$\Phi_q E_\eta(x;q,t) = t^{-\#\{i|i>1,\eta_i\le\eta_1\}}E_{\Phi\eta}(x;q,t) \tag{3.3}$$

and

$$T_iE_\eta = \begin{cases} \left(\dfrac{t-1}{1-\delta_{i\eta}^{-1}(q,t)}\right)E_\eta + tE_{s_i\eta}, & \eta_i<\eta_{i+1},\\ tE_\eta, & \eta_i=\eta_{i+1},\\ \left(\dfrac{t-1}{1-\delta_{i\eta}^{-1}(q,t)}\right)E_\eta + \dfrac{(1-t\delta_{i\eta}(q,t))(1-t^{-1}\delta_{i\eta}(q,t))}{(1-\delta_{i\eta}(q,t))^2}E_{s_i\eta}, & \eta_i>\eta_{i+1}. \end{cases} \tag{3.4}$$

Using these operators, it is simple to derive the following two identities by verifying that the respective quantities satisfy the same recursion relationships.

Proposition 3.3. [12] *Let $t^\delta := (1,t,\ldots,t^{n-1})$. We have*

$$E_\eta(t^\delta;q,t) = t^{l(\eta)}\frac{e_\eta(q,t)}{d_\eta(q,t)}. \tag{3.5}$$

Proof. Noting that, for any function $f = f(x)$,

$$(T_i f)(t^{\underline{\delta}}) = t f(t^{\underline{\delta}}) \tag{3.6}$$

shows that

$$\begin{aligned}(\Phi_q E_\eta(x;q,t))\big|_{x=t^{\underline{\delta}}} &= \left(t^{1-n}T_{n-1}\cdots T_1 x_1 E_\eta(x;q,t)\right)\big|_{x=t^{\underline{\delta}}} \\ &= E_\eta(t^{\underline{\delta}};q,t).\end{aligned} \tag{3.7}$$

Using (3.3), we then obtain

$$E_{\Phi\eta}(t^{\underline{\delta}};q,t) = t^{\#\{i|i>1,\eta_i\le\eta_1\}}E_\eta(t^{\underline{\delta}};q,t). \tag{3.8}$$

Supposing $\eta_i > \eta_{i+1}$ and applying (3.6) to (3.4) and rearranging, we also obtain

$$E_{s_i\eta}(t^{\underline{\delta}};q,t) = t\frac{1-\delta_{i,\eta}(q,t)}{1-t\delta_{i,\eta}(q,t)}E_\eta(t^{\underline{\delta}};q,t). \tag{3.9}$$

The relations (3.8) and (3.9) uniquely determine $E_\eta(t^{\underline{\delta}};q,t)$ given $E_0(t^{\underline{\delta}};q,t)$. Since Proposition 3.3 is obviously true for the case $\eta = 0$, all that remains is to show that the right-hand side of (3.5), RHS(η) say, obeys these relations. Using Lemma 2.1, we have

$$\frac{\mathrm{RHS}(\Phi\eta)}{\mathrm{RHS}(\eta)} = t^{\#\{i|i>1,\eta_i\le\eta_1\}}.$$

Supposing $\eta_i > \eta_{i+1}$ and again using Lemma 2.1, we have

$$\mathrm{RHS}(s_i\eta) = t\frac{1-\delta_{i,\eta}(q,t)}{1-t\delta_{i,\eta}(q,t)}\cdot\mathrm{RHS}(\eta).$$

∎

Proposition 3.4. *Write* $\mathcal{N}_\eta^{(E)}(q,t) := \langle E_\eta, E_\eta\rangle_{q,t}$. *We have*

$$\frac{\mathcal{N}_\eta^{(E)}(q,t)}{\mathcal{N}_0^{(E)}(q,t)} = \frac{d'_\eta(q,t)e_\eta(q,t)}{d_\eta(q,t)e'_\eta(q,t)}. \tag{3.10}$$

Remark. Macdonald [25] and Cherednik [12] have derived (3.10) although in a different form.

Proof. Using (3.3) we have

$$\begin{aligned}\langle E_{\Phi\eta}, E_{\Phi\eta}\rangle_{q,t} &= \langle t^{\#\{i|i>1,\eta_i\le\eta_1\}}\Phi_q E_\eta, t^{\#\{i|i>1,\eta_i\le\eta_1\}}\Phi_q E_\eta\rangle_{q,t} \\ &= \mathrm{CT}\Big(t^{\#\{i|i>1,\eta_i\le\eta_1\}}x_n\left(T_{n-1}^{-1}\cdots T_1^{-1}E_\eta(x;q,t)\right) \\ &\qquad \times t^{-\#\{i|i>1,\eta_i\le\eta_1\}}x_n^{-1}\left(T_{n-1}^{-1}\cdots T_1^{-1}E_\eta(x;q,t)\right)\big|_{\frac{1}{x},\frac{1}{q},\frac{1}{t}}W(x;q,t)\Big) \\ &= \langle T_{n-1}^{-1}\cdots T_1^{-1}E_\eta(x;q,t), T_{n-1}^{-1}\cdots T_1^{-1}E_\eta(x;q,t)\rangle_{q,t} \\ &= \langle E_\eta, E_\eta\rangle_{q,t}.\end{aligned}$$

In the last line, we have used the fact that T_i^{-1} is the adjoint operator of T_i with respect to $\langle \cdot, \cdot \rangle_{q,t}$ [25].

Supposing $\eta_i < \eta_{i+1}$ and using (3.4) we have

$$\begin{aligned}\langle E_{s_i\eta}, E_{s_i\eta}\rangle_{q,t} &= \langle t^{-1}T_iE_\eta - \frac{1-t^{-1}}{1-\delta_{i,\eta}^{-1}(q,t)}E_\eta, t^{-1}T_iE_\eta - \frac{1-t^{-1}}{1-\delta_{i,\eta}^{-1}(q,t)}E_\eta\rangle_{q,t} \\ &= \langle T_iE_\eta, T_iE_\eta\rangle_{q,t} - t^{-1}\frac{1-t}{1-\delta_{i,\eta}(q,t)}\langle T_iE_\eta, E_\eta\rangle_{q,t} \\ &\quad -t\frac{1-t^{-1}}{1-\delta_{i,\eta}^{-1}(q,t)}\langle E_\eta, T_iE_\eta\rangle_{q,t} \\ &\quad +\frac{(1-t)(1-t^{-1})}{(1-\delta_{i,\eta}(q,t))\left(1-\delta_{i,\eta}^{-1}(q,t)\right)}\langle E_\eta, E_\eta\rangle_{q,t}. \end{aligned} \tag{3.11}$$

Consider the right-hand side of this expression. The first term simplifies by again using the fact that that T_i^{-1} and T_i are adjoint operators, while the second and third terms simplify by making further use of (3.4) and then noting that, for $\eta_i \neq \eta_{i+1}$, E_η and $E_{s_i\eta}$ are orthogonal. After rearranging, we obtain

$$\langle E_{s_i\eta}, E_{s_i\eta}\rangle_{q,t} = \frac{\left(1-t\delta_{i,\eta}^{-1}(q,t)\right)\left(1-t^{-1}\delta_{i,\eta}^{-1}(q,t)\right)}{\left(1-\delta_{i,\eta}^{-1}(q,t)\right)^2}\langle E_\eta, E_\eta\rangle_{q,t}.$$

By replacing η by $s_i\eta$ and noting that if $\eta_i \neq \eta_{i+1}$, $\delta_{i,s_i\eta} = -\delta_{i,\eta}$, we see that in the case $\eta_i > \eta_{i+1}$

$$\langle E_{s_i\eta}, E_{s_i\eta}\rangle_{q,t} = \frac{(1-\delta_{i,\eta}(q,t))^2}{(1-t\delta_{i,\eta}(q,t))(1-t^{-1}\delta_{i,\eta}(q,t))}\langle E_\eta, E_\eta\rangle_{q,t}. \tag{3.12}$$

Using Lemma 2.1, it is clear that the right-hand side of (3.10) satisfies both the recursion relations (3.11) and (3.12). Since (3.10) is true in the trivial case $\eta = 0$, Proposition 3.4 is true by induction. ∎

We shall now show that the multivariable q-binomial theorem involving the non-symmetric Macdonald polynomials can be deduced using Propositions 3.1 and 3.3.

Proposition 3.5. [26]

$$\prod_{i=1}^{n}\frac{(ax_i;q)_\infty}{(x_i;q)_\infty} = \sum_\eta \frac{[a]_{\eta^+}^{q,t}}{u_\eta(q,t)d_\eta(q,t)}E_\eta(x;q,t). \tag{3.13}$$

Remark. The expression on the right-hand side of (3.13) will be able to be simplified using (4.27).

Proof. In (3.1), first replace n by kn for some $k \in \mathbb{Z}_{>0}$ and then substitute $y_j = t^{kn-j}$ and let $x_{n+1} = \cdots = x_{kn} = 0$. Since $E_\eta(cx) = c^{|\eta|}E_\eta(x)$, we can use Proposition 3.3 to obtain

$$\prod_{i=1}^{n}\frac{(qt^{kn}x_i;q)_\infty}{(x_i;q)_\infty}=\sum_\eta \frac{t^{(kn-1)|\eta|-l(\eta)}e_\eta(\frac{1}{q},\frac{1}{t})\Big|_{n\to kn}}{u_\eta(q,t)d_\eta(\frac{1}{q},\frac{1}{t})}\bar{E}_\eta(x_1,\dots,x_n,0,\dots,0;q,t).$$

Making use of (2.16), Lemma 2.1, the stability property (3.2), and the identity

$$\frac{e_\eta(q^{-1},t^{-1})\big|_{n\to kn}}{d_\eta(q^{-1},t^{-1})}=t^{(l(\eta)+l'(\eta)-(kn-1)|\eta|)}\frac{e_\eta(q,t)\big|_{n\to kn}}{d_\eta(q,t)}, \tag{3.14}$$

we obtain for $k\in\mathbb{Z}_{>0}$

$$\prod_{i=1}^{n}\frac{(qt^{kn}x_i;q)_\infty}{(x_i;q)_\infty}=\sum_\eta\frac{[qt^{kn}]^{q,t}_{\eta^+}}{u_\eta(q,t)d_\eta(q,t)}E_\eta(x;q,t). \tag{3.15}$$

Both sides of (3.15) are power series in x,q,t and t^k. Equating the coefficients with respect to q and x we can apply the following lemma to show that (3.15) is true for all $k\in\mathbb{R}$. We then have (3.13) by letting $a=qt^{kn}$. ∎

Lemma 3.6. [30] *Let $F(z,q)$ and $G(z,q)$ be formal power series in z and q. If $F(q^k,q)=G(q^k,q)$ for infinitely many integers $k\geq 0$, then $F=G$.*

4. A Generalization of the q-Selberg Integral

The q-Selberg integral, as formulated by Askey [2] and subsequently proved by Kadell [16] and Habseiger [15], has been extended by Kadell [16] and Kaneko [17] to involve the symmetric Macdonald polynomial as a factor in the integrand. An equivalent formulation of this result is as a constant term identity which generalizes the q-Morris identity [18]. Here this result will itself be extended in that the symmetric Macdonald polynomials will be replaced by the nonsymmetric Macdonald polynomials. The derivation of this identity will also yield a new derivation of the q-Selberg integral as well as allowing us to specify the constant $u_\eta(q,t)$ appearing in (3.1). The derivation is based on the multivariable q-binomial theorem (3.13).

Using Proposition 3.5, we have

$$\left\langle\prod_{i=1}^{n}\frac{1}{(x_i;q)_r},E_\eta(x;q,t)\right\rangle_{q,t}=\frac{[q^r]^{q,t}_{\eta^+}}{u_\eta(q,t)d_\eta(q,t)}\mathcal{N}^{(E)}_\eta(q,t). \tag{4.1}$$

Letting $x_i\mapsto x^{-1}_{n-i+1}$ inside the argument of the constant term function, an operation that leaves its value unchanged, we obtain

$$\mathrm{CT}\left(\prod_{i=1}^{n}\frac{1}{(x_i^{-1};q)_r}\hat{E}_\eta(x;q,t)W(x)\right)=\frac{[q^r]^{q,t}_{\eta^+}}{u_\eta(q,t)d_\eta(q,t)}\mathcal{N}^{(E)}_\eta(q,t). \tag{4.2}$$

Our first task is to manipulate (4.2) so that $\prod_{i=1}^{n}(x_i^{-1};q)_r^{-1}$ is replaced by

$$\prod_{i=1}^{n}(x_i;q)_a(qx_i^{-1};q)_b.$$

We require

Lemma 4.1. *We have*

$$x^pE_\eta(x;q,t)=E_{\eta+p}(x;q,t), \tag{4.3}$$

where $\eta+p=(\eta_1+p,\dots,\eta_n+p)$ *and* $x^p=(x_1\dots x_n)^p$.

Proof. From the definition of Y_i, we have $Y_ix^p=q^px^pY_i$. Using (2.10), we then obtain

$$Y_i\,(x^pE_\eta(x;q,t))=q^{\eta_i+p}t^{-l'_{\eta+p}(i)}x^pE_\eta(x;q,t),$$

where we have used $l'_\eta(i)=l'_{\eta+p}(i)$. From the defining properties (2.9) and (2.10), we then have the required conclusion. ∎

Corollary 4.2.

$$x^p\hat{E}_\eta(x;q,t)=\hat{E}_{\eta+p}(x;q,t).$$

Using the above proof, we can extend the nonsymmetric Macdonald polynomials to include Laurent polynomials. The defining properties of the Laurent polynomials E_η are the same as for the ordinary nonsymmetric Macdonald polynomials except that they are indexed by n-tuples which can have negative components. The nonsymmetric Macdonald Laurent polynomials can be expressed in terms of the ordinary nonsymmetric Macdonald polynomials by using (4.3). The dual nonsymmetric Macdonald polynomials can be similarly extended to include Laurent polynomials.

Consider (4.2) with η replaced by $\eta+a$. Using Lemma 4.2, we can write $\hat{E}_{\eta+a}=x^a\hat{E}_\eta$. Set $r=-a-b$ with $a,a+b\in\mathbb{Z}_{\geq 0}$. A brief calculation shows that

$$x^a\prod_{i=1}^{n}\frac{1}{(x_i^{-1};q)_r}=(-1)^{na}q^{-\frac{na}{2}(2b+a+1)}\prod_{i=1}^{n}(x'_i;q)_a(\frac{q}{x'_i};q)_b,$$

where $x'_i=q^{b+1}x_i$. Substituting into (4.2), we obtain

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a(\frac{q}{x_i};q)_b\hat{E}_\eta(q^{-(b+1)}x;q,t)W(x)\right)$$
$$=(-1)^{na}\frac{q^{\frac{na}{2}(2b+a+1)}\,[q^r]^{q,t}_{\eta^++a}}{u_{\eta+a}(q,t)\,d_{\eta+a}(q,t)}\mathcal{N}^{(E)}_{\eta+a}(q,t).$$

Since $\hat{E}_\eta(cx)=c^{|\eta|}\hat{E}_\eta(x)$ and $\mathcal{N}^{(E)}_{\eta+a}(q,t)=\mathcal{N}^{(E)}_\eta(q,t)$, we obtain

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a(\frac{q}{x_i};q)_b\hat{E}_\eta(x;q,t)W(x)\right)$$
$$=(-1)^{na}\frac{q^{\left(\frac{na}{2}(2b+a+1)+(b+1)|\eta|\right)}\,[q^r]^{q,t}_{\eta^++a}}{u_{\eta+a}(q,t)\,d_{\eta+a}(q,t)}\,\mathcal{N}^{(E)}_\eta(q,t).$$

The dependence on a in $1/u_{\eta+a}d_{\eta+a}$ can be determined using

Lemma 4.3. *We have*

$$E_\eta(\frac{1}{x}; q, t) = E_{-\underline{\eta}}(\underline{x}; q, t).$$

Proof. Let the star symbol * denote the involution on the ring of n-variable polynomials with coefficients in $\mathbb{Q}(q, t)$ which sends $x_i \to x_i^{-1}$, $q \to q^{-1}$, and $t \to t^{-1}$. Extend this operator to act on operators so that, for any operator T and polynomial f, $T^* f^* = (Tf)^*$. It follows from the respective definitions and (2.8) that $T_i^* = T_i^{-1}$ and $w^* = w$. Using these relations as well as $\hat{T}_i = T_{n-i}^{-1}$ and $\hat{w} = w^{-1}$, we have

$$(Y_{n-i+1}^*)^{-1} = t^{1-n}\hat{Y}_i. \tag{4.4}$$

From (2.10),

$$Y_i^{-1}E_\eta(x; q, t) = q^{-\eta_i}t^{l'_\eta(i)}E_\eta(x; q, t).$$

Applying the * operator and replacing i with $n - i + 1$, we obtain

$$(Y_{n-i+1}^*)^{-1}E_\eta(x^{-1}; q^{-1}, t^{-1}) = q^{\eta_{n-i+1}}t^{-l'_\eta(n-i+1)}E_\eta(x^{-1}; q^{-1}, t^{-1}).$$

Using (4.4), we obtain

$$\hat{Y}_i E_\eta(x^{-1}; q^{-1}, t^{-1}) = q^{\eta_{n-i+1}}t^{n-1-l'_\eta(n-i+1)}E_\eta(x^{-1}; q^{-1}, t^{-1}).$$

From the defining properties (2.18) and (2.19), it follows that $E_\eta(x^{-1}; q^{-1}, t^{-1})$ is a dual nonsymmetric Macdonald polynomial. Since $E_\eta(x^{-1}; q^{-1}, t^{-1})$ has the same leading term as $\hat{E}_{-\underline{\eta}}(x; q, t)$,

$$E_\eta(\frac{1}{x}; q^{-1}, t^{-1}) = E_{-\underline{\eta}}(\underline{x}; q^{-1}, t^{-1}).$$

The conclusion follows. ∎

Corollary 4.4. *We have*

$$\hat{E}_\eta(\frac{1}{x}; q, t) = \hat{E}_{-\underline{\eta}}(\underline{x}; q, t).$$

Now

$$\begin{aligned} &\mathrm{CT}\left(\prod_{i=1}^{n}(x_i; q)_a(\frac{q}{x_i}; q)_b\hat{E}_\eta(x; q, t)W(x)\right) \\ &= \mathrm{CT}\left(\prod_{i=1}^{n}(\frac{q}{x_i}; q)_a(x_i; q)_b\hat{E}_\eta(\frac{q}{\underline{x}}; q, t)W(\frac{q}{\underline{x}})\right) \\ &= q^{|\eta|}\mathrm{CT}\left(\prod_{i=1}^{n}(\frac{q}{x_i}; q)_a(x_i; q)_b\hat{E}_{-\underline{\eta}}(x; q, t)W(x)\right). \end{aligned} \tag{4.5}$$

To obtain the first equality we have used the invariance of the constant term identity under $x_i \mapsto \frac{q}{x_{n-i+1}}$, while to obtain the second equality, we have used Corollary 4.4 and $W(\frac{q}{\underline{x}}) = W(x)$. Applying (4.4) with η replaced by $-\underline{\eta}$ and interchanging a with b gives

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a(\frac{q}{x_i};q)_b\hat{E}_\eta(x;q,t)W(x)\right)$$

$$= (-1)^{nb}\frac{q^{\left(\frac{nb}{2}(2a+b+1)-a|\eta|\right)}[q^r]^{q,t}_{-\underline{\eta}^++b}}{u_{-\underline{\eta}+b}(q,t)\,d_{-\underline{\eta}+b}(q,t)}\mathcal{N}^{(E)}_\eta(q,t). \tag{4.6}$$

We write $\eta \le c$ if $\eta_i \le c$ for all $i = 1,\dots,n$. Equation (4.6) is valid for $\eta \le b$ while (4.4) is valid for $\eta \ge -a$. Equating the right-hand sides of (4.4) and (4.6) and setting $a = 0$, we obtain for $0 \le \eta \le b$

$$\frac{[q^{-b}]^{q,t}_{\eta^+}}{u_\eta(q,t)\,d_\eta(q,t)} = (-1)^{nb}q^{\left(\frac{nb}{2}(b+1)-(b+1)|\eta|\right)}\frac{[q^{-b}]^{q,t}_{-\underline{\eta}^++b}}{u_{-\underline{\eta}+b}(q,t)\,d_{-\underline{\eta}+b}(q,t)}. \tag{4.7}$$

We can use (4.7) to define $[q^{-b}]^{q,t}_{\eta^+}/u_\eta d_\eta$ when $\eta \not\ge 0$ and $\eta \le b$. It then follows that (4.7) is true for all $a, a+b \in \mathbb{Z}_{\ge 0}$ and either $\eta \le b$ or $\eta \ge 0$. This can be used to show that the right-hand sides of (4.6) and (4.4) are equal and hence (4.6) is true for this range of variables. Substituting (4.7) into (4.6), we obtain for this domain

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a(\frac{q}{x_i};q)_b\hat{E}_\eta(x;q,t)W(x)\right)$$

$$= \frac{q^{(nab+(b+1-a)|\eta|)}[q^{-b}]^{q,t}_{\eta^+}[q^{-a-b}]^{q,t}_{-\underline{\eta}^++b}}{u_\eta(q,t)\,d_\eta(q,t)\,[q^{-b}]^{q,t}_{-\underline{\eta}^++b}}\mathcal{N}^{(E)}_\eta(q,t). \tag{4.8}$$

It follows from the property

$$\Gamma_q[x+1] = [x]_q\Gamma_q[x], \quad \text{where} \quad [x]_q := \frac{1-q^x}{1-q}, \tag{4.9}$$

that, for all $k \in \mathbb{Z}$,

$$\frac{\Gamma_q[x+k]}{\Gamma_q[x]} = (-1)^k q^{kx+\frac{1}{2}k(k-1)}\frac{\Gamma_q[1-x]}{\Gamma_q[1-(x+k)]}. \tag{4.10}$$

Using (2.15) and (4.10), we can manipulate the right-hand side of (4.8) to obtain

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a(\frac{q}{x_i};q)_b\hat{E}_\eta(x;q,t)W(x)\right)$$

$$= \frac{q^{(b+1)|\eta|}[q^{-b}]^{q,t}_{\eta^+}}{u_\eta(q,t)\,d_\eta(q,t)}\mathcal{N}^{(E)}_\eta(q,t)\prod_{i=1}^{n}\frac{(q^{b+1}t^{i-1};q)_\infty(q^{a+1+\eta^+_{n-i+1}}t^{i-1};q)_\infty}{(q^{a+b+1}t^{i-1};q)_\infty(q^{1+\eta^+_{n-i+1}}t^{i-1};q)_\infty}. \tag{4.11}$$

Since both sides of (4.11) is a power series in q, q^a, q^b, and t, when $|q|$, $|t| < 1$, $\Re(a+b+1) > 0$ and $\eta \geq 0$, which includes the established domain of validity of both (4.11) and (4.8), we can use Lemma 3.6 to extend the validity of both (4.8) and (4.11) to this domain.

The identity (4.8) can be simplified by taking the limit $a \to \infty$ with $a+b=0$. For this purpose, we first take the ratio of (4.8) to that obtained with $\eta = 0$, thus obtaining

$$\frac{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b \hat{E}_\eta(x;q,t)W(x)\right)}{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b W(x)\right)}$$
$$= \frac{q^{(b+1-a)|\eta|}\,[q^{-b}]^{q,t}_{\eta^+}\,[q^{-b}]^{q,t}_{b}\,[q^{-a-b}]^{q,t}_{-\eta^++b}\,\mathcal{N}^{(E)}_\eta(q,t)}{u_\eta(q,t)\,d_\eta(q,t)\,[q^{-a-b}]^{q,t}_{b}[q^{-b}]^{q,t}_{-\eta^++b}\,\mathcal{N}^{(E)}_0(q,t)}, \tag{4.12}$$

where we used the facts that $[q^{-b}]^{q,t}_0 = d_0 = u_0 = 1$. Computing the the asymptotics requires

Lemma 4.5. [7] *For a Laurent polynomial $f(x_1,\ldots,x_n)$, we have*

$$\left(\frac{\Gamma_q[a+1]}{\Gamma_q[-b]\,\Gamma_q[a+b+1]}\right)^n \prod_{i=1}^n \int_0^1 d_q x_i x_i^{-b-1}\frac{(qx_i;q)_\infty}{(q^{a+b+1}x_i;q)_\infty} f(x_1,\ldots,x_n)$$
$$= \left(\frac{(q,q)_a(q,q)_b}{(q,q)_{a+b}}\right)^n \mathrm{CT}\left(\prod_{i=1}^n (x_i,q)_a(\frac{q}{x_i},q)_b f(q^{-(b+1)}x_1,\ldots,q^{-(b+1)}x_n)\right), \tag{4.13}$$

where $\int_0^1 f(x)d_qx := (1-q)\sum_{i=0}^\infty f(q^j)q^j$ is the q-integral.

Remark. There is a typing error in the statement of the above lemma in [7].

Lemma 4.6. *Suppose $t = q^\lambda$ where $\lambda \in Z_{\geq 0}$. Then*

$$\lim_{\substack{a\to\infty\\ a+b=0}} \frac{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b \hat{E}_\eta(x;q,t)W(x)\right)}{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b W(x)\right)} = q^{|\eta|(b+1)}\hat{E}_\eta(t^{\underline{\delta}};q,t). \tag{4.14}$$

Proof. Fixing $a+b=0$ and applying Lemma 4.5, we obtain

$$\frac{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b \hat{E}_\eta(x;q,t)W(x)\right)}{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b W(x)\right)}$$
$$= q^{(b+1)|\eta|}\frac{\prod_{i=1}^n \int_0^1 d_q x_i x_i^{-b-1}\hat{E}_\eta(x;q,t)W(x)}{\prod_{i=1}^n \int_0^1 d_q x_i x_i^{-b-1} W(x)}. \tag{4.15}$$

Using the definition of the q-integral, we have

$$\prod_{i=1}^{n}\int_0^1 d_qx_i x_i^{-b-1}\hat{E}_\eta(x;q,t)W(x)$$
$$= (1-q)^n \sum_{k_i\in\mathbb{Z}_{\geq 0}} q^{-b\Sigma_{i=1}^n k_i} W(q^{k_1},\dots,q^{k_n})\hat{E}_\eta(q^{k_1},\dots,q^{k_n};q,t).$$

Suppose $\lambda \in \mathbb{Z}_{\geq 0}$. Then

$$W(q^{k_1},\dots,q^{k_n}) = 0 \text{ if } k_{i+1} = k_i - \lambda,\dots,k_i+\lambda-1 \text{ while } W(1,q^\lambda,\dots,q^{\lambda(n-1)}) \neq 0.$$

It follows that in the limit $a \to \infty$ with $a+b=0$

$$\prod_{i=1}^{n}\int_0^1 d_qx_i x_i^{-b-1}\hat{E}_\eta(x;q,t)W(x) \sim (1-q)^n q^{-b\Sigma_{i=1}^n \lambda(i-1)}\hat{E}_\eta(t^{\underline{\delta}};q,t)W(t^{\underline{\delta}}). \quad (4.16)$$

Substituting (4.16) into (4.15) gives (4.14). ∎

Lemma 4.6 gives the asymptotics of the left-hand side of (4.12). We now seek the asymptotics of the right-hand side of (4.12). Use of (4.9) and (4.10), along with

$$\frac{\Gamma_q[x+a]}{\Gamma_q[x]} \sim [x]_q^a, \quad \text{as} \quad x \to \infty,$$

shows that, in the limit $a \to \infty$ with $a+b=0$,

$$[q^{-b}]_{\eta^+}^{q,t} \sim t^{l'(\eta^+)}(1-q)^{|\eta|}[-b]_q^{|\eta|}, \quad (4.17)$$

$$\frac{[q^{-a-b}]_{-\underline{\eta}^++b}^{q,t}}{[q^{-a-b}]_b^{q,t}} \sim (-1)^{|\eta|}(1-q)^{-|\eta|}q^{\left(a|\eta|+\frac{1}{2}\Sigma_{i=1}^n \eta_i^+(\eta_i^++1)\right)}[a]_q^{-|\eta|}, \quad (4.18)$$

$$\frac{[q^{-b}]_b^{q,t}}{[q^{-b}]_{-\underline{\eta}^++b}^{q,t}} = (-1)^{|\eta|}q^{-\frac{1}{2}\Sigma_{i=1}^n \eta_i^+(\eta_i^++1)}t^{-l'(\eta^+)}[q^{1+\lambda(n-1)}]_{\eta^+}^{q,t}. \quad (4.19)$$

Substituting these results into the right-hand side of (4.12), and using Lemma 4.6, we have in the limit $a \to \infty$ with $a+b=0$, $\lambda \in \mathbb{Z}_{\geq 0}$, and $\eta \geq 0$,

$$\hat{E}_\eta(t^{\underline{\delta}};q,t) = \frac{[q^{1+\lambda(n-1)}]_{\eta^+}^{q,t}}{u_\eta(q,t)\,d_\eta(q,t)}\frac{\mathcal{N}_\eta^{(E)}(q,t)}{\mathcal{N}_0^{(E)}(q,t)}. \quad (4.20)$$

Since both sides of this expression can be written as a power series in q and t for $0 < q,t < 1$, we can apply Lemma 3.6 to extend the validity of this result to all $\lambda > 0$. Using this result, (4.17) and

$$\frac{[q^{-a-b}]_{-\underline{\eta}^++b}^{q,t}}{[q^{-a-b}]_b^{q,t}} = \frac{(-1)^{|\eta|}t^{l'(\eta^+)}q^{\left(a|\eta|+\frac{1}{2}\Sigma_{i=1}^n \eta_i(\eta_i+1)\right)}}{[q^{1+a+\lambda(n-1)}]_{\eta^+}^{q,t}},$$

we can simplify (4.12) to obtain

$$\frac{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b \hat{E}_\eta(x;q,t)W(x)\right)}{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b W(x)\right)} = q^{(b+1)|\eta|}\hat{E}_\eta(t^{\underline{\delta}};q,t)\frac{[q^{-b}]^{q,t}_{\eta^+}}{[q^{1+a+\lambda(n-1)}]^{q,t}_{\eta^+}}. \tag{4.21}$$

It follows from (3.4) and Lemma 2.4 that $\{E_\eta\}_{\eta^+=\kappa}$ and $\{\hat{E}_\eta\}_{\eta^+=\kappa}$ span the same set of functions. In particular, we can write

$$E_\mu(x;q,t) = \sum_{\{\eta|\eta^+=\mu^+\}} c_{\mu\eta}\hat{E}_\eta(x;q,t)$$

for scalars $c_{\mu\eta}$. Multiplying both sides of (4.21) by $c_{\mu\eta}$ and summing over distinct permutations of μ^+, we then obtain

Proposition 4.7. *For all* $\Re(a+b+1)>0$, $|q|,|t|<1$, $t=q^\lambda$ *and* $\eta\geq 0$, *we have*

$$\frac{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b E_\eta(x;q,t)W(x)\right)}{\mathrm{CT}\left(\prod_{i=1}^n (x_i;q)_a(\frac{q}{x_i};q)_b W(x)\right)} = q^{(b+1)|\eta|}E_\eta(t^{\underline{\delta}};q,t)\frac{[q^{-b}]^{q,t}_{\eta^+}}{[q^{1+a+\lambda(n-1)}]^{q,t}_{\eta^+}}. \tag{4.22}$$

Note that by multiplying both sides of (4.22) by $d'_{\eta^+}(q,t)/d'_\eta(q,t)$, summing over distinct permutations of $\kappa=\eta^+$ and applying (2.35), we get back Proposition 4.7 with E_η replaced by the symmetric Macdonald polynomial P_κ.

Restraining λ to be a non-negative integer we can use Lemma 4.5 to transform (4.22) into a generalization of the q-Selberg integral.

Proposition 4.8. *For all* $\Re(\lambda_1),\Re(\lambda_2)>0$, $|q|<1$, $t=q^\lambda$, $\lambda\in\mathbb{Z}_{>0}$, *and* $\eta\geq 0$, *we have*

$$\begin{aligned}&\prod_{i=1}^n\int_0^1 d_qx_i x_i^{\lambda_1-1}\frac{(qx_i;q)_\infty}{(q^{\lambda_2}x_i;q)_\infty}E_\eta(x;q,t)\prod_{i<j}x_i^{2\lambda}(q^{1-\lambda}\frac{x_j}{x_i};q)_{2\lambda}\\ &= E_\eta(t^{\underline{\delta}};q,t)\frac{[q^{\lambda_1+\lambda(n-1)}]^{q,t}_{\eta^+}}{[q^{\lambda_1+\lambda_2+2\lambda(n-1)}]^{q,t}_{\eta^+}}\prod_{i=1}^n\int_0^1 d_qx_i x_i^{\lambda_1-1}\frac{(qx_i;q)_\infty}{(q^{\lambda_2}x_i;q)_\infty}\prod_{i<j}x_i^{2\lambda}(q^{1-\lambda}\frac{x_j}{x_i};q)_{2\lambda}.\end{aligned} \tag{4.23}$$

Proof. Apply (4.13) to (4.22) and write

$$W(x) = (-1)^{\lambda n(n-1)/2}q^{n(n-1)\lambda(\lambda-1)/4}\left(\prod_{i=1}^n x_i^{-\lambda(n-1)}\right)\prod_{i<j}x_i^{2\lambda}(q^{1-\lambda}\frac{x_j}{x_i};q)_{2\lambda}.$$

Then let $\lambda_1=-b-\lambda(n-1)$, $\lambda_2=a+b+1$. ∎

The above derivation of Propositions 4.7 and 4.8 has some further consequences in relation to the general theory. First, it allows new derivations of the q-Morris identity and the q-Selberg integral.

Proposition 4.9. [27] *For all* $\Re(a+b+1)>0$, $|q|,|t|<1$, $t=q^\lambda$, *and* $\eta\geq 0$ *we have*

$$\mathrm{CT}\left(\prod_{i=1}^{n}(x_i;q)_a\left(\frac{q}{x_i};q\right)_b W(x)\right) = \prod_{i=1}^{n}\frac{\Gamma_q[1+a+b+\lambda(i-1)]\Gamma_q[1+\lambda i]}{\Gamma_q[1+a+\lambda(i-1)]\Gamma_q[1+b+\lambda(i-1)]\Gamma_q[1+\lambda]}. \tag{4.24}$$

Proof. Let $\eta=0$ in (4.11). Using (2.15) and the evaluation [1,9],

$$\mathcal{N}_0^{(E)}(q,t)=\frac{\Gamma_q[\lambda n+1]}{\Gamma_q[\lambda+1]^n}.$$

∎

The q-Selberg integral can be evaluated as in [7] by applying Lemma 4.5 to the q-Morris identity and making some manipulations.

Proposition 4.10. [2] *For all* $\Re(\lambda_1),\Re(\lambda_2)>0$ *and* $\lambda\in\mathbb{Z}_{>0}$

$$\prod_{i=1}^{n}\int_0^1 d_qx_i x_i^{\lambda_1-1}\frac{(qx_i;q)_\infty}{(q^{\lambda_2}x_i;q)_\infty}\prod_{i<j}x_i^{2\lambda}\left(q^{1-\lambda}\frac{x_j}{x_i};q\right)_{2\lambda} = q^{\lambda\lambda_1\binom{n}{2}+2\lambda^2\binom{n}{3}}\prod_{i=1}^{n}\frac{\Gamma_q[\lambda_1+\lambda(i-1)]\Gamma_q[\lambda_2+\lambda(i-1)]\Gamma_q[1+\lambda i]}{\Gamma_q[\lambda_1+\lambda_2+\lambda(n+i-2)]\Gamma_q[\lambda+1]}. \tag{4.25}$$

We can use (4.25) to simplify Proposition 4.8.

Proposition 4.11. *For all* $\Re(\lambda_1),\Re(\lambda_2)>0$, $|q|<1$, $t=q^\lambda$, $\lambda\in\mathbb{Z}_{>0}$, *and* $\eta\geq 0$, *we have*

$$\prod_{i=1}^{n}\int_0^1 d_qx_i x_i^{\lambda_1-1}\frac{(qx_i;q)_\infty}{(q^{\lambda_2}x_i;q)_\infty}E_\eta(x;q,t)\prod_{i<j}x_i^{2\lambda}\left(q^{1-\lambda}\frac{x_j}{x_i};q\right)_{2\lambda} = q^{\lambda\lambda_1\binom{n}{2}+2\lambda^2\binom{n}{3}}E_\eta(t^\delta;q,t)\prod_{i=1}^{n}\frac{\Gamma_q[\lambda i+1]\Gamma_q[\lambda_1+\lambda(n-i)+\eta_i^+]\Gamma_q[\lambda_2+\lambda(n-i)]}{\Gamma_q[\lambda+1]\Gamma_q[\lambda_1+\lambda_2+\lambda(2n-i-1)+\eta_i^+]}. \tag{4.26}$$

This formula is a generalization of the integration formula of Kadell [16] and Kaneko [17]. The formula of [17] can be reclaimed by multiplying both sides of (4.26) by $d'_{\eta^+}(q,t)/d'_\eta(q,t)$ and summing over distinct permutations of $\kappa=\eta^+$ using (2.35). Writing $E_{\eta+\lambda_1}=x^{\lambda_1}E_\eta$ and noting that $E_{\eta+\lambda_1}(t^\delta;q,t)=q^{\lambda\lambda_1\binom{n}{2}}E_\eta(t^\delta;q,t)$, it is clear that (4.26) is true for all $\eta\in\mathbb{Z}^n$ as long as it is defined.

The second consequence of the derivation of Proposition 4.8 is that it allows us to calculate the normalization constant $u_\eta(q,t)$ appearing in (3.1).

Proposition 4.12. [26]

$$u_\eta(q,t) = \frac{d'_\eta(q,t)}{d_\eta(q,t)}. \tag{4.27}$$

Proof. Using (2.16), (3.5), (3.14) and $E_\eta(cx) = c^{|\eta|}E_\eta(x)$ we obtain $u_\eta(q,t) = d'_\eta(q,t)/d_\eta(q,t)$ for $\lambda \in \mathbb{Z}_{\geq 0}$. Since both sides of this expression can be written as formal power series in q and t if $0 < q, t < 1$, we can apply Lemma 3.6 to show that this result is true for all $\lambda > 0$. ∎

5. Symmetric Macdonald Polynomial Theory

In this section we will deduce symmetric analogs of Propositions 3.1–3.5 and 4.12. This will be done by exploiting the relationships between the symmetric, t-antisymmetric, and nonsymmetric Macdonald polynomials.

In order to deduce the analog of Proposition 3.1, we need to derive the following two results. The first reveals the relationship between the symmetric and t-antisymmetric Macdonald polynomials.

Lemma 5.1.

$$S_{\kappa+\delta}(x;q,t) = t^{-\frac{n(n-1)}{2}}\Delta_t(x)P_\kappa(x;q,qt). \tag{5.1}$$

Proof. Consider

$$\langle \Delta_t(x)P_\kappa(x;q,qt), \Delta_t(x)P_\lambda(x;q,qt)\rangle_{q,t} = \mathrm{CT}\left(\prod_{i<j}(x_i - \frac{1}{t}x_j)h(x)\right), \tag{5.2}$$

where

$$h(x) := \prod_{i<j}(\frac{1}{x_i} - \frac{1}{x_j})\frac{(q\frac{x_i}{x_j};q)_\infty}{(qt\frac{x_i}{x_j};q)_\infty}\frac{(q\frac{x_j}{x_i};q)_\infty}{(qt\frac{x_j}{x_i};q)_\infty}P_\kappa(x;q,qt)P_\lambda(\frac{1}{x};\frac{1}{q},\frac{1}{qt})$$

is an antisymmetric polynomial. Let $[n]_a! := \prod_{j=1}^n [j]_a$. It follows from the identity

$$\mathcal{A}\left(\prod_{i<j}(x_i - ax_j)\right) = [n]_a!\prod_{i<j}(x_i - x_j),$$

that

$$\mathrm{CT}\left(\prod_{i<j}(x_i - ax_j)h(x)\right) = \frac{[n]_a!}{n!}\mathrm{CT}\left(\prod_{i<j}(x_i - x_j)h(x)\right). \tag{5.3}$$

Applying this result twice to the left-hand side of (5.2) gives

$$\begin{aligned}\langle \Delta_t(x)P_\kappa(x;q,qt),\Delta_t(x)P_\lambda(x;q,qt)\rangle_{q,t} &= \frac{[n]_{t^{-1}}!}{[n]_{tq}!}\mathrm{CT}\left(\prod_{i<j}(x_i-qtx_j)h(x)\right)\\ &= \frac{[n]_{t^{-1}}!}{[n]_{tq}!}\langle P_\kappa(x;q,qt),P_\lambda(x;q,qt)\rangle_{q,qt}\\ &= \frac{[n]_{t^{-1}}!}{[n]_{tq}!}\langle P_\kappa(x;q,qt),P_\kappa(x;q,qt)\rangle_{q,qt}\,\delta_{\kappa\lambda}.\end{aligned}$$

The polynomials $t^{-n(n-1)/2}\Delta_t(x)P_\kappa(x;q,qt)$ then form an orthogonal set with respect to $\langle\cdot,\cdot\rangle_{q,t}$. Since they also satisfy (2.33) with leading term $m'_{\kappa+\delta}$, we obtain (5.1). ∎

Lemma 5.2. *We have*

$$U^{-(x)}\left(\prod_{i<j}(1-tx_iy_j)(1-x_jy_i)\right)=\Delta_t(x)\Delta_{t^{-1}}(y). \tag{5.4}$$

Proof. It follows from (2.26) that (5.4) is equivalent to

$$\Delta_t^{-1}(x)\mathcal{A}^{(x)}\left(\prod_{i<j}(1-tx_iy_j)(1-x_jy_i)\right)=\Delta_{t^{-1}}(y), \tag{5.5}$$

where $\mathcal{A}$ is the antisymmetrization operator with respect to x. Arising out of the theory of the Schubert polynomials, we have the reproducing kernel (see, e.g., [23] (2.10), (5.15), (5.2))

$$\Delta_t^{-1}(x)\mathcal{A}^{(x)}\left(f(x)C(x,z)\right)=f(z), \tag{5.6}$$

where $C(x,z):=\prod_{i<j}(z_i-x_j)$ and f is any linear combination of monomials x^η where $\eta_i<n-i$ for each i. Substituting $f(x):=f(x,y):=y^\delta\prod_{i<j}(1-x_iy_j)$, and $z:=1/y$, we obtain (5.5). ∎

We can now give a new derivation of the symmetric analog of Proposition 3.1.

Proposition 5.3. [24] *We have*

$$\begin{aligned}\Pi(x,y;q,t) &= \sum_\kappa \frac{1}{v_\kappa(q,t)}P_\kappa(x;q,t)P_\kappa(y;q,t),\\ \Pi(x,y;q,t) &:= \prod_{i,j=1}^{n}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\end{aligned} \tag{5.7}$$

for scalars $v_\kappa(q,t)$ independent of n.

Proof. To derive (5.7), we apply $U^{-(x)}$ to both sides of (3.1). On the left-hand side, write

$$\Omega(x,y;q,t) = \Pi(x,y;q,qt)\prod_{i<j}(1-tx_iy_j)(1-x_jy_i).$$

Since U^- commutes with symmetric functions Lemma 5.2, we have

$$U^{-(x)}\Omega(x,y;q,t) = \Delta_{t^{-1}}(y)\,\Delta_t(x)\Pi(x,y;q,qt). \tag{5.8}$$

On the right-hand side, we use (2.34) and Lemma 5.1. This gives

$$\Delta_{t^{-1}}(y)\,\Delta_t(x)\Pi(x,y;q,qt) = \sum_{\rho}{}^{*}\frac{\Upsilon_{\bar\rho}(q,t)}{u_\rho(q,t)}\Delta_t(x)\,P_{\rho^+-\delta}(x;q,qt)\,E_\rho(x;\frac{1}{q},\frac{1}{t}), \tag{5.9}$$

where $*$ denotes that the sum is restricted to ρ with distinct components. Now, applying $U^{-(y)}|_{t\to t^{-1}}$ to both sides of (5.9), we see that $\Delta_{t^{-1}}(y)$ is replaced by $[n]_t!\Delta_{t^{-1}}(y)$ on the left-hand side, while on the right-hand side $E_\rho(x;q^{-1},t^{-1})$ is replaced by using Lemma 5.1. Cancelling $\Delta_{t^{-1}}(y)\Delta_t(x)$ from both sides and summing over permutations of ρ which give the same κ gives (5.7). The stability property of the symmetric Macdonald polynomials [24]

$$P_\kappa(x_1,\dots,x_{n-1},0;q,t) = \begin{cases} P_{(\kappa_1,\dots,\kappa_{n-1})}(x_1,\dots,x_{n-1};q,t), & \kappa_n = 0, \\ 0, & \kappa_n > 0 \end{cases} \tag{5.10}$$

applied to (5.7) shows that the $v_\kappa(q,t)$ are independent of n. ∎

Define the polynomials $g_\kappa(x;q,t)$ by [24]

$$\Pi(x,y;q,t) := \sum_\kappa g_\kappa(x;q,t)m_\kappa(y).$$

Corollary 5.4. (cf. [24, pp. 310, 311, 313]) *Define an scalar product by*

$$\langle P_\kappa(x;q,t), P_\mu(x;q,t)\rangle_g := \nu_\kappa(q,t)\delta_{\mu\kappa}.$$

We have

$$\langle g_\mu(x;q,t), m_\kappa(x)\rangle_g = \delta_{\mu\kappa}$$

and hence, the $g_\mu(x;q,t)$ are a basis for the ring of symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$.

Proof. Similar to the proof of Corollary 3.2. ∎

In order to proceed further with the development of the symmetric theory we require the following symmetrization formulas.

Lemma 5.5. *Let* $\eta^R := \underline{(\eta^+)}$ *and* $f_j := f_j(\eta) := \#\{i \mid \eta_i = j\}$. *Then*

(a) $$P_{\eta^+}(x;q,t) = t^{-n(n-1)/2} \prod_{j=0}^{\eta_1^+} \frac{1}{[f_j]_{t^{-1}}!} U^+ E_{\eta^R}(x;q,t), \tag{5.11}$$

(b) $$S_{\eta^+}(x;q,t) = (-1)^{n(n-1)/2} U^- E_{\eta^R}(x;q,t). \tag{5.12}$$

Remark. Using the theory of the symmetric Macdonald polynomials, Baker and Forrester [5, (5.8), (5.18)] derived a more general formula for the constant relating to U^+E_η and P_{η^+}. Their expression is not in the same form as (5.11), although they can be shown to be equal using the first equality of (5.14).

Proof. We shall only consider (a) as the proof of (b) is similar. Since U^+f is symmetric for all f, it follows from (2.3) and the triangular structure of E_{η^R} that

$$U^+E_{\eta^R}(x;q,t) = U^+x^{\eta^R} + \sum_{\mu<\eta^+} a_\mu m_\mu(x)$$

for scalars a_μ. From (2.30), we know that $U^+E_{\eta^R}$ is a scalar multiple of P_{η^+}. To find the scalar multiple, we need to determine $[m_{\eta^+}]U^+x^{\eta^R}$. We first determine $[m_{\eta^+}]U^+x^{\eta^+}$. It follows from (2.3) that

$$T_\sigma x^{\eta^+} = c_\sigma x^{\sigma\eta^+} + \sum_{\mu|\mu^+<\eta^+} c_{\sigma,\mu} x^\mu,$$

so

$$[x^{\eta^+}]U^+x^{\eta^+} = [x^{\eta^+}] \sum_{\sigma|\sigma\eta^+=\eta^+} T_\sigma x^{\eta^+}.$$

Since $T_i x_i^a x_{i+1}^a = t x_i^a x_{i+1}^a$, we have

$$[x^{\eta^+}]U^+x^{\eta^+} = \sum_{\sigma|\sigma\eta^+=\eta^+} t^{l(\sigma)},$$

and so

$$[x^{\eta^+}]U^+x^{\eta^+} = \prod_{i=0}^{\eta_1^+} \sum_{\sigma^{(i)}\in S_{f_i(\eta)}} t^{l(\sigma^{(i)})} = \prod_{i=0}^{\eta_1^+} [f_i(\eta)]_t!.$$

Since $U^+x^{\eta^+}$ is symmetric, we then have

$$[m_{\eta^+}]U^+x^{\eta^+} = \prod_{i=0}^{\eta_1^+} [f_i(\eta)]_t!.$$

Hence, from Lemma 2.7, we have

$$\begin{aligned} [m_{\eta^+}]U^+x^{\eta^R} &= t^{n(n-1)/2} [m_{\eta^+}] \widehat{U^+x^{\eta^+}} \\ &= t^{n(n-1)/2} \prod_{j=0}^{\eta_1^+} [f_j(\eta)]_{t^{-1}}!. \end{aligned} \tag{5.13}$$

∎

We can now deduce the symmetric analog of Proposition 3.5.

Proposition 5.6. [24]

$$P_{\eta^+}(t^{\underline{\delta}};q,t)=\frac{t^{l(\eta^R)}\,[n]_{t^{-1}}!}{\prod_{i=0}^{\eta_1^+}[f_j]_{t^{-1}}!}\frac{e_{\eta^R}(q,t)}{d_{\eta^R}(q,t)}=t^{l(\eta^+)}\frac{b_{\eta^+}(q,t)}{h_{\eta^+}(q,t)}, \tag{5.14}$$

where

$$h_\kappa(q,t):=\prod_{s\in\kappa}(1-q^{a_\eta(s)}t^{l_\eta(s)+1}).$$

Proof. Applying Lemma 5.5(a), we have

$$P_{\eta^+}(t^{\underline{\delta}};q,t)=\frac{t^{-n(n-1)/2}}{\prod_{j=0}^{\eta_1^+}[f_j]_{t^{-1}}!}\sum_{\sigma\in S_n}\left(T_\sigma E_{\eta^R}(x;q,t)\right)\big|_{x=t^{\underline{\delta}}}.$$

Using (3.6), we obtain

$$P_{\eta^+}(t^{\underline{\delta}};q,t)=\frac{t^{-n(n-1)/2}\sum_{\sigma\in S_n}t^{l(\sigma)}}{\prod_{j=0}^{\eta_1^+}[f_j]_{t^{-1}}!}E_{\eta^R}(t^{\underline{\delta}};q,t).$$

Since $t^{-n(n-1)/2}\sum_{\sigma\in S_n}t^{l(\sigma)}=\sum_{\sigma\in S_n}t^{l(\underline{\sigma})}=[n]_{t^{-1}}!$, we obtain the first equality in (5.14) by using Proposition 3.3.

The second equality follows immediately from the identities

$$\frac{[n]_t!}{\prod_{j=0}^{\eta_1^+}[f_j]_t!}\frac{e_{\eta^R}(q,t)}{d_{\eta^R}(q,t)}=\frac{b_{\eta^+}(q,t)}{h_{\eta^+}(q,t)}, \tag{5.15}$$

$$t^{l(\eta^R)-l(\eta^+)}=\frac{[n]_t!}{[n]_{t^{-1}}!}\prod_{j=0}^{\eta_1^+}\frac{[f_j]_{t^{-1}}!}{[f_j]_t!}. \tag{5.16}$$

For the first identity, we use (2.16) and (4.9) to obtain

$$\begin{aligned}\frac{e_{\eta^R}(q,t)}{b_{\eta^+}(q,t)}&=\frac{1}{[n]_t!}\prod_{i=1}^{n}[\alpha\eta_i^+ +n-i+1]_t\\&=\frac{[f_0(\eta)]_t!}{[n]_t!}\prod_{i=1}^{n-f_0(\eta)}[\alpha\eta_i^+ +n-i+1]_t.\end{aligned} \tag{5.17}$$

It suffices then to show that

$$\frac{h_{\eta^+}(q,t)}{\prod_{j=0}^{\eta_1^+}[f_j]_t!}=\frac{d_{\eta^R}(q,t)}{\prod_{i=1}^{n-f_0(\eta)}[\alpha\eta_i^+ +n-i+1]_t}. \tag{5.18}$$

This is an easy consequence of a natural q-generalization of the argument used in [6] to prove the corresponding identity in the Jack polynomial theory.

We now turn to the second identity. Noting that $\frac{[m]_t!}{[m]_{t^{-1}}!} = t^{m(m-1)/2}$, we have

$$\frac{[n]_t!}{[n]_{t^{-1}}!}\prod_{j=0}^{\eta_1^+}\frac{[f_j]_{t^{-1}}!}{[f_j]_t!} = t^{n(n-1)/2-\sum_{j=0}^{\eta_1^+} f_j(f_j-1)/2}.$$

It follows from Lemma 2.1 that

$$l(\eta^R) = l(\sigma) + l(\eta^+),$$

where σ is the permutation of minimum length for which $\eta^+ = \sigma(\eta^R)$. Since the minimum of such a length is $l(\sigma) = n(n-1)/2 - \sum_{j=0}^{\eta_1^+} f_j(f_j-1)/2$, we obtain (5.16). ∎

Proposition 5.7. [24] *Let* $\mathcal{N}_\kappa^{(P)}(q,t) := \langle P_\kappa(x;q,t), P_\kappa(x;q,t)\rangle_{q,t}$. *With* $\eta^+ = \kappa$, *we have*

$$\frac{\mathcal{N}_{\eta^+}^{(P)}(q,t)}{\mathcal{N}_0^{(P)}(q,t)} = \frac{[n]_t!}{\prod_{j=0}^{\eta_1^+}[f_j]_t!}\frac{d'_{\eta^+}(q,t)\, e_{\eta^R}(q,t)}{d_{\eta^R}(q,t)\, e'_{\eta^R}(q,t)} = \frac{b_{\eta^+}(q,t)\, d'_{\eta^+}(q,t)}{h_{\eta^+}(q,t) e'_{\eta^+}(q,t)}. \tag{5.19}$$

Proof. We have

$$\begin{aligned}\langle U^+E_{\eta^R}, U^+E_{\eta^R}\rangle_{q,t} &= \sum_{\sigma\in S_n}\langle U^+E_{\eta^R}, T_\sigma E_{\eta^R}\rangle_{q,t} = \sum_{\sigma\in S_n}\langle T_\sigma^{-1}U^+E_{\eta^R}, E_{\eta^R}\rangle_{q,t}\\ &= \sum_{\sigma\in S_n} t^{-l(\sigma)}\langle U^+E_{\eta^R}, E_{\eta^R}\rangle_{q,t} = [n]_{t^{-1}}!\,\langle U^+E_{\eta^R}, E_{\eta^R}\rangle_{q,t}.\end{aligned}$$

In the second equality, we use the fact that T_i^{-1} is the adjoint operator of T_i, while in the third equality, we use (2.23). Multiplying each side of (5) by $\prod_{i=0}^{\eta_1^+} 1/[f_j]_t![f_j]_{t^{-1}}!$ and using Lemma 5.5 we obtain

$$\langle P_{\eta^+}, P_{\eta^+}\rangle_{q,t} = \frac{t^{n(n-1)/2}[n]_{t^{-1}}!}{\prod_{i=0}^{\eta_1^+}[f_j]_t!}\langle P_{\eta^+}, E_{\eta^R}\rangle_{q,t}.$$

Using (2.35) and the orthogonality of the nonsymmetric Macdonald polynomials, we obtain

$$\langle P_{\eta^+}, P_{\eta^+}\rangle_{q,t}\frac{[n]_t!}{\prod_{i=0}^{\eta_1^+}[f_j]_t!}\frac{d'_{\eta^+}(q,t)}{d'_{\eta^R}(q,t)}\langle E_{\eta^R}, E_{\eta^R}\rangle_{q,t}.$$

Dividing each side by $\mathcal{N}_0^{(P)}(q,t) = \mathcal{N}_0^{(E)}(q,t)$ and using Proposition 3.4, we obtain the equality on the right-hand side of (5.19). The second identity follows from using the identity (5.15). ∎

It remains to establish the analog of Proposition 3.5 and to specify the constant $\nu_\kappa(q,t)$ appearing in Proposition 5.3. We proceed as in the derivation of Proposition 3.5 using (5.7), (5.14), and the identity

$$\frac{b_{\eta^+}(\frac{1}{q},\frac{1}{t})}{h_{\eta^+}(\frac{1}{q},\frac{1}{t})} = t^{(l(\eta^+)+l'(\eta^+)-(n-1)|\eta|)}\frac{b_{\eta^+}(q,t)}{h_{\eta^+}(q,t)}.$$

We obtain

$$\prod_{i=1}^{n}\frac{(ax_i;q)_\infty}{(x_i;q)_\infty} = \sum_{\eta^+}\frac{[a]^{q,t}_{\eta^+}}{\nu_{\eta^+}(q,t)h_{\eta^+}(q,t)}P_{\eta^+}(x;q,t). \tag{5.20}$$

Now substituting (2.35) for P_{η^+} and comparing the results with (3.13), we can read off the value of $\nu_{\eta^+}(q,t)$.

Proposition 5.8. [26]

$$\nu_\kappa(q,t) = \frac{d'_\kappa(q,t)}{h_\kappa(q,t)}.$$

Substituting this result back into (5.20), we obtain the q-binomial theorem involving the symmetric Macdonald polynomials.

Proposition 5.9. [17]

$$\prod_{i=1}^{n}\frac{(ax_i;q)_\infty}{(x_i;q)_\infty} = \sum_{\kappa}\frac{[a]^{q,t}_{\kappa}}{d'_\kappa(q,t)}P_\kappa(x;q,t). \tag{5.21}$$

As an application of the above theory, we can derive the value of the constant $\gamma^-_\eta(q,t)$ appearing in (2.34).

Lemma 5.10.

$$\gamma^-_{\sigma(\rho^+)}(q,t) = (-1)^{l(\sigma)}\frac{d'_{\sigma(\rho^+)}(q,t)\,h_{(\kappa)}(q,qt)}{d_{\rho^+}(q,t)\,d'_{(\kappa)}(q,qt)}, \tag{5.22}$$

where $\rho^+ = \kappa+\delta$.

Proof. Substitute (5.7) with $t\mapsto qt$ into the left-hand side of (5.9) and cancel $\Delta_t(x)$ from both sides. Noting that $P_\kappa(x;q,t) = P_\kappa(x,q^{-1},t^{-1})$, we obtain

$$\sum_\kappa \frac{1}{\nu_\kappa(q,qt)}P_\kappa(x;q,qt)\Delta_{t^{-1}}(y)P_\kappa\left(y;\frac{1}{q},\frac{1}{qt}\right)$$
$$=\sum_\rho{}^{*}\frac{\gamma^-_\rho(q,t)}{u_\rho(q,t)}P_{(\kappa)}(x;q,qt)E_\rho\left(y;\frac{1}{q},\frac{1}{t}\right). \tag{5.23}$$

Using (2.35) and (5.1), we can write

$$\Delta_{t^{-1}}(y)P_\kappa\left(y;\frac{1}{q},\frac{1}{qt}\right) = \sum_{\sigma\in S_n}(-t)^{l(\sigma)}\frac{d_{\sigma(\rho^+)}(\frac{1}{q},\frac{1}{t})}{d_{\rho^+}(\frac{1}{q},\frac{1}{t})}E_{\sigma(\rho^+)}\left(y;\frac{1}{q},\frac{1}{t}\right). \tag{5.24}$$

Substituting (5.23) into the left-hand side of (5.23) and equating the coefficients of $P_\kappa(y; \frac{1}{q}, \frac{1}{qt}) E_{\sigma(\rho^+)}(y; \frac{1}{q}, \frac{1}{q})$, we obtain

$$\gamma^-_{\sigma(\rho^+)}(q,t) = (-t)^{l(\sigma)} \frac{u_{\sigma(\rho^+)}(q,t)}{v_{(\kappa)}(q,qt)} \frac{d_{\sigma(\rho^+)}(\frac{1}{q},\frac{1}{t})}{d_{\rho^+}(\frac{1}{q},\frac{1}{t})}. \tag{5.25}$$

To obtain (5.22), we simplify (5.25) using the identity

$$\frac{d_{\rho^+}(q,t)\, d_{\sigma(\rho^+)}(\frac{1}{q},\frac{1}{t})}{d_{\sigma(\rho^+)}(q,t) d_{\rho^+}(\frac{1}{q},\frac{1}{t})} = t^{-l(\sigma)}.$$

∎

The expression for the constant $\gamma^-_{\sigma(\eta^+)}(q,t)$ in Lemma 5.10 can be simplified using a natural q-generalization of the argument in [6]. The simplification gives

Lemma 5.10′

$$\gamma^-_{\sigma(\eta^+)}(q,t) = (-1)^{l(\sigma)} \frac{d'_{\sigma(\eta^+)}(q,t)}{d'_{\eta^R}(q,t)}.$$

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