A STUDY OF PLASMA FILLED PARALLEL PLATE WAVEGUIDE WITH ONE BOUNDARY CORRUGATED

S. K. Ghosh, S. P. Pal

Department of Mathematics, University of North Bengal, Darjeeling — 734 430, India

Warm plasma theory is used to investigate the propagation of electromagnetic waves in a plasma filled parallel plate waveguide with one boundary plate corrugated. Dispersion relations for TE- and TM-modes are derived and it is found that the propagation of TE-modes is unaffected by corrugation of one boundary plane while the propagation of TM-modes is affected by it. For TM-modes the wave number k depends on the frequency ω as well as on the distance through which the wave is propagated.

1. INTRODUCTION

Wave propagations in a cold plasma between two infinite conducting plates were investigated by Schumann [1], Dawson and Oberman [2] and Weber [3]. Recently, the wave propagation in a warm plasma between two infinite conducting plates was investigated by Azakami et al. [4].

In the present paper, we discuss the wave propagation in a warm plasma between two conducting plates whose one plate is plane and the other is corrugated. In this paper we derive dispersion relation (characteristic equation) for transverse electric (TE) modes and transverse magnetic (TM) modes. We see that the propagation of TE-modes is unaffected by the corrugation of one conducting plate whereas the propagation of TM-modes is affected by it. Here the frequency ω of wave propagation depends on wave number k as well as on the distance z. If the corrugated sheet becomes plane the result reduces to the result obtained by Azakami et al. [4].

2. BASIC EQUATIONS

To explain the warm plasma motion inside a waveguide we take following set of equations (Krall and Trivelpiece [5])

$$U^2 Nm \, \nabla . \boldsymbol{V} = - \frac{\partial p}{\partial t}, \tag{1}$$

$$Nm \left(\frac{\partial \boldsymbol{V}}{\partial t} + \nu \boldsymbol{V} \right) = -Ne\boldsymbol{E} - \nabla p, \tag{2}$$

$$\nabla \times \boldsymbol{H} = -Ne\boldsymbol{V} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \tag{3}$$

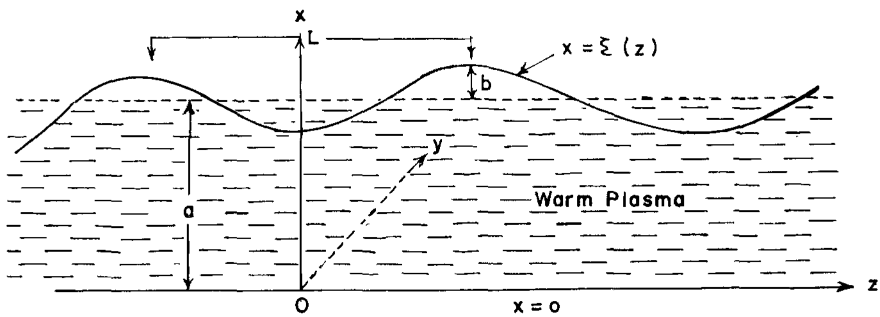
and

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \tag{4}$$

where $U = (\gamma kT/m)^{1/2}$ is the velocity of sound wave propagation in warm plasma,
N – electron density,
m – electron mass,
γ – ratio of specific heats,
k – Boltzmann's constant,
T – temperature of the warm plasma,
ν – electron-collision frequency,
p – pressure,
$\mathbf{V}$ – velocity of the electron,
$\mathbf{E}$ – electric intensity and
$\mathbf{H}$ – magnetic intensity.

3. GEOMETRY OF THE PROBLEM

We shall apply the basic equations of the last section to discuss the following boundary value problem.



Let us take $x = 0$ as the lower boundary surface and $x = \xi(z)$ as the upper boundary surface, where $\xi(z)$ is a periodic function of z and z-axis is taken in the direction of wave propagation. We consider the upper surface as cosine-function, then

$$x = \xi(z) = a\left(1 - \varepsilon \cos \frac{2\pi z}{L}\right), \tag{5}$$

where $\varepsilon = b/a$ and a is the average distance between two plates, b and L are the amplitude and wave length of the periodic function. The system is taken to be infinite in y- and z-directions. We assume that all physical quantities depend on x and z only. Further, we assume that the amplitude of the upper surface is small in comparison with the average thickness a of the waveguide, i.e. ε is very small.

The boundary conditions satisfied by $\mathbf{E}$ and $\mathbf{V}$ at the plate $x = 0$ are

$$E_y = 0, \quad E_z = 0 \quad \text{and} \quad V_x = 0 \tag{6}$$

and the boundary conditions at the upper surface require that the tangential component of electric field vector and the normal component of electron velocity vector vanish. Thus at the upper surface $x = \xi(z)$

$$E_z + \xi' E_x = 0\,, \qquad V_x - \xi' V_z = 0\,. \tag{7}$$

4. SOLUTION OF THE PROBLEM

To solve the problem we take the time dependence of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{V}$ and p in the following form

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \sin \omega t\,, \\ \mathbf{H} &= \mathbf{H}_0 \cos \omega t\,, \\ \mathbf{V} &= \mathbf{V}_0 \cos \omega t\,, \\ p &= p_0 \sin \omega t\,. \end{aligned} \tag{8}$$

For typographical convenience from now on we shall write $\mathbf{E}$, $\mathbf{H}$, $\mathbf{V}$, p for the amplitudes $\mathbf{E}_0$, $\mathbf{H}_0$, $\mathbf{V}_0$ and p_0, and the z dependence of the amplitudes $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{V}$ are taken in the form

$$\begin{aligned} \mathbf{E} &= \hat{x} E_x \sin kz + \hat{y} E_y \sin kz + \hat{z} E_z \cos kz\,, \\ \mathbf{H} &= \hat{x} H_x \cos kz + \hat{y} H_y \cos kz + \hat{z} H_z \sin kz\,, \\ \mathbf{V} &= \hat{x} V_x \sin kz + \hat{y} V_y \sin kz + \hat{z} V_z \cos kz\,, \end{aligned} \tag{9}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors. Elimination of $\mathbf{V}$ from equation (3) with the help of equations (1) and (2) gives

$$\nabla \times \mathbf{H} = \omega \varepsilon_0 \varepsilon_p \mathbf{E} + \frac{\varepsilon_0 U^2}{\omega} \nabla(\nabla \cdot \mathbf{E}) \tag{10}$$

where

$$\varepsilon_p = 1 - \frac{\omega_p^2}{\omega^2} \tag{11}$$

and

$$\omega_p = \left(\frac{Ne^2}{m\varepsilon_0}\right)^{1/2} = \text{plasma frequency}\,, \tag{12}$$

we have assumed that $(\nu/\omega) \ll 1$.

After elimination, from equations (4) and (10), H_z and E_z are found to satisfy the following equations

$$\frac{d^2 H_z}{dx^2} + k_e^2 H_z = 0\,, \tag{13}$$

$$\frac{d^4 E_z}{dx^4} + \left(k_e^2 + k_s^2\right) \frac{d^2 E_z}{dx^2} + k_e^2 k_s^2 Ez = 0\,, \tag{14}$$

where

$$k_e^2 = \frac{\omega^2}{c^2}\,\varepsilon_p - k^2\,, \tag{15}$$

$$k_s^2 = \frac{\omega^2}{U^2}\,\varepsilon_p - k^2\,. \tag{16}$$

The other components of field vectors and electron velocity in terms of H_z and E_z are as follows:

$$E_x = -\,\frac{U^2/c^2}{k k_e^2 (1 - U^2/c^2)} \left[\frac{d^3 E_z}{dx^3} + \{k_e^2 - k^2(1 - c^2/U^2)\} \frac{dE_z}{dx}\right], \tag{17}$$

$$E_y = -\,\frac{\omega \mu_0}{k_e^2} \frac{dH_z}{dx}\,, \tag{18}$$

$$H_x = \frac{k}{k_e^2} \frac{dH_z}{dx}\,, \tag{19}$$

$$H_y = -\,\frac{U^2/c^2}{\omega \mu_0 k_e^2 (1 - U^2/c^2)} \left[\frac{d^3 E_z}{dx^3} + \left\{k_e^2 \left(2 - \frac{c^2}{U^2}\right) - k^2 \left(1 - \frac{c^2}{U^2}\right)\right\} \frac{dE_z}{dx}\right], \tag{20}$$

$$V_x = \frac{e U^2/c^2}{m \omega k k_e^2 (1 - U^2/c^2)} \left[\frac{U^2}{\omega_p^2} \frac{d^5 E_z}{dx^5} - \left\{1 - \frac{U^2 k_e^2}{\omega_p^2} + \frac{U^2 k^2}{\omega_p^2} \left(1 - \frac{c^2}{U^2}\right)\right\} \frac{d^3 E_z}{dx^3} - \right.$$
$$\left. - \left\{k_e^2 + k^2 \left(1 - \frac{c^2}{U^2}\right) \left(\frac{U^2 k_e^2}{\omega_p^2} - 1\right)\right\} \frac{dE_z}{dx}\right], \tag{21}$$

$$V_y = -\,\frac{e \mu_0}{m k_e^2} \frac{dH_z}{dx}\,, \tag{22}$$

$$V_z = \frac{e}{m \omega} \cdot \frac{U^4}{\omega_p^2 c^2 k_e^2 (1 - U^2/c^2)} \left[\frac{d^4 E_z}{dx^4} + \left\{k_e^2 - k^2 \left(1 - \frac{c^2}{U^2}\right)\right\} \frac{d^2 E_z}{dx^2} - \right.$$
$$\left. - \frac{\omega_p^2 k_e^2}{U^2} \left(1 - \frac{c^2}{U^2}\right) \left(1 + \frac{U^2 k^2}{\omega_p^2}\right) E_z\right]. \tag{23}$$

5. DISPERSION RELATIONS

Applying the above set of equations together with the boundary conditions we obtain dispersion relations for TE- and TM-modes:

5.1. TE-modes

In case of TE-modes $E_z = 0$ and from equation (13)

$$H_z = A \cos k_e x + B \sin k_e x \,, \tag{24}$$

where A and B are integrating constants. The equations to determine other components of field vectors and electron velocity will take form

$$E_x = 0 \,, \tag{25}$$

$$E_y = -\frac{\omega\mu_0}{k_e^2} \frac{\mathrm{d}H_z}{\mathrm{d}x} \,, \tag{26}$$

$$H_x = \frac{k}{k_e^2} \frac{\mathrm{d}H_z}{\mathrm{d}x} \,, \tag{27}$$

$$H_y = 0 \,, \tag{28}$$

$$V_x = 0 \,, \tag{29}$$

$$V_y = -\frac{e\mu_0}{mk_e^2} \frac{\mathrm{d}H_z}{\mathrm{d}x} \,, \tag{30}$$

$$V_z = 0 \,. \tag{31}$$

The boundary conditions (6) and (7) together with the above equations give the following relation between ω and k

$$\omega^2 = \omega_p^2 + k^2 c^2 \,. \tag{32}$$

5.2. TM-modes

In case of TM-modes $H_z = 0$ and from equation (14)

$$E_z = A_1 \cos k_s x + A_2 \cos k_e x + B_1 \sin k_s x + B_2 \sin k_e x \tag{33}$$

where A_1, B_1, A_2, B_2 are integrating constants. The other exponents of field vectors and electron velocity will be determined from the following equations

$$E_x = -\frac{U^2/c^2}{kk_e^2(1 - U^2/c^2)} \left[\frac{\mathrm{d}^3 E_z}{\mathrm{d}x^3} + \left\{k_e^2 - k^2\left(1 - \frac{c^2}{U^2}\right)\right\} \frac{\mathrm{d}E_z}{\mathrm{d}x}\right], \tag{34}$$

(35) $$E_y = 0\,,$$

(35a) $$H_x = 0\,,$$

(36) $$H_y = -\frac{U^2/c^2}{\omega\mu_0 k_e^2(1 - U^2/c^2)}\left[\frac{d^3E_z}{dx^3} + \left\{k_e^2\left(2 - \frac{c^2}{U^2}\right) - k^2\left(1 - \frac{c^2}{U^2}\right)\right\}\frac{dE_z}{dx}\right],$$

(37) $$V_x = \frac{e}{m\omega}\cdot\frac{U^2/c^2}{kk_e^2(1 - U^2/c^2)}\left[\frac{U^2}{\omega_p^2}\frac{d^5E_z}{dx^5} - \left\{1 - \frac{U^2k_e^2}{\omega_p^2} + \frac{U^2k^2}{\omega_p^2}\left(1 - \frac{c^2}{U^2}\right)\right\}\cdot\right.$$
$$\left.\cdot\frac{d^3E_z}{dx^3} - \left\{k_e^2 + k^2\left(1 - \frac{c^2}{U^2}\right)\left(\frac{U^2k_e^2}{\omega_p^2} - 1\right)\right\}\frac{dE_z}{dx}\right],$$

(38) $$V_y = 0\,,$$

(39) $$V_z = \frac{e}{m\omega}\cdot\frac{U^4}{\omega_p^2c^2k_e^2(1 - U^2/c^2)}\left[\frac{d^4E_z}{dx^4} + \left\{k_e^2 - k^2\left(1 - \frac{c^2}{U^2}\right)\right\}\frac{d^2E_z}{dx^2} - \right.$$
$$\left. - \frac{\omega_p^2k_e^2}{U^2}\left(1 - \frac{c^2}{U^2}\right)\left(1 + \frac{U^2k^2}{\omega_p^2}\right)E_z\right].$$

we have from the boundary conditions (6) and (7) together with the equations (33) to (39) by using $(\omega_p/\omega) < 1$ and neglecting second and higher order terms in ε the following characteristic equation

(40) $$4\chi - (\chi + 1)^2\cos(ak_s + ak_e) + (\chi - 1)^2\cos(ak_s - ak_e) -$$
$$- \varepsilon[P\sin(ak_s + ak_e) - Q\sin(ak_s - ak_e)] = 0\,,$$

where

(41) $$\chi = \frac{k^2}{k_ek_s}(\omega_p/\omega)^2\,,$$

(42)
$$P = 2(\chi + 1)\left[(\chi + 1)(ak_s + ak_e)\cos\frac{2\pi z}{L} - \frac{2\pi}{kL}\chi\left(1 - \frac{\omega^2}{\omega_p^2}\right)(ak_s - ak_e)\sin\frac{2\pi z}{L}\right],$$

(43)
$$Q = 2(\chi - 1)\left[(\chi - 1)(ak_s - ak_e)\cos\frac{2\pi z}{L} - \frac{2\pi}{kL}\chi\left(1 - \frac{\omega^2}{\omega_p^2}\right)(ak_s + ak_e)\sin\frac{2\pi z}{L}\right],$$

when we take $(\omega_p/\omega) > 1$, equation (40) becomes

(44) $$4\chi' + (\chi' - 1)^2\cosh(ak_s' + ak_e') - (\chi' + 1)^2\cosh(ak_s' - ak_e') -$$
$$- \varepsilon[P'\sinh(ak_s' + ak_e') - Q'\sinh(ak_s' - ak_e')] = 0$$

where

(45) $$\chi' = \frac{k^2}{k_s'k_e'}\left(\frac{\omega_p}{\omega}\right)^2,$$

(46) $$k_s' = \sqrt{\left(k^2 - \frac{\omega^2}{U^2}\varepsilon_p\right)}, \quad k_e' = \sqrt{\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_p\right)},$$

(47) $$P' = 2(1-\chi')\left[(1-\chi')(ak_s' + ak_e')\cos\frac{2\pi z}{L} + \frac{2\pi}{kL}\chi'\left(1 - \frac{\omega^2}{\omega_p^2}\right) . (ak_s' - ak_e')\sin\frac{2\pi z}{L}\right],$$

(48) $$Q' = 2(1+\chi')\left[(1+\chi')(ak_s' - ak_e')\cos\frac{2\pi z}{L} - \frac{2\pi}{kL}\chi'\left(1 - \frac{\omega^2}{\omega_p^2}\right) . (ak_s' + ak_e')\sin\frac{2\pi z}{L}\right].$$

6. CONCLUSION

In this paper the characteristic equations for TE- and TM-modes are derived for a waveguide whose lower boundary is a plane conducting plate but the upper boundary is corrugated. In case of TE-modes we see that the characteristic equation (32) is same in case of parallel plane waveguide. But for the TM-modes the characteristic equations (40) and (44) express the fact that the wave number k depends upon the frequency ω as well as distance z in the manner which is determined by the form of roots of these equations. The characteristic equation can thus be used to determine frequency as a function of the wave number for each value of z. As the wavelength of the periodic surface approaches to infinity or the amplitude of the surface tends to zero the characteristic equations (40) and (44) take the same form as was obtained by Azakami et al. [4].

The authors take the opportunity to thank Professor M. L. Ghosh, Department of Mathematics, North Bengal University for suggesting the problem and one of the authors (S. K. Ghosh) wishes to express his thanks to the University of North Bengal for financial support.

Received 11. 9. 1979.

References

[1] Schuman W. O.: Z. Angew. Phys. *8* (1956), 482.
[2] Dawson J., Oberman C.: Phys. Fluids *2* (1959), 103.
[3] Weber H. N.: J. Plasma Phys. (GB) *14* (1975), 399.
[4] Azakami T., Narita H., Thein U Aye: Bull. Nagoya Inst. of Technol. *24* (1972), 201.
[5] Krall N. A., Trivelpiece A. W.: Principles of Plasma Physics, McGraw Hill Inc., New York, 1973, p. 82.