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Scattering States and Bound States in $\lambda\mathscr{P}(\phi)_2$*

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Abstract. By analyzing the Bethe-Salpeter equation for even $\lambda\mathscr{P}(\phi)_2$ models we show that for weak coupling the mass spectrum is discrete and of finite multiplicity below $2m$. Moreover on even states of energy less than $4(m-\varepsilon)$ we show that the S matrix is unitary. Here m is the physical mass and $\varepsilon=\varepsilon(\lambda)\to 0$ as $\lambda\to 0$. Our results rely essentially only on a simple assumption about the analyticity of the Bethe-Salpeter kernel which has been verified for weak coupling. For the interaction $\lambda\phi^4$, $(\lambda/m_0^2 \ll 1)$ we show that there are no even bound states of energy less than $4(m-\varepsilon)$.

Introduction

We investigate the energy-momentum spectrum for even $\lambda\mathscr{P}(\phi)_2$ models via the Euclidean Bethe-Salpeter equation. Let $P=(P^0, P^1)$ be the energy-momentum operator acting on the Hilbert space of states $\mathscr{H}$ and define $\Omega\in\mathscr{H}$ to be the vacuum. The first results concerning the spectrum of P were established by Glimm et al. [1, 2]. By using a weak coupling cluster expansion, they showed that the closure of the span of

$$\Omega, e^{ix^0P^0}\phi_0(f_1)\Omega, \ldots, e^{ix^0P^0}\prod_i^N \phi_0(f_i)\Omega, f_i\in C_0^\infty(\mathbb{R})$$

contains all states of energy less than $(N+1)(m-\varepsilon)$ for λ (depending on N) sufficiently small. Here $\varepsilon(\lambda)\to 0$ as $\lambda\to 0$ and $\phi_0(f_i)$ denotes the time zero field smeared with f_i. It was also shown that for even $\mathscr{P}$ the mass operator restricted to the odd subspace of $\mathscr{H}$ has exactly one eigenvalue m on the interval $[0, 3(m-\varepsilon)]$. As a result the Haag-Ruelle theory [3] yields the existence of an isometric S matrix. It has recently been shown that $S\neq 1$ and is asymptotic in λ [4, 5, 13]. For the special case of $\lambda\phi^4$, bound states of energy less than $2m$ were excluded by using correlation inequalities [2].

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The Bethe-Salpeter kernel in quantum field theory is analogous to a non-local potential in quantum mechanics. The Bethe-Salpeter equation takes the form

$$R(k)=R_0(k)-R_0(k)K(k)R(k) \tag{1.1}$$

where $k=(k_0, k_1)$ denotes energy-momentum vector. R_0 and R correspond to the resolvents of the free and interacting two body hamiltonian in quantum mechanics. See § 2 for precise definitions. Unlike quantum mechanics K is not given in closed form. In perturbation theory the kernel K is the sum of all two particle irreducible diagrams in a particular channel. There is a vast literature in physics which investigates (1.1) in an approximation where K is replaced by its first order contribution in perturbation theory. This approximation is referred to as the ladder approximation and is frequently used by physicists to compute bound state energies and wave functions.

In this paper we study the qualitative structure of the energy momentum spectrum assuming strong exponential decay estimates on K in position space (or equivalently analyticity in momentum space). These estimates have been verified in [6] for weakly coupled $\lambda\mathscr{P}(\phi)_2$ models by extending the cluster expansion of [1, 2]. In §3 we prove that these results imply that K is compact relative to R_0. The compactness of K enables us to show that on the even subspace of energy less than $4(m-\varepsilon)$ the spectrum of P is equivalent to that of a free theory of mass m apart from possible bound states. (See § 5.) By the theory of spectral multiplicity and the Haag-Ruelle theory, it is easy to establish a restricted form of unitarity for the S matrix. For the case of weakly coupled $\lambda(\phi^4)_2$ we exclude bound states of energy less than $4(m-\varepsilon)$. Here we use the fact that K is repulsive to first order for $\lambda\phi^4$. Bound states are expected to occur for $\lambda(\phi^6-\phi^4)$ for weak coupling because of the infrared singulatities in one or two space dimensions and the fact that $-\lambda\phi^4$ is attractive.

Next we wish to explain the connection between our work and that of Bros[1]. Several years ago J. Bros [7] made a study of the Bethe-Salpeter equation in the framework of axiomatic field theory. He assumes asymptotic completeness in addition to a number of technical assumptions in order to obtain decay and regularity properties of the Bethe-Salpeter kernel. These properties are slightly stronger than those proved for $\lambda\mathscr{P}(\phi)_2$ in [6]. Now using the Bethe-Salpeter equation and compactness techniques, he shows that the four point function has a double sheeted meromorphic continuation in the energy across the cut $[2m, 4m]$. The poles on the second sheet should correspond to resonances. Thus, although Bros does not analyze the energy momentum spectrum (since in fact he assumes asymptotic completeness), some of our results are implicit in his work.

This paper has only started to answer the fundamental questions concerning scattering in constructive quantum field theory. There are many techniques of potential scattering theory such as dilatation analytic methods which may be useful in the study of the S matrix. The most obvious open problem is to study three body scattering for $\lambda\phi^4$ with the three body equations presented in [8]. The spectral multiplicity techniques used to prove the main theorems of this paper fail to yield unitarity in the three body region[2]. Hence the solution of the

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[1] We wish to thank Professor A. Wightman for pointing out this connection

[2] See Remark 2 in § 2

three body problem may give some insight into the problem of full unitarity for $\lambda\phi^4$. Another problem is to obtain our results away from the weak coupling region. This requires new estimates on the Bethe-Salpeter kernel. We expect that for strong bare coupling such estimates can be obtained by extending the recent cluster expansion of [12].

§ 2. The Bethe-Salpeter Equation

The Euclidean Bethe-Salpeter equation for an even $\lambda\mathscr{P}(\phi)_2$ quantum field model has the following form

$$D(x_1,x_2,x_3,x_4)=D_0(x_1,x_2,x_3,x_4) - \int D_0(x_1,x_2,y_1,y_2)\hat{K}(y_1,y_2,y_3,y_4)D(y_3,y_4,x_3,x_4)dy \quad (2.1)$$

where $x_i=(x_i^0\,x_i^1)$ and $y_i=(y_i^0\,y_i^1)$ are Euclidean coordinates and $\hat{K}$ is the Bethe-Salpeter kernel. Here

$$D(x_1,x_2,x_3,x_4)=S^{(4)}(x_1,x_2,x_3,x_4)-S^{(2)}(x_1,x_2)S^{(2)}(x_3,x_4)$$

and

$$D_0(x_1,x_2,x_3,x_4)=S^{(2)}(x_1,x_3)S^{(2)}(x_2,x_4)+S^{(2)}(x_1,x_4)S^{(2)}(x_2,x_3)$$

and $S^{(2)}$, $S^{(4)}$ denote the two and four point Schwinger functions respectively. The Bethe-Salpeter kernel and (2.1) are defined whenever the physical mass is positive [8].

We reexpress (2.1) with the following change of variables:

$$\xi=\frac{x_2-x_1}{2} \qquad \eta=\frac{x_4-x_3}{2} \qquad \tau=\frac{x_4+x_3}{2}-\frac{x_1+x_2}{2}$$

$$\xi'=\frac{y_2-y_1}{2} \qquad \eta'=\frac{y_4-y_3}{2} \qquad \tau'=\frac{x_1+x_2}{2}-\frac{y_1+y_2}{2}$$

$$\tau''=\frac{y_4+y_3}{2}-\frac{x_3+x_4}{2}$$

with $\xi=(\xi_0,\xi_1)$, $\eta=(\eta_0,\eta_1)$, etc. Using the translation invariance of D, D_0, and $\hat{K}$ and the $\tau\to-\tau$ symmetry Equation (2.1) can be written

$$D(\tau,\xi,\eta)=D_0(\tau,\xi,\eta)-\int D_0(\tau,\xi,\xi')\hat{K}(\tau-\tau'-\tau'',\xi',\eta') \cdot D(\tau'',\eta,\eta')d\xi'd\eta'd\tau'd\tau''\,. \quad (2.3)$$

Let p, q, and k be the momentum conjugate variables of ξ, η, and τ respectively and set $\tilde{k}=(ik_0,k_1)$. We define

$$R(k,p,q)=\int D(\tau,\xi,\eta)e^{i(\tau\cdot\tilde{k}+p\cdot\xi+q\cdot\eta)}d\tau d\xi d\eta$$

$$\tilde{D_0}(k,p,q)=\int D_0(\tau,\xi,\eta)e^{i(\tau\cdot\tilde{k}+p\cdot\xi+q\cdot\eta)}d\tau d\xi d\eta$$

$$K(k,p,q)=\int \hat{K}(\tau,\xi,\eta)e^{i(\tau\cdot\tilde{k}+p\cdot\xi+q\cdot\eta)}d\tau d\xi d\eta\,.$$

If we consider (2.1) as an equation for integral kernels of operators acting on a space of functions invariant under $\xi \to -\xi$ we have

$$D_0^\frown(k, p, q) = \delta(p+q) R_0(k, p)$$

where

$$R_0(k, p) = 2S^\frown\left(\frac{p-\tilde{k}}{2}\right) S^\frown\left(\frac{p+\tilde{k}}{2}\right), \tag{2.5}$$

$S^\frown$ denotes the Fourier transform of the two point function. Using the fact that the integral (2.2) is a convolution in the τ variables, it is easy to show that (2.2) may be written in the following form

$$R(k, p, q) = R_0(k, p)\delta(p+q) - \int R_0(k, p) K(k, -p, q') R(k, q', q) dq' \tag{2.6}$$

or as operators

$$R(k) = R_0(k) - R_0(k) K(k) R(k) .$$

The Feynman-Kac formula connects the analytic properties of $R(k)$ with the spectrum of the energy momentum operator $P = (P_0, P_1)$. Let $dE(p)$ be the joint spectral resolution of P given by the SNAG Theorem [9] so that

$$e^{ix \cdot P} = \int e^{ix \cdot p} dE(p) .$$

For $\xi_0 = \eta_0 = 0$ note that

$$D(\tau, \xi, \eta) = \langle \theta(\xi_1), e^{-|\tau_0| P_0} e^{i\tau_1 P_1} \theta(\eta_1) \rangle .$$

The inner product on the right is in $\mathscr{H}$ and

$$\theta(\xi_1) = \phi_0(\xi_1)\phi_0(-\xi_1)\Omega - \langle \Omega, \phi_0(-\xi_1)\phi_0(\xi_1)\Omega \rangle \Omega$$

where ϕ_0 denotes the time zero field, Ω is the vacuum. Hence for $f^\frown$ and $g^\frown$ belonging to $C_0^\infty(\mathbb{R}^1)$ we have

$$\begin{aligned} &\int d\tau e^{i\tau \tilde{k}} \int \delta(\xi_0)\delta(\eta_0) f^\frown(\xi_1) g^\frown(\xi_1) D(\tau, \xi, \eta) d\eta d\xi \\ &= \int R(k, p, q) f(p_1) g(q_1) dp dq \\ &= \int \langle \theta(f) e^{-|\tau_0| P_0} e^{i\tau_1 P_1} \theta(g) \rangle e^{i\tau \tilde{k}} d\tau \\ &= \int \delta(p_1 - k_1) \left[\frac{1}{p_0 - k_0} + \frac{1}{p_0 + k_0} \right] d\langle \theta(f) E(p_0, p_1) \theta(g) \rangle . \end{aligned} \tag{2.7}$$

The convergence of the $d\tau$ integration follows from a mass gap which we assume throughout the paper. In § 3 and 4 we study the analytic properties in k of

$$\int R(k, p, q) f(p) g(q) dp dq .$$

In § 5 we discuss the consequences of these properties for the spectral measure $dE(p)$.

Now we state the conditions we assume throughout the paper. Let $\mathscr{H}^a$ be the subspace of even states in $\mathscr{H}$ of energy less than a.

Condition (*):

A. The two point Schwinger function S has the form

$$S(p)=\frac{1}{p^2+m^2}+\int_{m+2\delta_0}(p^2+a^2)^{-1}d\rho(a), \quad 0\leqq\delta_0\leqq m. \tag{2.8}$$

B. The closed subspace of $\mathscr{H}$ spanned by

$$\Omega,\, e^{ix\cdot P}\theta(f)$$

for $x\in R^2$ and $f\in C_0^\infty(R^1)$ contains $\mathscr{H}^{2(m+\delta_0)}$.

C. The Bethe-Salpeter kernel K is bounded and analytic in the region

$$\begin{aligned}&|\mathrm{Im}\,q_0|,\,|\mathrm{Im}\,p_0|\leqq 2\delta_0\\&|\mathrm{Im}\,q_1|,\,|\mathrm{Im}\,p_1|\leqq 2\delta_1\\&|\mathrm{Re}\,k_0|\leqq 2(m+\delta_0)\,.\end{aligned} \tag{2.9}$$

Condition (*) has been established for the case of weakly coupled even $\mathscr{P}(\phi)_2$ models in [1, 6] with $\delta_0=m-\varepsilon(\lambda)$ and $\delta_1=\varepsilon(\lambda)/2$, $\varepsilon(\lambda)\to 0$ as $\lambda\to 0$. The following additional condition enables us to eliminate bound states in the weak coupling region by isolating first order repulsive δ functions.

Condition (**): For sufficiently small $\lambda>0$ the Bethe-Salpeter kernel has the form

$$K\hat{}(x_1,\ldots,x_4)=\lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)+\lambda^2K\hat{}_1(x_1,x_2,x_3,x_4) \tag{2.10}$$

where $K_1(k,p,q)$ is analytic and uniformly bounded in λ for k,p,q satisfying (2.9). This has been verified for weakly coupled $\lambda(\phi^4+Q(\phi))_2$ models in [6] where $Q(\phi)$ is a positive even polynomial of degree greater than four.

Let $\mathscr{H}_{\mathrm{in(out)}}$ be the in(out) states constructed by the Haag Ruelle theory. Hence $\mathscr{H}_{\mathrm{in(out)}}$ is the closure of states

$$\lim_{\substack{t\to-\infty\\(+\infty)}}\prod^{n}\phi(f_t^{(i)})\Omega$$

where

$$f_t^{(i)}(p)=e^{it(p_0-\sqrt{p_1^2+m^2})}f^{(i)}(p)$$

and $f^{(i)}(p)$ are smooth functions whose supports are disjoint and contained in neighborhood G of the one particle mass hyperboloid. G is defined so that its intersection with the energy momentum spectrum consists only of the one particle hyperboloid.

Let $\mathscr{H}_d$ denote the closed subspace of $\mathscr{H}$ spanned by the eigenvectors of the mass operator $M=(P_0^2-P_1^2)^{1/2}$. We shall restrict our attention to vectors of energy less than $2(m+\delta_0)$ so we define

$$\mathscr{H}_d^{2(m+\delta_0)}=\mathscr{H}_d\cap\mathscr{H}^{2(m+\delta_0)}$$

and also

$$\mathscr{H}^{2(m+\delta_0)}_{\binom{\mathrm{out}}{\mathrm{in}}}=\mathscr{H}_{\binom{\mathrm{out}}{\mathrm{in}}}\cap\mathscr{H}^{2(m+\delta_0)}\,.$$

The following Theorems will be established in § 5.

Theorem 2.1. *If condition (*) holds, then the spectrum of* $M \upharpoonright \mathscr{H}_d^{(2m+\delta_0)}$ *has at most one point of accumulation at the threshold 2m. The unitary representation of the Lorentz group carried by each eigenspace of M of mass less than 2m is a finite sum of irreducible representations.*

Theorem 2.2. *If condition (*) holds, then*

$$\mathscr{H}^{2(m+\delta_0)} = \mathscr{H}_d^{2(m+\delta_0)} + \mathscr{H}_{\binom{\text{out}}{\text{in}}}^{2(m+\delta_0)}.$$

Theorem 2.3. *Suppose both conditions (*) and (**) hold. Then for sufficiently small* λ *(and fixed bare mass)*

$$\mathscr{H}^{2(m+\delta_0)} = \mathscr{H}_{\binom{\text{in}}{\text{out}}}^{2(m+\delta_0)}.$$

Remarks. 1) To establish the discreteness of bound states below the threshold $2m$, it suffices to take $\delta = (0, 0)$.

2) To analyze the absolutely continuous part of the spectrum, we show that the absolutely continuous spectrum of P has multiplicity 1. Let $\mathscr{H}_{\text{ac}}^{2(m+\delta_0)}$ be the corresponding invariant subspace. Since the in and out states are absolutely continuous of multiplicity one, we have

$$\mathscr{H}_{\text{ac}}^{2(m+\delta_0)} = \mathscr{H}_{\text{in}}^{2(m+\delta_0)} = \mathscr{H}_{\text{out}}^{2(m+\delta_0)}.$$

This yields unitarity of the S matrix for energy less than $2(m+\delta_0)$.

3) Note that we have excluded singular continuous spectrum for the mass operator, but we have not excluded eigenvalues embedded in the continuum except in Theorem 2.3.

4) If K satisfies condition C, then there are no bound states below $m+\delta_0$. To see this, we observe that a bound state of lower energy would create an additional subspace of absolutely continuous spectrum of energy less than $2(m+\delta_0)$ corresponding to a pair of bound states. This violates Theorem 2.2.

§ 3. $K(k)R_0(k)$ is Compact

In order to obtain information about the spectral resolution of the energy momentum operator $dE(p)$ we shall analyze the singularities of $R(k_0 + i\varepsilon, k_1)$ where k_0 and k_1 are real and $\varepsilon \to 0$. We shall define a Hilbert space A_δ such that $K(k)R_0(k)$ has an analytic continuation as a family of compact operators for fixed real k_1 and for k_0 belonging to

$$\begin{aligned}&\mathscr{D}(\alpha, \delta_0, \delta_1, k_1)\\&= \{k_0 : |\text{Re}\, k_0| \leqq 2(m+\delta_0), \quad \text{and} \quad \arg(k_0 - \sqrt{k_1^2 + 4m^2}) \mp \alpha\}\end{aligned} \tag{3.1}$$

for small α.

Thus matrix elements of

$$R(k) = R_0(k)(1 + K(k)R_0(k))^{-1}$$

for fixed k_1 are meromorphic functions of k_0 for $k_0 \in \mathscr{D}(\alpha, \delta_0, \delta_1, k_1)$. The choice of α is equivalent to the choice of a cut whose branch point is the threshold $\sqrt{k_1^2 + 4m^2}$.

The compactness and analyticity of $K(k)R_0(k)$ rely on the choice of a rather complicated Hilbert space. To motivate our choice of Hilbert space, we fix $k_1=0$ and consider $R_0(k_0, 0)$ as a bilinear form on $L^2(R^2)\times L^2(R^2)$. For $|\mathrm{Re}\,k_0|<2m$, $R_0(k_0, 0)$ is analytic in k_0.

However, $R_0(k_0, 0)$ is not analytic when $|\mathrm{Re}\,k_0|\geqq 2m$ even when $\mathrm{Im}\,k_0\neq 0$. This is because

$$(p_0-ik_0)^2+p_1^2+4m^2=0$$

has a real solution $p\in R^2$ whenever $|\mathrm{Re}\,k_0|\geqq 2m$. We can regain analyticity in k_0 (for $\mathrm{Im}\,k_0\neq 0$) if we replace L^2 by a space of analytic functions. To illustrate this idea, note that if $f(p_0)$ is analytic and bounded for $|\mathrm{Im}\,p_0|\leqq a$

$$\int_{-\infty}^{+\infty} f(p_0)[(p_0-ik_0)^2+p_1^2+4m^2]^{-1}dp_0$$

is analytic in k_0 for $0<\mathrm{Re}\,k_0<2m+a$. We eliminate singularities of the integrand by shifting to contour of integration to $\mathrm{Im}\,p_0=\pm a$.

We now turn to the definition of our Hilbert space. Let $w(p)=[p_0^2+p_1^2+16m^2]^{-2/3}$. Let A_δ be the Hardy class of functions analytic in the region

$$|\mathrm{Im}\,p_0|\leqq 2\delta_0;\ |\mathrm{Im}\,p_1|\leqq 2\delta_1$$

and such that $f(p)=f(-p)$ with the norm given by

$$\|f\|_{A_\delta}^2=\sup_{\alpha\in I_\delta}\int_{-\infty}^{+\infty}|(wf)(p+i\alpha)|^2dp\,. \tag{3.2}$$

Here $\delta=(\delta_0, \delta_1)$ and $I_\delta=\{\alpha=(\alpha_0, \alpha_1)\,|\,|\alpha_0|\leqq 2\delta_0;|\alpha_1|\leqq 2\delta_1\}$ and $dp=dp_0dp_1$.

In x space the norm is equivalent to

$$\|f\|_{A_{\delta}}^2=\int|[wf]\check{}(x)|^2e^{4\delta_0|x_0|}e^{4\delta_1|x_1|}dx \tag{3.3}$$

where $[wf]\check{}(x)$ is the Fourier transform of $[wf](p)$ and $dx=dx_0dx_1$. We shall show that $K(p)R_0(k)$ maps A_δ to A_δ.

Theorem 3.1. *Let $f=g_1\cdot g_2$ where $g_i\in A_\delta$ and let k_1 be real. Then as a function of k_0*

$$\int R_0(k, p)f(p)dp \tag{3.4}$$

has an analytic continuation for k_0 in $\mathscr{D}(\alpha, \delta_0, \delta_1, k_1)$ for α and δ_1 sufficiently small. Moreover if $0<k_0<2(m+\delta_0)$ we have

$$\lim_{\varepsilon\downarrow 0}\int \mathrm{Im}\,R_0(k_0+i\varepsilon, k_1, p)f(p)dp$$
$$=\begin{cases}0 & \text{for}\quad 0<k_0<\sqrt{k_1^2+4m^2}\\ 16\pi^2(k_0^2-k_1^2)^{-1/2}(k_0^2-k_1^2-4m^2)^{-1/2}f(\Lambda_k^{-1}(0,(k_0^2-k_1^2-4m^2)^{1/2}) & \\ \quad\text{for}\quad \sqrt{k_1^2+4m^2}<k_0<2(m+\delta_0)\,. & \end{cases} \tag{3.5}$$

Here Λ_k is the complex rotation defined so that

$$\Lambda_k(ik_0, k_1)=(i(k_0^2-k_1^2)^{1/2}, 0)\,. \tag{3.6}$$

Proof. Since R_0 is invariant under the complex rotations Λ and $\det\Lambda = 1$ we have

$$\int R_0(k_0, k_1, p) f(p) dp = \int R_0((k_0^2 - k_1^2)^{1/2}, 0, p) f(\Lambda_k^{-1} p) dp .$$

Hence we need only establish the theorem for $k_1 = 0$. We represent $R_0(k, p)$ using (2.5) and (2.8). Let R_{00} be defined by

$$R_{00}(k, p) = 32[(p - \tilde{k})^2 + 4m^2]^{-1}[(p + \tilde{k})^2 + 4m^2]^{-1}$$

and set $\delta R_0 = R_0 - R_{00}$. Recall $\tilde{k} = (ik_0, k_1)$. For $k_1 = 0$ note that

$$R_{00}(k, p) = \frac{8}{ik_0 p_0} \{[(p + \tilde{k})^2 + 4m^2]^{-1} - [(p - \tilde{k})^2 + 4m^2]^{-1}\} . \tag{3.7}$$

In order to establish the analyticity of $\int R_{00}(k, p) f(p) dp$ we consider the region $Q = \{(p_0, p_1) | |p_1| \leqq 4m\}$ and its complement $R^2 \sim Q$. Then we have

$$\int R_{00}(k,p) f(p) = \int_{Q'} R_{00}(k,p) \delta f(p) + \int_{Q} R_{00}(k,p) \delta f(p) dp + \int R_{00}(k,p) f(0,p_1) dp \tag{3.8}$$

where $\delta f(p) = f(p) - f(0, p_1)$. The first integral on the right is clearly analytic in k since R_{00} has no singularity for $p \in Q'$. By (3.7) the second term on the right of (3.8) equals

$$8 \int_{Q} [(p + \tilde{k})^2 + 4m^2]^{-1} \frac{\delta f(p)}{ik_0 p_0} dp - 8 \int_{Q} [(p - \tilde{k})^2 + 4m^2]^{-1} \frac{\delta f(p)}{ik_0 p_0} dp . \tag{3.9}$$

Since $(ik_0 p_0)^{-1} \delta f(p)$ is analytic for $|\mathrm{Im}\, p_0| < 2(m + \delta_0)$ we can shift the contour of integration to $\mathrm{Im}\, p_0 = -2\delta'_0, +2\delta'_0$, in each of the two integrals in (3.9) respectively. Here $\delta'_0 < \delta_0$. This yields analyticity in k for $|\mathrm{Re}\, k_0| < 2(m + \delta_0)$. The region Q is used to ensure the convergence for large p of the integrals in (3.9). Because

$$\lim_{\varepsilon \downarrow 0} \mathrm{Im}\, R_{00}(k_0 + i\varepsilon, k_1, p) = 0 \tag{3.10}$$

uniformly in p when $0 < \mathrm{Re}\, k_0 < 2m$, the two terms of (3.9) do not contribute to (3.5) for $0 < \mathrm{Re}\, k_0 < 2m$. Moreover because they are analytic in the region $0 < \mathrm{Re}\, k_0 < 2(m + \delta_0)$, they don't contribute to (3.5) in the same region.

The analyticity of

$$\int_{\mathbb{R}^2} \delta R_0(k, p) f(p) dp$$

also follows by shifting the contour of the p_0 integration and again it is easy to see that δR_0 makes no contribution to (3.5).

Now it remains to analyze the final term of (3.9). We compute the p_0 integral explicitly by the method of residues. This yields

$$\int_{-\infty}^{+\infty} R_{00}(k, p) dp_0 = 8\pi \{[k_0 \mu(p_1)(\mu(p_1) - k_0)]^{-1} - [k_0 \mu(p_1)(\mu(p_1) + k_0)]^{-1}\} \tag{3.12}$$

where $\mu(p_1) = (p_1^2 + 4m^2)^{1/2}$. To compute the limit of (3.5) as $\varepsilon \downarrow 0$ we use the identity

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{(a,b)} \frac{h(v) dv}{(v - r)^2 + \varepsilon^2} = h(r) \quad \text{when} \quad r \in (a, b) . \tag{3.13}$$

Let $v=\mu(p_1)$ then

$$p_1=\sqrt{v^2-4m^2} \quad \text{and} \quad dp^1=\frac{v}{\sqrt{v^2-4m^2}}dv\,.$$

By (3.12), (3.13) and the symmetry $f(p)=f(-p)$, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^2} \operatorname{Im} R_{00}(k_0+i\varepsilon, k_1, p)f(0,p_1)dp \\
&=\lim_{\varepsilon\downarrow 0} 8\pi \int_{-\infty}^{+\infty} [k_0\mu(p_1)(\mu(p_1)-k_0+i\varepsilon)]^{-1} f(0,p_1)dp_1 \\
&=\lim_{\varepsilon\downarrow 0} 8\pi \int_{2m}^{\infty} \varepsilon[k_0 v((v-k_0)^2+\varepsilon^2)]^{-1}(v^2-4m^2)^{-1/2} v f(0,(v^2-4m^2)^{1/2})dv \\
&=16\pi^2 f(0,(k_0-4m^2)^{1/2})k_0^{-1}(k_0^2-4m^2)^{-1/2} \qquad (3.14)
\end{aligned}$$

which yields (3.5) for the case $k_1=0$. Note that the second term on the right of (3.10) does not contribute since $\operatorname{Re} k_0>0$.

To establish the analyticity of

$$\int \frac{f(0,p_1)dp_1}{\mu(p_1)(\mu(p_1)-k_0)k_0} \qquad (3.15)$$

across the cut $(2m, 2(m+\delta_0))$ we deform the contour of the p_1 integration to the contour given by

$$t(s)=\begin{cases} s+is & |s|<\delta_1' \\ s+i\delta_1' & s>\delta_1' \\ s-i\delta_1' & s<-\delta_1' \end{cases}$$

Along this contour

$$\begin{aligned}
\mu(t(s)) &= 2m+is^2/2m+\mathcal{O}(\delta_1'^3) && |s|<\delta_1' \\
&=\sqrt{4m^2+s^2}+i|s|\delta/\sqrt{4m^2+s^2}+\mathcal{O}(\delta_1'^2) && |s|>\delta_1'\,.
\end{aligned}$$

Hence for δ_1' and $\alpha>0$ sufficiently small (3.15) has an analytic continuation for k_0 in $\mathcal{D}(\alpha,\delta_0,\delta_1',0)$. This completes the proof.

Theorem 3.2. *If condition (*) holds, then for fixed k_1 and $g\in A_\delta$ the operator*

$$K(k)R_0(k): g\to \int K(k,p,q)R_0(k,q)g(q)dq$$

has an analytic continuation as a Hilbert-Schmidt operator from A_δ to A_δ for $k_0\in\mathcal{D}(\alpha,\delta,k_1)$.

Proof. Let U be the unitary map from A_δ to $L^2(\mathbb{R}^2, dx)$ defined by

$$[U(g)](x)=e^{2(\delta_0|x_0|+\delta_1|x_1|)}[wg]^\wedge(x)\,.$$

Then the Hilbert-Schmidt norm of $UK(k)R_0(k)U^{-1}$ is bounded by

$$\begin{aligned}
&\int dxdy\,|e^{2(\delta_0|x_0|+\delta_1|x_1|)}[wK(k)R_0(k)w^{-1}]^\wedge(x,y)e^{-2(\delta_0|y_0|+\delta_1|y_1|)}|^2 \\
&\leq \text{const}\int w(p)^2[\int K(k,p+i2\delta,q')R_0(k,q')w^{-1}(q')B_\delta(q'-q)dq']^2 dpdq
\end{aligned} \qquad (3.16)$$

where

$$B_\delta(q) = \frac{1}{q_0^2 + 4\delta_0^2} \frac{1}{q_1^2 + 4\delta_1^2}.$$

To bound the q' integral we follow Theorem 3.1 where f in (3.4) is replaced by

$$f(q'; q, p, k) \equiv K(k, p + i2\delta, q')w^{-1}(q')B_\delta(q' - q).$$

From the proof of this theorem it is easy to show that the integral has an analytic continuation in $k_0 \in \mathscr{D}$ such that

$$|\int R_0(k, q')f(q', q, p, k)dq'| \leqq \text{const}\, w(q).$$

Hence (3.16) is bounded by

$$\int w(q)^2 w(p)^2 dp dq < \infty.$$

Theorem 3.3. *Let $T(k_0)$ be an analytic family of compact operators for k_0 belonging to a domain $\mathscr{D}$. Then either*

(i) $(1 - T(k_0))^{-1}$ *does not exist for all* $k_0 \in \mathscr{D}$

or

(ii) $(1 - T(k_0))^{-1}$ *is meromorphic in $\mathscr{D}$ i.e., there is a discrete set $S \subset \mathscr{D}$ of poles such that $T(k_0)v = v$ has a solution for $k_0 \in S$. For $k_0 \in \mathscr{D} \setminus S$, $[1 - T(k_0)]^{-1}$ exists and is analytic.*

The proof follows from well known facts about compact operators [10]. The above formulation of the theorem has been frequently used in quantum mechanics. See [11] for example.

Theorem 3.4. *For fixed k_1 the operator $[1 + K(k)R_0(k)]^{-1}$ is meromorphic for $k_0 \in \mathscr{D}(\alpha, \delta_0, \delta_1, k_1)$.*

Proof. We exclude conclusion (i) of Theorem 3.3 by noting that if $\operatorname{Im} k_0$ is sufficiently large the norm of the operator $K(k)R_0(k)$ goes to zero so that $[1 + K(k)R_0(k)]^{-1}$ exists.

§ 4. The Absence of Bound States

If condition (**) is fulfilled, the Bethe-Salpeter kernel is given by repulsive δ function to first order in perturbation theory. This condition enables us to exclude bound states for weak coupling. The basic idea of the section is to eliminate the infrared singularity of $R_0(k)$ for k near the treshold by considering the operator

$$R_0'(k) \equiv [R_0(k)^{-1} + \lambda\delta]^{-1} \tag{4.1}$$

where $\delta = \delta(x_1 - x_2)\delta(x_2 - x_3)\delta(x_3 - x_4)$. Let $R_0'(k, p, q)$ the integral kernel of $R_0'(k)$ acting on functions $g \in A_\delta$ in the following way

$$[R_0'(k)g](p) = \int R_0'(k, p, q)g(q)dq. \tag{4.2}$$

Note that $R_0(k)g\in A_\delta^*$ because $R_0'(k)$ is a continuous form on $A_\delta\times A_\delta$, where A_δ^* is the strong dual of A_δ with respect to the L_2 inner product. Let

$$\mathscr{E}=\{k_0|\mathrm{Re}\,k_0\leqq 2(m+\delta_0)\}\setminus\{k_0|\mathrm{Im}\,k_0=0,\,2m\leqq \mathrm{Re}\,k_0\leqq 2(m+\delta_0)\}$$

and set

$$\mathscr{E}^{\sim}=\bigcup_{c\in\mathscr{E}}\{k|k_0^2-k_1^2=c^2\}.$$

Lemma 4.1. *Let $R_0(k)$, $R_0(k)$, $R_0'(k,p,q)$, and $R_0(k,p)$ be as above and $k\in\mathscr{E}^{\sim}$. Then*

$$R_0'(k,p,q)=R_0(k,p)\delta(p-q)-\frac{\lambda d^2(k)}{\lambda d^2(k)+1}\,\frac{R_0(k,p)}{d(k)}\,\frac{R_0(k,q)}{d(k)}\tag{4.3}$$

where

$$d^2(k)=\int dpR_0(k,p).$$

Proof. A straightforward computation for $\psi_1\in A_\delta$ using (4.3) yields:

$$(R_0^{-1}(k)+\lambda\delta)R_0'(k)\psi_1(p)=\psi_1(p)+\left(\lambda-\frac{\lambda}{d^2(k)+1}-\frac{\lambda^2d^2(k)}{\lambda d^2(k)+1}\right)c(k)=\psi_1(p)$$

where $c(k)=\int dpR_0(k,p)\psi_1(p)$. And for ψ_2 belonging to the range of $R_0'(k)$ we have

$$\begin{aligned}R_0'(k)(R_0^{-1}(k)+\lambda\delta)\psi_2(p)&=\psi_2(p)+\left(\lambda-\frac{\lambda}{\lambda d^2(k)+1}-\frac{\lambda^2d^2(k)}{\lambda d^2(k)+1}\right)\hat{\psi}_2(0)R_0(k,p)\\&=\psi_2(p).\end{aligned}$$

This completes the proof.

For $\mathrm{Im}\,k_0$ sufficiently large it is easy to verify by Neumann series the following identity

$$R(k)=R_0'(k)(1+\lambda^2K_1(k)R_0'(k))^{-1}\tag{4.4}$$

where K_1 is defined by (2.10). By analyticity (4.4) holds in $\mathscr{D}$.

We shall show that for sufficiently small λ

$$(1+\lambda^2K_1(k)R_0'(k))^{-1}$$

has no poles in $\{(k_0,k_1)|0\leqq\mathrm{Re}\,k_0\leqq 2(m+\delta_0),\,\mathrm{Im}\,k_1=0\}$.

It suffices to show that for $0<\lambda\leqq 1$

$$\|K_1(k)R_0'(k)\|_{\mathrm{H.S.}}\leqq\mathrm{const}\,\lambda^{-1}\tag{4.5}$$

where $\|\cdot\|_{\mathrm{H.S.}}$ denotes the Hilbert-Schmidt norm in A_δ. From Theorem 3.2 it is easy to show that $\|K_1(k)R_0'(k)\|_{\mathrm{H.S.}}$ is uniformly bounded (independent of λ) when $k_0\in\mathscr{D}$ is bounded away from the threshold $k_0=\sqrt{k_1^2+4m^2}$. In the following theorem we establish (4.5) for k_0 near the threshold using the fact that

$$\begin{aligned}\int R_0(k,p,q)dpdq&=d^2(k)-\lambda d^4(k)[\lambda d^2(k)+1]^{-1}\\&=d^2(k)[\lambda d^2(k)+1]^{-1}=\mathcal{O}(\lambda^{-1}).\end{aligned}\tag{4.6}$$

Note that this bound holds even as $d^2(k)\to\infty$.

Theorem 4.2. *Suppose conditions* (*) *and* (**) *hold, and* $R'_0(k)$ *is defined by* (4.3). *Then* (4.5) *holds for* $k_1=0$ $0<\mathrm{Re}\,k_0<2(m+\delta_0)$, *uniformly as* $\mathrm{Im}\,k_0\to 0$.

Proof. As in the proof of Theorem 3.2, it suffices to show that

$$|\int f_1(p')R'_0(k,p',q')f_2(q';q)dpdq'| \leqq \frac{\text{const}}{\lambda} w(q) \tag{4.7}$$

where

$$f_1(p')=K_1(k,p+i2\delta,p')$$
$$f_2(q';q)=w^{-1}(q')B_\delta(q'-q).$$

By (4.3) and (4.6) we see that (4.7) follows from bounds of the form

$$|\int R_0(k,p)[g(p;q)-g(0;q)]dp| \leqq \text{const}\, w(q) \tag{4.8}$$

where $g(p;q)=f_1(p)f_2(p;q)$ or $g(p;q)=f_2(p;q)$ etc. Here we follow the proof of Theorem 3.1. First we replace R_0 by R_{00} in (4.8) since by complex translation it is easy to show that δR_0 has no singularity near $k=(2m,0)$. We set

$$\delta g(p,q)=g(p,q)-g((0,p_1),q),$$

so that reasoning as in (3.8) and (3.9) one can show

$$|\int R_{00}(k,p)\delta g(p,q)dp| \leqq \text{const}\, w(q).$$

Using (3.12) the key estimate reduces to a bound on

$$\int R_{00}(k,p)[g((0,p_1),q)-g(0,q)]dp$$
$$=4\pi\int [\mu(p_1)(\mu(p_1)^2-k_0^2)]^{-1}[g((0,p_1),q)-g(0,q)]dp_1.$$

We split the dp_1 integral into $|p_1|\leqq 1$ and $|p_1|\geqq 1$. For $|p_1|\geqq 1$ there is no singularity near $k_0=2m$ and so

$$\left|\int_{|p_1|\geqq 1} [\mu(p_1)(\mu(p_1)^2-k_0^2)]^{-1}[p_1^2+16m^2]^{2/3}B_\delta(q_0,p_1-q_1)dp_1\right| \leqq \text{const}\, w(q).$$

For $|p_1|\leqq 1$ let $g''(p_1;q)$ be the smooth function defined by

$$g((0,p_1);q)=g((0,0),q)+p_1\left[\frac{\partial}{\partial p_1}g((0,p_1);q)\right]\Bigg|_{p_1=0}+p_1^2 g''(p_1;q)$$

using the $p_1\to -p_1$ symmetry, we see that the integral over $|p_1|\leqq 1$ reduces to

$$\left|\int_{|p_1|\leqq 1} \frac{p_1^2 g''(p_1,q)}{\mu(p_1)[p_1^2+4m^2-k_0^2]}dp_1\right| \leqq \text{const}\, w(q).$$

This estimate is elementary using the factor of p_1^2 to cancel the infrared divergence of the denominator when $k_0\to 2m$.

§ 5. The Energy-Momentum Spectrum

In this section we show how the results of § 3 and § 4 can be applied to give information about the energy momentum spectrum. Let A_δ^1 be the subspace of functions in A_δ which are independent of p_0. Note that the span of $e^{ix\cdot P}\theta(f)$ for $f \in A_\delta^1$ and $x \in R^2$ is dense in $\mathscr{H}^{2(m+\delta_0)}$ because A_δ^1 contains the Fourier transform of functions in $C_0^\infty(R^1)$. We recall the formula (2.7) which connects $R(k)$ with the spectral resolution of P,

$$\langle f, R(k)f\rangle_{L^2} = \int \delta(p_1 - k_1)\left[\frac{1}{p_0 - k_0} + \frac{1}{p_0 + k_0}\right] d\langle\theta(f), E(p)\theta(f)\rangle \tag{5.1}$$

for $f \in A_\delta^1$ and $\operatorname{Im} k_0 \neq 0$. Let h be a smooth function. Then for real k_0 we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \int \langle f, \operatorname{Im} R(k_0 + i\varepsilon, k_1) f\rangle h(k_0) dk_0 \\
&= \lim_{\varepsilon\downarrow 0} \int \delta(p_1 - k_1) \operatorname{Im}[(p_0 - k_0 + i\varepsilon)^{-1} + (p_0 + k + i\varepsilon)^{-1}] h(k_0) dk_0 d\langle\theta(f), E(p)\theta(f)\rangle \\
&= \int \delta(p_1 - k_1) h(p_0) d\langle\theta(f), E(p)\theta(f)\rangle
\end{aligned} \tag{5.2}$$

where $\operatorname{supp} h \subset \mathbb{R}^+$. For the last identity we have used (3.13). Now let us consider the spectrum of P for energy less than $2m$.

Lemma 5.1. *The spectrum of the mass operator below $2m$ is contained in the poles of* $(1 + K(k_0, 0)R_0(k_0, 0))^{-1}$.

Proof. If $b < 2m$ is not a pole of $[1 + K(k_0, 0)R_0(k_0, 0)]^{-1}$ then since $R(k)$ is analytic in a neighborhood of $(b, 0)$ (with no cut) we have for k_0 near b and k_1 small

$$\lim_{\varepsilon\downarrow 0} \langle f, \operatorname{Im} R(k_0 + i\varepsilon, k_1) f\rangle = 0\,, \qquad f \in A_\delta^1\,.$$

By (5.2) we have

$$\begin{aligned}
&\int_{|p-(b,0)|<\varepsilon} dE(p) e^{ix\cdot P}\theta(f) \\
&= \int_{|p-(b,0)|<\varepsilon} e^{ix\cdot p} dE(p)\theta(f) \\
&= 0\,.
\end{aligned}$$

Lorentz covariance implies that $dE(p)$ has no support in a neighborhood of the hyperboloid of mass b.

Lemma 5.2. *On each eigenspace of mass less than $2m$ the representation of the Lorentz group is a finite sum of irreducible representations.*

Proof. Let $b < 2m$ be an eigenvalue of M. Then from (5.1) we have for small $\varepsilon > 0$

$$\begin{aligned}
&(2\pi i)^{-1} \oint_{|k_0 - b| = \varepsilon} \langle f, R(k_0, 0) f\rangle dk_0 \\
&= \int_{|b - p_0| < \varepsilon} \delta(p_1) d\langle\theta(f) E(p)\theta(f)\rangle\,.
\end{aligned} \tag{5.3}$$

Note that since the poles of $[1 + K(k_0, 0)R_0(k_0, 0)]^{-1}$ are simple [by (5.1)] we have

$$0 = [1 + K(b, 0)R_0(b, 0)] \oint_{|k_0 - b| = \varepsilon} [1 + K(k_0, 0)R_0(k_0, 0)]^{-1} dk_0\,. \tag{5.4}$$

The compactness of $K(b,0)R_0(b,0)$ and (5.4) imply that the range of $\oint[\ ]^{-1}$ is finite dimensional, say of dimension n. Hence (5.3) as a bilinear form is of rank n.

Now suppose the Lorentz representation is $n+1$ reducible. Then there exist $n+1$ vectors v_i belonging to distinct components of the representation such that

$$d\langle v_i E(p) v_j\rangle = \delta_{ij}\delta(p_0^2 - p_1^2 - b^2)dp$$

for small p_1. Since the v_i can be approximated by vectors $e^{ix\cdot P}\theta$ it follows that the form (5.3) has rank $n+1$ which is a contradiction.

Next we analyze the spectrum above $2m$. We shall show that apart from possible bound states the energy momentum spectrum is absolutely continuous and of multiplicity 1.

Let c_i be the poles of $[1+K(k_0,0)R_0(k_0,0)]^{-1}$ above $2m$. For a $C\subset[2m,2(m+\delta_0))$, let

$$\tilde{C} = \bigcup_{e\in C} \{(k_0,k_1)|k_0^2 - k_1^2 = e^2\}\cap\Omega$$

where

$$\Omega = \{(k_0,k_1)|k_0^2 - k_1^2 \geqq 4m, 0 \leqq k_0 \leqq 2(m+\delta_0)\}.$$

Lemma 5.3. *Let $C=\{c_i\}$. Then for $k\in\Omega/\tilde{C}$ and $f\in A_\delta$ we have*

$$\lim_{\varepsilon\downarrow 0}\langle f, \operatorname{Im} R(k_0+i\varepsilon, k_1)f\rangle_{L^2} = \lim_{\varepsilon\downarrow 0}\langle W(k)f, \operatorname{Im} R_0(k_0+i\varepsilon, k)W(k)f\rangle_{L^2} \tag{5.5}$$

where

$$W(k) = \lim_{\varepsilon\downarrow 0}[1+K(k_0+i\varepsilon,k_1)R_0(k_0+i\varepsilon,k_1)]^{-1}. \tag{5.6}$$

Remark. By Theorem 3.4 $W(k)$ is well defined and bounded for $k\notin\tilde{C}$.

Proof. Let $z=(k_1+i\varepsilon,k_1)$ and for an operator $T(z)$ let

$$\operatorname{Im} T(z) \equiv (2i)^{-1}[T(z)-T(\bar{z})].$$

Note that

$$\operatorname{Im} R(z) = \operatorname{Im}\{R_0(z)W(z)\} = (\operatorname{Im} R_0(z))W(z) - R_0(\bar{z})\operatorname{Im}(W(\bar{z})),$$

and

$$\operatorname{Im} W(\bar{z}) = -W(z)\operatorname{Im}(K(\bar{z})R_0(\bar{z}))W(z).$$

Since

$$\begin{aligned} R_0(z)W(z)K(z) &= 1-[1+R_0(z)K(z)]^{-1} \\ &= 1-W(\bar{z})^*, \end{aligned}$$

the above identities yield

$$\operatorname{Im} R(z) = W(z)^*\operatorname{Im} R_0(z)W(z) - R_0(\bar{z})W(\bar{z})(\operatorname{Im} K(z))R_0(z)W(z). \tag{5.7}$$

As $\varepsilon\downarrow 0$, $(\operatorname{Im} K(z))R_0(z)\to 0$ in the strong operator topology on A_δ hence the second term on the right side of (4.7) vanishes as $\varepsilon\downarrow 0$, and the lemma follows.

Remark. Lemma 5.2 and (5.2) imply that for $p \notin \tilde{C}$ and $f \in A_\delta^1$

$$d\langle \theta(f) E(p) e^{ix\cdot P} \theta(f) \rangle$$

is absolutely continuous with respect to Lebesgue measure.

Next we show that $e^{ix\cdot P}$ acting on the absolutely continuous spectrum of P is unitarily equivalent to multiplication by $e^{ix\cdot P}$ on $L^2(\Omega, d^{(2)}p)$ i.e. the absolutely continuous spectrum is of multiplicity 1.

By Theorem 3.1 we can compute the right side of (5.5). For $f \in A_\delta$ and $k \in \Omega/\tilde{C}$ we define

$$(Lf)(k) = \pi(k_0^2 - k_1^2)^{1/4}(k_0^2 - k_1^2 - 4m^2)^{1/4}(Wf)[\Lambda_k(0, (k_0^2 - k_1^2 - 4m^2)^{1/2})].$$

Then we have

$$\lim_{\varepsilon \downarrow 0} \langle g, \operatorname{Im} R(k_0 + i\varepsilon, k) f \rangle = (Lf)(k)\overline{(Lg)}(k)$$

and so by (5.2)

$$d\langle \theta(g), E(p) e^{ix\cdot p} \theta(f) \rangle = e^{ix\cdot p}(Lf)(p)\overline{(Lg)}(p)dp.$$

Let N be an open set containing $\tilde{C}$, $(c = \{c_i\})$. Then the above identity enables us to define a unitary map

$$U : E(\Omega \sim N)\mathscr{H} \to L^2(\Omega \sim N, dp)$$

which extends the map

$$U : E(\Omega \sim N) e^{ix\cdot P} \theta(f) \to e^{ix\cdot p}(Lf)(p).$$

We have used the fact that $e^{ix\cdot P}\theta(f)$ is dense in $\mathscr{H}^{2(m+\delta_0)}$.

Proof of Theorems 2.1 and 2.2. By the Haag Ruelle theory, P acts on $E(\Omega)\mathscr{H}_{\substack{\text{in}\\ \text{out}}}$ as a free energy momentum operator. Hence the multiplicity of $e^{ix\cdot P}$ is one on $E(\Omega)\mathscr{H}_{\substack{\text{in}\\ \text{out}}}$. Since

$$E(\Omega \sim N)\mathscr{H}_{\substack{\text{in}\\ \text{out}}} \subset E(\Omega \sim N)\mathscr{H}$$

inclusion must be equality for each neighborhood N of $\tilde{C}$. The proof of Theorem 2.3 follows immediately from Theorem 2.2 and Theorem 4.3.

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