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On Cauchy's Equations of Motion

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1. In continuum mechanics it is usual to postulate equations of motion and momentum, an equation of energy and an equation concerning the rate of production of entropy. Subsequently, in discussing constitutive equations for quantities such as the components of stress, considerable use is made of invariance properties under superposed rigid body motions. Elsewhere, in a more general theory than that usually considered in continuum mechanics, the present writers have shown that the equations of motion and momentum can be *deduced* from the equation of energy by making full use of invariance conditions under superposed rigid body motions[1]). The result seems to be of sufficient interest to be reproduced separately with particular reference to classical continuum mechanics, since it may be overlooked in a paper which is primarily concerned with other ideas.

2. Using rectangular cartesian coordinates and cartesian tensor notation we postulate an energy balance in the form

$$\int_V \varrho\, v_i\, \dot{v}_i\, dV + \int_V \varrho\, \dot{U}\, dV = \int_V (\varrho\, r + \varrho\, F_i\, v_i)\, dV - \int_A h\, dA + \int_A t_i\, v_i\, dA\,, \tag{1}$$

where v_i is velocity, U is internal energy per unit mass, F_i is body force per unit mass, r is the heat supply function per unit mass and unit time, V is an arbitrary material volume bounded by a surface A at time t. Also h is the heat flux across the surface A, per unit area, and t_i is the stress vector across this surface, the unit outward normal to A being n_i. A dot denotes material derivative with respect to time and ϱ is density at time t.

We suppose that the body has arrived at the given state at time t through some prescribed motion. We consider a second motion which differs from the given motion only by a *constant* superposed rigid body translational velocity, the body occupying the same position at time t, and we assume that $\dot{U}$, t_i, F_i, h, and r are unaltered by such superposed rigid body velocity. Equation (1) is valid for all velocity fields and in particular for a velocity field $v_i + a_i$, where a_i is constant (in space and time). Thus

$$\int_V \varrho\, (v_i + a_i)\, \dot{v}_i\, dV + \int_V \varrho\, \dot{U}\, dV = \int_V [\varrho\, r + \varrho\, F_i\, (v_i + a_i)]\, dV - \int_A h\, dA + \int_A t_i\, (v_i + a_i)\, dA\,, \tag{2}$$

and since $\dot{U}$, r, F_i, h, and t_i are the same as in (1) it follows that

$$\left[\int_V \varrho\, \dot{v}_i\, dV - \int_V \varrho\, F_i\, dV - \int_A t_i\, dA\right] a_i = 0 \tag{3}$$

for all arbitrary constant a_i. Since the quantities in the square brackets in (3) are independent of a_i it follows that

$$\int_V \varrho\, F_i\, dV + \int_A t_i\, dA = \int_V \varrho\, \dot{v}_i\, dV\,, \tag{4}$$

[1]) Since writing this paper Professor W. Noll has sent us a proof copy of a paper, written in 1960 and to be published in the proceedings of 'Colloque sur l'axiomatique' in which he obtains the classical equations of motion and moments for forces from other postulates, but his ideas do not appear to be the same as those used here.

the classical equation of motion. If the components of stress across the coordinate planes are σ_{ji} it follows from (4), by the usual methods, that

$$\sigma_{ji,j} + \varrho F_i = \varrho \dot{v}_i , \tag{5}$$

$$t_i = n_j \sigma_{ji} , \tag{6}$$

where a comma denotes partial derivative with respect to x_j.

With the help of (5) and (6) Equation (1) becomes

$$\int_V \varrho \dot{U} \, dV = \int_V (\varrho r + \sigma_{ji} v_{i,j}) \, dV - \int_A h \, dA . \tag{7}$$

We now consider a motion of the body which differs from the given motion only by a superposed *uniform* rigid body angular velocity, the body occupying the same position at time t, and we assume that $\dot{U}$, r, σ_{ji}, and h are unaltered by such motions. Equation (7) holds for all velocity fields so it holds when $v_{i,j}$ is replaced by $v_{i,j} + \Omega_{ij}$, where Ω_{ij} is a constant skew symmetric tensor representing a constant rigid body angular velocity. It follows that

$$\Omega_{ij} \int_V \sigma_{ji} \, dV = 0 \tag{8}$$

for all arbitrary skew symmetric tensors Ω_{ij}. Since $\int_V \sigma_{ji} \, dV$ is independent of Ω_{ij} it follows that

$$\int_V (\sigma_{ij} - \sigma_{ji}) \, dV = 0$$

for all arbitrary volumes, so that

$$\sigma_{ij} = \sigma_{ji} . \tag{9}$$

If we use (6) and apply (1) to an arbitrary tetrahedron bounded by coordinate planes through the point x_i and by a plane whose unit normal is n_j, we obtain the result

$$h = n_j Q_j , \tag{10}$$

where Q_j are components of the heat flux vector across the x_j-planes. With the help of (10) and (9), we have, from (7),

$$\varrho \dot{U} = \varrho r + \sigma_{ji} d_{ij} - Q_{i,i} , \tag{11}$$

where

$$d_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) . \tag{12}$$

In addition to Equation (11) we must add a postulate about entropy production.

In discussing constitutive equations it is assumed that U is unaltered by superposed rigid body motions and that σ_{ij}, Q_j, and h are unaltered by such motions, apart from orientation at time t. These assumptions include those already made in deriving Equations (4) and (9) except for the additional assumptions that F_i and r are unaltered by a superposed constant rigid body translational velocity, and r is unaltered by a constant rigid body angular velocity, the body occupying the same position at time t.

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Zusammenfassung

Cauchys Bewegungs- und Momentengleichungen der klassischen Kontinuumsmechanik werden, mit Hilfe von Invarianz-Bedingungen gegenüber starren Zusatzbewegungen, aus dem Energietheorem hergeleitet.

Note added 9th March 1964. During discussions with Professor P. M. NAGHDI we have found that the equation of continuity may also be derived from the energy equation and invariance conditions. We need an additional term

$$\int_V \left(U + \frac{1}{2} v_i v_i\right) (\dot{\varrho} + \varrho v_{m,m})\, dV$$

on the left-hand side of (1) with a corresponding term

$$a_i \int_V v_i (\dot{\varrho} + \varrho v_{m,m})\, dV + \frac{1}{2} a_i a_i \int_V (\dot{\varrho} + \varrho v_{m,m})\, dV$$

added to the left-hand side of (3). This leads to the equation of continuity

$$\dot{\varrho} + \varrho v_{m,m} = 0$$

and then Equation (4) as before. We have assumed that U and ϱ are unaltered by constant superposed rigid body translational velocity, the body occupying the same position at time t.

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Zur Berechnung der aerodynamischen Koeffizienten von Rotationskörpern mit Tragflächen im Überschallbereich[1])

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1. Einleitung

Die Arbeit von PITTS et al. [1][2]) erlaubt die Berechnung von Grösse und Lage der resultierenden Normalkraft an Flugkörpern, deren Steuer- und Tragflächen in der Meridianebene eines kreiszylindrischen Rumpfes liegen. Der Rechenaufwand ist relativ bescheiden und für ingenieurmässige Anwendung gut geeignet. Die erwähnte Arbeit berücksichtigt als einzigen Wirbeleffekt jenen der abgehenden Wirbel der vorderen auf die hinteren Tragflächen (Flügel, bzw. Leitwerk genannt), was ihren Anwendungsbereich auf kleine Winkel beschränkt.

Jedoch schon da zeigt sich eine Diskrepanz zwischen der berechneten und der gemessenen Normalkraftlage für Körper mit einem langen Heck [1]. Ein Versuch, die Anwendbarkeit der Methode auf grössere Winkel auszudehnen unter Berücksichtigung der Querkräfte am Rumpf infolge Wirbelbildung [2] sowie der Effekte dieser Wirbel auf die Tragflächen zeigte erneut, dass die Lage der Resultierenden nicht befriedigend berechnet werden kann, wenn die Tragflächen weit vorne am Rumpf angebracht sind.

Im Bestreben, ein einfaches Rechenmodell der Rumpf-Tragflächen-Kombination für die Praxis zu entwickeln, haben PITTS et al. die von der Tragfläche auf den Rumpf induzierten Normalgeschwindigkeiten vernachlässigt. Der Einfluss der endlichen Spannweite der Tragfläche wurde ebenfalls nicht berücksichtigt. Alle diese Effekte kommen erst zur Geltung, wenn das Heck lange genug ist. Da nur spärliche Angaben über dieses wichtige Problem in der Literatur zu finden sind, entschloss sich die Contraves AG., Zürich, eine Untersuchung durchzuführen.

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[1]) Vorgetragen an der Tagung der Schweizerischen Physikalischen Gesellschaft am 4. Mai 1963 in Bern.

[2]) Die Ziffern in eckigen Klammern verweisen auf das Literaturverzeichnis, Seite 299.