Journal of Applied Mathematics and Physics (ZAMP)
Vol. 28/6, 1977 Birkhäuser Verlag Basel

On the Curvature of Compressible Boundary Layer Flows Near Separation [1])

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Introduction

The need to estimate the slope of the flow streamlines at special points often arises in practical fluid mechanics problems. An example is the well-known formula derived by Oswatitsch [1] giving the separation streamline slope of a 2-D viscous flow as a function of the local streamwise gradients of wall pressure and shear stress. However, there also arises the need to evaluate streamline *curvature*, as for example in the study of Görtler-type instability mechanisms which can occur in regions of local concave streamline curvature [2]. A particular case in point is the possibility of spanwise-periodic disturbances near separation of a nominally 2-D boundary layer flow [3] where one needs an analytical relation for the flow curvature above the separation streamline. In the present paper, such an expression is derived in a form analogous to Oswatitsch's result by a direct extension of his approach, for either laminar or turbulent compressible boundary layer flows. Following this, we give an example of its practical application.

Analysis

Consider steady compressible laminar or turbulent compressible flow with arbitrary heat transfer and Prandtl number past an impermeable non-ablating surface of given but arbitrary wall temperature. We invoke a boundary layer approximation by neglecting the very small $\partial(\mu\, \partial u/\partial x)/\partial x$ viscous term in the x-momentum equation; however, the turbulence model is arbitrary except for the general physical restriction that the turbulent eddy viscosity μ_τ vanish at the wall including the separation point [here defined as $\tau_w \equiv \mu_w(\partial u/\partial y)_w = 0$].

We follow Oswatitsch [1] and further assume the separation point x_s, 0 to be regular in the sense that it is free from singularities and hence that streamwise and normal flow velocity components u, v in its neighborhood ($\sim \delta_{BL}$ in size) can be represented by Taylor Series expansions in x-x_s and y (an abundance of evidence has accumulated to support this and the aforementioned assumption *provided* the local

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pressure gradient field is not prescribed but is rather a truly interactive one, i.e., coupled with the boundary layer displacement thickness distribution and unknown *a priori* [4, 5]). Then noting from the compressible continuity equation that

$$
\begin{aligned}
(\partial v/\partial y)_w &= (\partial^2 v/\partial x\,\partial y)_w = (\partial^3 v/\partial x^2\,\partial y)_w = 0,\\
(\partial^2 v/\partial y^2)_w &= -\,\partial(\tau_w/\mu_w)/\partial x - (\tau_w/\mu_w)(\partial p_w/\partial x)/p_w,\\
(\partial^3 v/\partial x\,\partial y^2)_w &= -\,\partial^2/\partial x^2(\tau_w/\mu) - \partial(\tau_w/\mu_w)/\partial x(\partial p_w/\partial x)/p_w - (\tau_w/\mu_w)\\
&\qquad \times\,\partial/\partial x[(\partial p_w/\partial x)/p_w]
\end{aligned}
$$

and

$$
\begin{aligned}
(\partial^3 v/\partial y^3)_w &= \partial/\partial x(\partial^2 u/\partial y^2)_w - (\partial^2 u/\partial y^2)_w(\partial p_w/\partial x)/p_w\\
&\quad - 2(\tau_w/\mu_w)\partial/\partial y[(\partial p/\partial x)/p] - (\partial^2 v/\partial y^2)_w(\partial \rho/\partial y)_w/\rho_w
\end{aligned}
$$

in the absence of mass transfer, we have

$$
\frac{v}{u} = \frac{\left(\dfrac{\partial^2 v}{\partial y^2}\right)_w \dfrac{y^2}{2} + \left(\dfrac{\partial^3 v}{\partial y^3}\right)_w \dfrac{y^3}{6} + \left(\dfrac{\partial^3 v}{\partial x\,\partial y^2}\right)_w \dfrac{(x - x_s)y^2}{2} + \cdots}{\left(\dfrac{\tau_w}{\mu_w}\right) y + \left(\dfrac{\partial^2 u}{\partial y^2}\right)_w \dfrac{y^2}{2} + \dfrac{\partial}{\partial x}\left(\dfrac{\tau_w}{\mu_w}\right)(x - x_s)y + D + \cdots} \tag{1}
$$

where

$$
\begin{aligned}
D \equiv (y^3/6)(\partial^3 u/\partial y^3)_w &+ [(x - x_s)y^2/2]\,\partial/\partial x(\partial^2 u/\partial y^2)_w\\
&+ [(x - x_s)^2 y/2]\,\partial^2/\partial x^2(\tau_w/\mu_w)
\end{aligned}
$$

Furthermore, from the x-momentum equation and its y-derivative evaluated at the wall,

$$
(\partial^2 u/\partial y^2)_w = (\partial p/\partial x)/\mu_w - [\partial(\mu + \mu_T)/\partial y]_w \tau_w/\mu_w^2
$$

and

$$
\begin{aligned}
(\partial^3 u/\partial y^2)_w = &- 2[\partial(\mu + \mu_T)/\partial y]_w(\partial^2 u/\partial y^2)_w/\mu_w\\
&- [\partial^2(\mu + \mu_T)/\partial y^2]_w \tau_w/\mu_w^2 + 1/\mu_w[\partial/\partial x(\partial p/\partial y)]_w.
\end{aligned}
$$

Away from the separation point ($\tau_w \neq 0$) approaching the surface, Eqn. (1) thus yields $v/u \to 0$ as $y \to 0$ (i.e., the flow becomes parallel to the wall), whereas approaching the surface point $\tau_w = 0$ along the separation streamline $y = (x - x_s)\tan\theta_s$ as $y \to 0$ gives the well-known result that

$$
\tan\theta_s = v/u = -\frac{(\partial \tau_w/\partial x)}{(\partial p_w/\partial x) + 2(\partial \tau_w/\partial x)\cot\theta_s}
$$

or

$$
\tan\theta_s = 3\,\frac{(-\partial \tau_w/\partial x)_s}{(\partial p_w/\partial x)_s}. \tag{2}
$$

This result, which applies equally well to a point of reattachment, is independent of the compressibility, viscosity law, $\partial p/\partial y$, heat transfer or state of turbulence.[3]) It is reemphasized that Eqn. (2) pertains to the interacted values of the *wall* gradients involved. Moreover, for this reason, the contribution of $\partial p_w/\partial x$ to $\partial \rho_w/\partial x$ terms must be taken into account in the subsequent curvature analysis for the compressible case.

Now the corresponding curvature of the flow just above this separation point can be straightforewardly derived by differentiation of Eqn. (1) and examination of the result near the wall. Since both experiment and Eqn. (2) show that θ_s is small (5–10°), as in fact it must to be consistent with the boundary layer approximation, the flow curvature may be accurately calculated as

$$K = |R|^{-1} \simeq \partial(v/u)/\partial x = [(\partial v/\partial x) - (v/u)\partial u/\partial x]u^{-1}$$

(being positive for convex streamlines) where R is the corresponding local radius of curvature. Thus differentiating Eqn. (1) with respect to x and substituting the aforementioned velocity derivative values we obtain above any point on the surface not a separation point that

$$K \simeq \frac{(\partial^3 v/\partial x\,\partial y^2)_w \dfrac{y^2}{2} + 0(y^3)}{(\tau_w/\mu)y + 0(y^2) + \cdots} - \left(\frac{\partial^2 v}{\partial y^2}\right)_w \frac{\left[\dfrac{\partial(\tau_w/\mu_w)}{\partial x}\dfrac{y^3}{2} + 0(y^4) + \cdots\right]}{[(\tau_w/\mu_w)y + 0(y^2) + \cdots]^2} \tag{3a}$$

$$\simeq \frac{y}{2}\left\{\frac{\left[\dfrac{\partial(\tau_w/\mu_w)}{\partial x}\right]^2}{(\tau_w/\mu_w)^2} - \frac{\dfrac{\partial^2}{\partial x^2}(\tau_w/\mu_w)}{\tau_w/\mu_w} - \frac{\partial}{\partial x}\left(\frac{\partial p_w/\partial x}{p_w}\right)\right\} + 0(y^2) + \cdots \tag{3b}$$

which is seen to vanish smoothly as $y \to 0$ upstream or downstream of separation since the flow becomes parallel either to the flat surface or the straight separation streamline, respectively. Away from separation above the surface the streamline curvature increases linearly with distance[4]). On the other hand, if we now consider a small distance right above the separation point $x = x_s$, we find instead that

$$K \simeq \frac{(\partial^3 v/\partial x\,\partial y^2)_w}{(\partial^2 u/\partial y^2)_w} - \frac{2\tan\theta_s}{y}\left[\frac{\dfrac{\partial(\tau_w/\mu_w)}{\partial x} + \dfrac{y}{2}\dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial y^2}\right)_w}{(\partial^2 u/\partial y^2)_w + (y/3)(\partial^3 u/\partial y^3)_w}\right] \tag{4a}$$

$$K \simeq \frac{2}{3}\frac{\tan^2\theta_s}{y} - \frac{\left[\begin{array}{l}\dfrac{\partial^2 \tau_w}{\partial x^2} - \tan\theta_s \dfrac{(\partial p/\partial x)_w^2}{p_w} \\ + \mu_w \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial y^2}\right)_w \tan\theta_s + \dfrac{2}{9}\mu_w \tan^2\theta_s \left(\dfrac{\partial^3 u}{\partial y^3}\right)_w\end{array}\right]}{\partial p_w/\partial x} + 0(y) \tag{4b}$$

[3]) It can also be shown applicable to curved as well as flat walls.
[4]) Note that the last term in Eqn. (3b) is absent in an incompressible flow.

$$K \simeq \frac{2}{3}\frac{\tan^2\theta_s}{y} - \frac{\left\{ \begin{aligned} &\frac{\partial^2 \tau_w}{\partial x^2} + \tan\theta_s \left[\left(\frac{\partial^2 p}{\partial x^2}\right)_s - \frac{(\partial p/\partial x)_s^2}{p_w}\right]_w \\ &- \frac{1}{9}\tan^2\theta_s \left[\left(\frac{\partial(\mu+\mu_\tau)}{\partial y}\right)_w \left(\frac{\partial p_w/\partial x}{\mu_w}\right)_s - 2\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)_{w,s}\right] \end{aligned} \right\}}{\partial p_w/\partial x} \tag{4c}$$

Right at the separation point on the surface, this relation indicates infinite streamline curvature as indeed it must owing to the streamline slope jump across x_s. A short distance away from the surface, however, being proportional to the very small quantity $\tan^2\theta_s$, the effect of this singularity dies out quickly leaving the y-independent value given by the second term on the right of Eqn. (4c) which is the main result of practical interest here [5]). We note that while it does depend explicitly on the compressibility, heat transfer, $\partial p/\partial y$ and laminar-turbulent viscosity law through the last term, these effects are negligible in practice unless the heat transfer pressure gradient product or $\partial/\partial x\,(\partial p/\partial y)_w$ is very large $[\gtrsim 0\,(\cot^2\theta_s)]$ such as might be the case in strongly-interactive hypersonic boundary layer separation from a highly cooled wall. Dropping the first term on the RHS of (4c) that merely represents the far field of the wall surface curvature singularity, we thus have the following final approximate relation for the average flow curvature above separation (or reattachment):

$$-K \simeq \frac{\left(\dfrac{\partial^2\tau_w}{\partial x^2}\right) + \tan\theta_s\left[\dfrac{\partial^2 p}{\partial x^2} - \dfrac{(\partial p/\partial x)^2}{p}\right]_w}{(\partial p/\partial x)_{s,w}}. \tag{5}$$

It is noted that the last term here is derived from a $\partial p_w/\partial x$ contribution and hence is not present in purely incompressible flow; however, since its effect is $\sim \tan\theta_s$, it is probably important only under hypersonic flow conditions.

The foregoing analysis indicates that a short distance above the wall the flow streamline curvature undergoes a rapid streamwise change approaching the separation point. The curvature at a fixed $y > 0$ somewhat away from x_s is according to Eqn. (3b) proportional to τ_w^{-2} and hence increases without bound as $x \to x_s$ with a rate proportional to y. However, according to Eqn. (4) this increase must stop and become bounded at $x = x_s$ by a value which quickly decays rapidly in y to that given by Eqn. (5) (see Fig. 1).

Application

Although strictly speaking it should be retained, $(\partial^2 p_w/\partial x^2)_s$ in many cases is negligibly small (see, e.g., Fig. 3) and it is sufficient to neglect all but the first numer-

[5]) The work of Stewartson and Williams [4] also formally justifies neglecting $y^{-1}\tan^2\theta_s$ in Eqn. 4c for $Re^{-5/8} \leq y \leq Re^{-1/2}$ in the laminar case, since there $\theta_s \sim 0\,(Re^{-1/4})$, $\Delta p \sim 0\,(Re^{-1/4})$, $\Delta\tau_w \sim 0\,(Re^{-1/2})$ and $\Delta x \sim 0\,(Re^{-3/8})$.

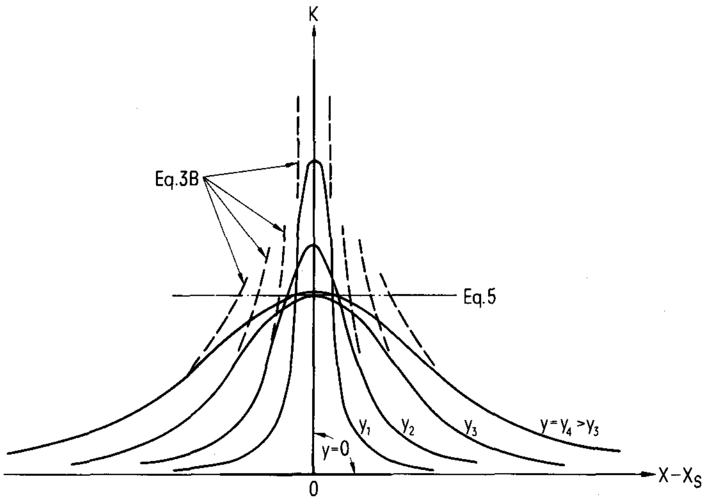


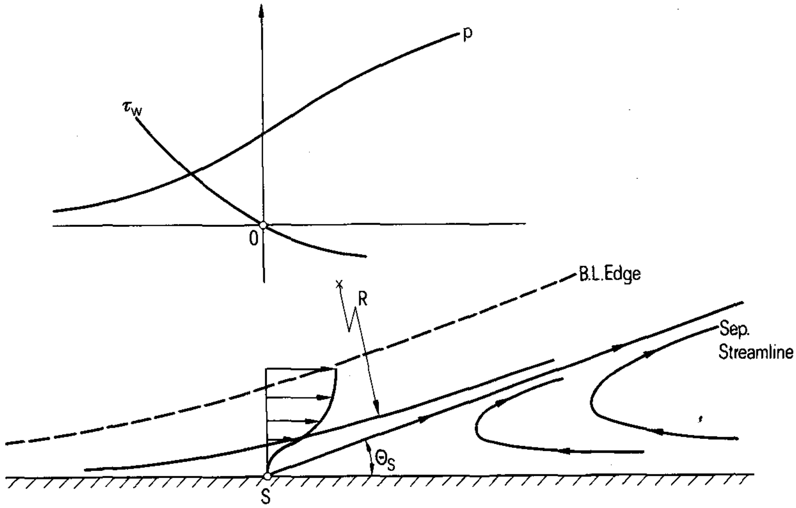
Figure 1
Behavior of streamline curvature near separation point.

ator term in Eqn. (5) to get approximately

$$|K^{-1}| = R_{AV,s} \simeq \frac{(\partial p/\partial x)_s}{(\partial^2 \tau_w/\partial x^2)_s} \simeq -\frac{3(\partial \tau_w/\partial x)_s}{(\partial^2 \tau_w/\partial x^2)_s} \cot \theta_s. \tag{6}$$

This predicts that $R_{AV,s}$ is directly proportional to the radius of curvature of the $\tau_w(x)$ plot at separation and that it also decreases as the separation angle increases (i.e., in agreement with physical expectation, the curvature vanishes as $\theta_s \to 0$, Fig. 2). To apply Eqn. (6) in practice thus requires data on the streamwise pressure gradient and *second* derivative of the streamwise wall shear distribution at the separation (or reattachment) point.

Fortunately, there exists some relatively detailed experimental data at conditions satisfying the assumptions underlying Eqn. (6) from which these quantities can be estimated: Figures 3–5 illustrate the pressure, wall shear distributions and flow geometry measured by Sfeir [6] for a $M_\infty = 2.64$, $Re_L = 1.4 \times 10^5$ laminar flow separating and reattaching on a small angle ramp compression corner. Let us first check the Oswatitsch formula (2) for the separation point streamline slope against these data. From Fig. 3 we readily estimate that $[\partial(p/p_\infty)/\partial(x/L)]_s \equiv 1.67$; however, the corresponding value of $(\partial \tau_w/\partial x)_s$ is more difficult to determine accurately from Fig. 4, because only two points (one at x_s) on the $\tau_w(x)$ curve near separation are given and it is implied that the slope varies rapidly as $x \to x_s$ (to which the $\tan \theta_s$ estimate from Eqn. (3) is quite sensitive). As a reasonable average value for engineering purposes, we use the slope defined by the two experimental points *D* and *S*, which gives $\partial(\tau/\tau_{ref})/\partial(x/L) \simeq 2.6$ where $\tau_{ref} = \rho_\infty U_\infty^2 [Re_{\infty x}(T_w/T_\infty)^{0.25}]^{-1/2}$ is the Blasius reference value (defined at $x/L = 0.25$) and $T_w/T_\infty \simeq T_{w,AD}/T_\infty \simeq 2.17$. Then substituting these values into Eqn. (3) yields $\tan \theta_s \simeq 0.074$ which agrees fairly



$$\tan\theta_s \simeq -3\,\frac{(\partial\tau_w/\partial x)_s}{(\partial p/\partial x)_s}$$

$$R \simeq -3\,\frac{(\partial\tau_w/\partial x)_s}{(\partial^2\tau_w/\partial x^2)_s}\cot\theta_s \simeq \frac{(\partial p/\partial x)_s}{(\partial^2\tau_w/\partial x^2)_s}$$

Figure 2
Schematic illustration of separation region.

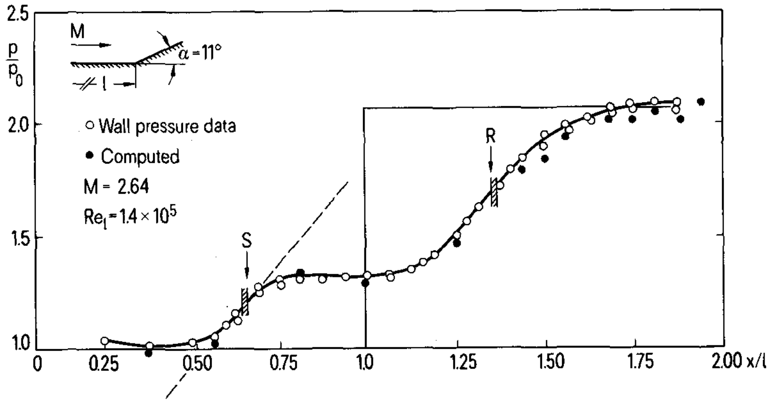


Figure 3
Pressure distribution along a ramp-induced separation [6].

well with the experimental value 0.107 obtained directly from the streamline geometry shown in Fig. 5. A similar calculation at the reattachment point gives comparable agreement: Eqn. (3) yields $\tan\theta_R \simeq 0.091$ compared with the measured value 0.112 from Fig. 5.

We now evaluate the corresponding streamline curvature. Near separation there are sufficient data points on Fig. 4 (including the reversed flow region) to enable a

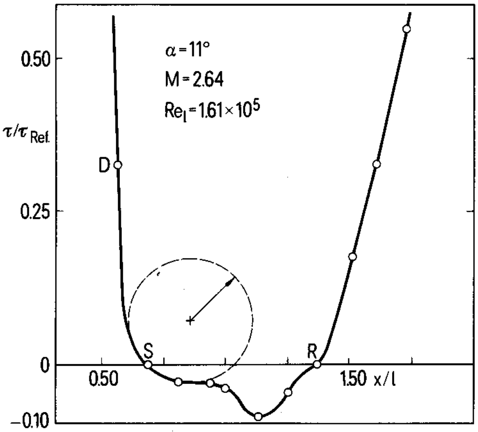


Figure 4
Wall shear stress distribution [6].

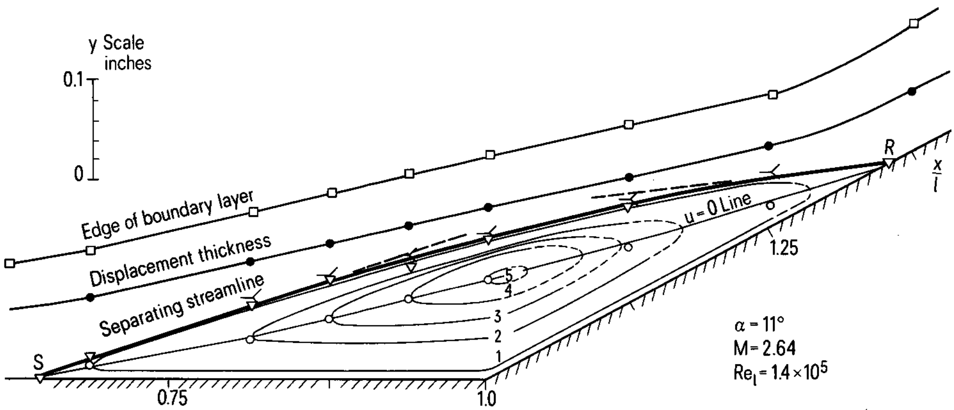


Figure 5
Flow detail within separation bubble [6].

reasonable circular fit to the $\tau_w(x)$ curvature and thus obtain

$$[\partial^2 \tau_w / \partial(x/L)^2]_s \simeq 4\{1 + [\partial(\tau_w/\tau_{ref})/\partial(x/L)]_s^2\}^{3/2} \tau_{ref} \simeq 5\tau_{ref}$$

where to be self-consistent we have used here the corresponding circle slope value $[\partial(\tau_w/\tau_{ref})/\partial(x/L)]_s \simeq 0.40$. Thus substituting into our curvature radius formula (6) we obtain $(R_{AV}/L)_s \simeq 0.4 Re_L^{1/2}/M_\infty^2 \simeq 21.2$. A similar calculation at reattachment where the curvature of the $\tau_w(x)$ curve is considerably larger yields $(R_{AV}/L)_R \simeq 5.3$.

An immediate and interesting application of these estimates is to examine the possibility that spanwise-periodic disturbances (Görtler-type vortices) may form in the concave streamlines above the separation and reattachment points. This can be done approximately by a local application of Görtler's instability theory results for curved wall flows, these vortices likely occurring if the prevailing conditions appreciably exceed his instability criterion. According to Görtler [2, 7], a concave boundary layer flow having an average longitudinal radius of curvature R will be susceptible to amplified streamwise vortex disturbances whenever the following criterion is met:

$$\frac{\theta^* U_e}{\nu}\sqrt{\frac{\theta^*}{R}} = (\theta^*/L)^{3/2}\left(\frac{Re_L}{\sqrt{R/L}}\right) \gtrsim 0.25 \tag{7}$$

where θ^* is the momentum thickness at the local curvature region under consideration[6]). Applying this to Sfeir's reattachment conditions at $x_R/L = 1.375$ (extrapolating his given value $\theta_s^* = 0.07$ assuming $\theta^* \sim x^{1/2}$ and using the foregoing estimate of R/L there, we obtain

$$(\theta^* U_e/\nu)\sqrt{\theta^*/R} \simeq 248 \quad \text{(Reattachment).} \tag{8a}$$

Since this exceeds the critical value by three orders of magnitude, one might expect streamwise vortex disturbances to occur in reattaching flows. Indeed this has been confirmed by numerous experiments under comparable conditions, especially the detailed studies of Ginoux [9], and in fact an approximate theory of the resulting steady state disturbance flow has already been developed by Inger [10, 11].

At the separation point using the foregoing values, we get the somewhat smaller stability number

$$[(\theta^* U_e/\nu)\sqrt{\theta^*/R}]_{\text{sep}} \approx 70.4. \tag{8b}$$

Since this still exceeds the critical value by *two* orders of magnitude,[7]) we are led to expect the possible presence of streamwise vortex instabilities at separation also, as indicated in Fig. 6. Indeed, some experimental evidence of this has arisen in a recent

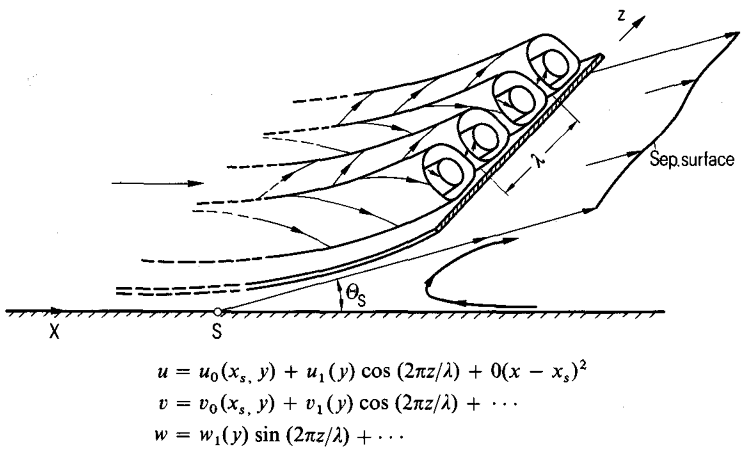


Figure 6
Schematic of three-dimensional disturbance field originating at separation [3].

[6]) Since Eqn. (7) is based on an incompressible flat plate boundary layer profile, its application near separation or reattachment here is only a rough approximation because it is at present not known exactly how vanishing skin friction, adverse pressure gradient and compressibility all affect the right side of Eqn. (7). However, since this value decreases with decreasing boundary layer wall slope [7] and increasing Mach number [8] for fixed θ^*, Eqn. (7) provides an upper limit and conservative stability estimate.

[7]) Under other conditions where the separation angle and hence the flow curvature is too small, however, the instability criterion (7) may not be satisfied and these vortices will not occur.

experimental study of $M = 2.85$ ramp-induced turbulent boundary layer separation [12]: a careful examination of the surface flow pattern at the separation line reveals evidence of a spanwise waviness with a typical wave-length at 2–4 boundary layer thicknesses [11]. Accordingly, a theoretical study of the problem has been initiated [3] for the non-parallel, vanishingly-small skin friction type of mean flow occurring at separation.

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Summary

Oswatitsch's analytical expression for the slope of a viscous flow separation or reattachment streamline is shown to be consistent with detailed experimental data on a supersonic boundary layer flow past a compression corner. An extension of his analysis is then given which yields a comparable new relation for the streamline *curvature* just above separation in either laminar or turbulent two-dimensional compressible boundary layer flows. This result is applied to examine the possible occurrence of Görtler streamwise vortices due to such curvature.

Zusammenfassung

Es wird gezeigt, daß die analytische Beziehung von Oswatitsch für den Neigungswinkel einer zähflüssigen Strömungsablösung (oder das Wiederanlegen), in Übereinstimmung ist mit detaillierten Versuchsergebnissen an einer Überschall-Grenzschichtströmung längs einer Kompressionsecke. Darüber hinaus wird eine Erweiterung dieser Theorie angegeben, welche eine vergleichbare Beziehung für die Stromlinienkrümmung im Bereich über dem Ablösepunkt, sowohl für laminare als auch turbulente zweidimensionale Grenzschichtströmung gibt. Dieses Ergebnis wird zu einer Abschätzung möglichen Auftretens von Görtlerschen Wirbeln in Strömungsrichtung infolge dieser Krümmung verwendet.

(Received: December 22, 1976; revised: June 16, 1977)