Thermoelastic Problem for an Isotropic Sphere with Temperature Dependent Properties[1])

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Introduction

Modern structural elements are often subjected to temperature fields of such intensity that their mechanical behavior approaches that of viscoelastic bodies, and a representation by means of a classical Hookean model does not adequately describe their actual properties.

In spite of this fact, it was shown recently that the elastic thermostress if based on temperature dependent properties of a material corresponds to the upper value of stress in an inelastic problem for the incompressible viscoelastic Maxwell body [1][3]). The results obtained for rotationally symmetrical plane states of stress and strain with steady-state radial temperature gradient make it plausible that also in other cases the elastic thermostress, defining the initial conditions of a viscoelastic thermal process, might serve as an estimate of the decisive thermostress in actual structural elements.

Such a postulate will be also adopted in the sequel for metallic elements under consideration.

On the other hand, experimental data for common steels evidence that at elevated temperatures Poisson's ratio approaches its upper bound for isotropic materials namely 0·5 [5]. Of course, this is only approximately true and the respective values oscillate between 0·4 or 0·45 and 0·5.

The assumption of the elastic incompressibility of a body at elevated temperatures, however, permits us to obtain a simple solution to the problem, in a closed form convenient for numerical computations. Moreover, the solution may pertain to any polarly symmetrical temperature field $T = T(r)$, r being a position vector, as well as to any variation of Young's modulus $E = E(T)$, shear modulus $G = G(T)$ and coefficient of thermal expansion $\alpha = \alpha(T)$, with temperature.

This assumption will be, therefore, adopted in subsequent arguments as the second basic postulate.

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[3]) Numbers in brackets refer to References, page 575.

As far as the influence of the temperature dependent properties of a material on the thermostress is concerned, it has been investigated, to the present writer's knowledge, in the case of an infinite thick-walled tube and a circular ring, only in four papers [1–4]. The first two papers treat elastic incompressible material, the last two an arbitrary but constant Poisson ratio[4]).

The scope of the present paper is confined to an extension of the previous solutions to the case of a solid or hollow sphere subjected to spherically symmetrical temperature fields. Formulae have been derived for thermal stress dealing with arbitrary variation of temperature as well as arbitrary variation of YOUNG's modulus, coefficient of thermal expansion and thermal conductivity with temperature.

Strain and Stress Fields

Consider an isotropic elastic sphere, with the pole of spherical coordinates at the center of the sphere, subjected to spherically symmetrical external tractions and spherically symmetrical temperature field $T = T(r)$, with r as position vector. Assume that YOUNG's modulus E and coefficient of thermal expansion of the material of the sphere show marked dependence on temperature, and that POISSON's ratio ν is constant and equal to 0·5.

With usual notation, we obtain, in the case considered, the Hooke-Duhamel equations

$$\varepsilon_r = \frac{1}{E(T)} (\sigma_r - \sigma_\varphi) + \int_0^{T(r)} \alpha(\tau)\, d\tau\,, \quad \varepsilon = \frac{1}{2\,E(T)} (\sigma_\varphi - \sigma_r) + \int_0^{T(r)} \alpha(\tau)\, d\tau\,, \tag{1}$$

whereas all other components of stress and strain tensors vanish. In fact, $T(r)$ denotes the excess of temperature above the uniform ambient conditions at which thermostress disappears, and elastic and thermal coefficients have their usual values.

With u as radial displacement we have for the only non-vanishing unit elongations in the radial and tangent directions,

$$\varepsilon_r = \frac{du}{dr}\,, \quad \varepsilon_\varphi = \frac{u}{r}\,. \tag{2}$$

Since the elastic dilatation of the body is zero, by hypothesis, equations (1)

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4) *Addendum by Proofreading:* The author wishes to express his thanks to Professor CHANG, University of Minnesota, for drawing attention to the paper *Stresses in a Metal Tube Under Both High Radial Temperature Variation and Internal Pressure* by C. C. CHANG and W. H. CHU, J. appl. Mech. *21*, 101 (1954), in which a related problem has been investigated on the basis of experimental data. Also a recent paper by R. TROSTEL, *Stationäre Wärmespannungen mit temperaturabhängigen Stoffwerten*, Ing.-Archiv *26*, 416 (1958), giving a general solution to the thermoelastic problem for temperature dependent material properties, using the perturbation method, should be mentioned.

and (2) yield simply

$$\frac{d}{dr}\,(r^2\,u) = 3\,r^2\,\Phi\,, \tag{3}$$

with a notation

$$\Phi = \int_0^{T(r)} \alpha(\tau)\,d\tau\,. \tag{4}$$

An integration of the differential equation (3) yields the radial displacement

$$u = \frac{3}{r^2}\,\chi_{a1}(r) + \frac{C_1}{r^2}\,, \tag{5}$$

where C_1 is a constant of integration to be determined later from the boundary conditions, and

$$\chi_{a1}(r) = \int_a^r \varrho^2\,\Phi\,d\varrho; \tag{6}$$

a is any convenient lower limit for the integral, such as the inner radius of a hollow sphere.

Clearly now

$$\varepsilon_r = -\frac{2}{r^3}\left[3\,\chi_{a1}(r) - \frac{3}{2}\,\Phi\,r^3 + C_1\right], \quad \varepsilon_\varphi = -\frac{1}{r^3}\,[3\,\chi_{a1}(r) + C_1]\,, \tag{7}$$

and the second equation (1) gives

$$\sigma_r - \sigma_\varphi = -\frac{2\,E(T)}{r^3}\,[3\,\chi_{a1}(r) - r^3\,\Phi + C_1]\,. \tag{8}$$

In addition to the foregoing equation, in the absence of body forces, we have the following well-known stress equation of equilibrium

$$\frac{d\sigma_r}{dr} + \frac{2}{r}\,(\sigma_r - \sigma_\varphi) = 0\,. \tag{9}$$

By virtue of these two equations a trivial integration gives finally the radial and hoop stresses

$$\sigma_r = 4\,[3\,\Psi_a(r) - \psi_a(r) + C_1\,\chi_{a2}(r)] + C_2\,, \tag{10}$$

$$\left.\begin{aligned}\sigma_\varphi = 6\left[2\,\Psi_a(r) + \frac{E(T)}{r^3}\,\chi_{a1}(r)\right] - 2\,[2\,\psi_a(r) + E(T)\,\Phi]\\ + 2\left[2\,\chi_{a2}(r) + \frac{E(T)}{r^3}\right] C_1 + C_2\,,\end{aligned}\right\} \tag{11}$$

with the notation

$$\left.\begin{aligned} \Psi_a(r) &= \int\limits_a^r \frac{E(T)}{\sigma^4} \int\limits_a^\sigma \varrho^2 \Phi \, d\varrho \, d\sigma \,, \quad \psi_a(r) = \int\limits_a^r E(T) \frac{\Phi}{\varrho} \, d\varrho \,, \\ \chi_{a2}(r) &= \int\limits_a^r \frac{E(T)}{\varrho^4} \, d\varrho ; \end{aligned}\right\} \tag{12}$$

and C_2 as constant of integration.

Thus we have obtained, as was announced earlier, a solution in a closed form and valid for any symmetrical temperature distribution as well as for any variation of $T(r)$ and $\alpha(r)$.

Let us now investigate two characteristic cases.

Solid Sphere

In this particular case the lower limit a of the integrals may be put equal to zero.

For $r = 0$, because of symmetry, wo ought to have $u = 0$. This yields $C_1 = 0$ in equation (5) since by virtue of the theorem of the mean value of an integral,

$$\lim_{r \to 0} \frac{1}{r^2} \chi_{01}(r) = 0 \,, \tag{13}$$

with 0 substituted for the subscript a in χ_{a1}. A similar notation will be subsequently used, for $a = 0$, in the symbols Ψ_a, ψ_a and χ_{a2}.

We proceed now to show that σ_r and σ_φ also remain bounded, for $a \to 0$ and $r \to 0$, if we take $C_1 = 0$. In fact, since, for $a = 0$, the integrands in $\Psi_a(r)$ and $\chi_{a2}(r)$ increase infinitely by approaching the lower limit of the interval of integration, we have to compute the difference of the improper integrals $\Psi_a(r)$ and $\psi_a(r)$,

$$D_{ar} = 3\, \Psi_a(r) - \psi_a(r) \tag{14}$$

assuming $a \neq 0$, and then pass to the limit by allowing $a \to 0$. This yields

$$\lim_{\substack{a \to 0 \\ r \to 0}} D_{ar} = -\frac{1}{3} E(0)\, \Phi(0) \,. \tag{15}$$

Now, in order that χ_{a2} in equation (10) and the last bracketed term in (11) remain bounded as $a \to 0$ and $r \to 0$, it suffices to put $C_1 = 0$, as was asserted.

Thus, finally, we obtain for the stress at the center of a solid sphere the bounded expression

$$\sigma_r(0) = \sigma_\varphi(0) = -\frac{4}{3} E(0)\, \Phi(0) + C_2 \,. \tag{16}$$

Let us write explicitly the difference $\sigma_\varphi - \sigma_r$, for $C_1 = 0$ and $a = 0$, from (8) in the form

$$\sigma_\varphi - \sigma_r = 2\,E(T)\left(\Phi - \frac{\int\limits_0^r 4\,\pi\,\varrho^2\,\Phi\,d\varrho}{\frac{4}{3}\,\pi\,r^3}\right). \tag{17}$$

Clearly the second bracketed term in the foregoing equation represents the mean unit thermal expansion of the sphere with radius r. Hence, the following theorem holds: In a solid sphere with temperature dependent properties subjected to radial temperature gradient, the difference between the local hoop and radial stresses at a point r is in direct proportion to the difference between the local value of the thermal expansion and its mean value within the sphere with radius r. The factor of proportionality is double the local value of YOUNG's modulus.

Since half a difference between each two principal normal stresses at a point designates a principal shear stress at the point, the foregoing theorem provides us with direct data for the Tresca-Saint Venant or Huber-Mises-Hencky criterion of plastic yielding at the point. Furthermore, the factor of proportionality being equal to the modulus of elasticity, we can at once perceive in what degree the temperature dependent properties of a material influence its tendency to plastic yielding. It appears, at this early stage of our investigation, that the assumption of a temperature-independent elastic modulus and the use of its value corresponding to the ambient conditions lead to an overestimation of the intensity of stress, provoking plastic yielding at elevated temperatures.

The constant of integration C_2 can be now computed from the boundary conditions at the surface of the sphere $r = b$. For a tractionfree surface, for instance, we get

$$\left.\begin{aligned} \sigma_r &= 12\,[\Psi_0(r) - \Psi_0(b)] - 4\,[\psi_0(r) - \psi_0(b)]\,, \\ \sigma_\varphi &= 12\,[\Psi_0(r) - \Psi_0(b)] - 4\,[\psi_0(r) - \psi_0(b)] + 2\left[\frac{3\,E(T)}{r^3}\,\chi_{01}(r) - E(T)\,\Phi\right]. \end{aligned}\right\} \tag{18}$$

Sphere with a Central Hole

Let us denote, in this particular case, the inner and outer radii of the hollow sphere by a and b, respectively. Suppose, furthermore, that in addition to a steady-state radial temperature gradient, the spherical container is submitted to an action of an internal uniform pressure p, and embedded in an elastic medium of Winkler type characterized by the modulus $\varkappa$. These assumptions clearly yield,

$$\sigma_r(a) = -p\,, \quad \sigma_r(b) = -\varkappa\,u(b)\,. \tag{19}$$

From (5) and (10) a trivial computation yields the following values of the constant of integration:

$$C_1 = -\frac{3\,\Psi_{ab} - \psi_{ab} + \dfrac{3\,\varkappa}{4\,b^2}\,\chi_{a1b} - \dfrac{p}{4}}{\chi_{a2b} + \dfrac{\varkappa}{4\,b^2}}\,, \qquad C_2 = -p\,, \tag{20}$$

an additional subscript in the notation of the functions (6) and (12) being required to indicate their value on the outer surface $r = b$.

For $\varkappa = 0$, one deals with an instance of a sphere free from external load on the outer surface.

For $\varkappa = \infty$, an instance of a container embedded in an absolutely rigid medium is obtained. In this particular case

$$C_1 = -3\,\chi_{a1b}\,, \tag{21}$$

and $u(b) = 0$.

The elastic medium which, in a general case, surrounds the container may be, of course, represented by a second hollow sphere, constituting an outer layer of a composite double layer sphere. It is easy then to compute the respective modulus of the medium $\varkappa$.

To this purpose, let us assume that the medium is also temperature sensitive, and denote all quantities concerning the medium by an asterisk. We can adapt all the foregoing equations by substituting b for a, for the inner radius of the outer sphere, and c for b, for the outer radius of that sphere. Moreover, we have to take $\varkappa = 0$ and denote inner pressure p by $\varkappa$. This procedure leads to the following equations:

$$\left.\begin{aligned} u^* &= \frac{3}{r^2}\int_b^r \varrho^2\,\Phi^*\,d\varrho + \frac{C_1^*}{r^2}\,,\\ \sigma_r^* &= 12\int_b^r \frac{E^*(T)}{\sigma^4}\int_b^\sigma \varrho^2\,\Phi^*\,d\varrho\,d\sigma - 4\int_b^r E^*(T)\,\frac{\Phi^*}{\varrho}\,d\varrho\\ &\qquad + 4\,C_1^*\int_b^r \frac{E^*(T)}{\varrho^4}\,d\varrho + C_2^*\,, \end{aligned}\right\} \tag{22}$$

with $\Phi^* = \int_0^{T(r)} \alpha^*(\tau)\,d\tau$.

Using notation similar to that used previously one gets from (20) the constants of integration,

$$C_1^* = -\frac{3\,\Psi_{bc}^* - \psi_{bc}^* - \dfrac{\varkappa}{4}}{4\,\chi_{b2c}^*}\,, \quad C_2^* = -\varkappa\,. \tag{23}$$

Now the condition $u^*(b) = 1$, which expresses the true nature of the modulus $\varkappa$, as being a pressure which provokes a unit radial displacement on the inner surface $r = b$ of the outer sphere, yields

$$\varkappa = 4\,(3\,\Psi^*_{bc} - \psi^*_{bc} + \chi^*_{b2c}\,b^2)\,. \tag{24}$$

If the improper integrals in the last equation converge for $c \to \infty$, then the limit of the right-hand member in (24) will represent a unit response of a temperature sensitive infinite elastic space produced by an expansion of the (inner) sphere. It follows that we shall be concerned with a thermo-elastic problem for a temperature sensitive container embedded in an ideal temperature sensitive elastic medium.

Let us now take the situation as it often occurs in actuality, namely that of a medium at zero excess temperature, $T^* = 0$. Then $\Phi^* = 0$, $E^*(0) = E^*_0$, and one obtains simply

$$\varkappa = \frac{4\,E^*_0}{3\,b}\,, \tag{25}$$

which is a known result [6].

Temperature Field

Let us suppose that the thermal conductivity of a material k has a marked dependence on local temperature. Then the general equation of the temperature field, for a steady-state conduction in a system free of sources and sinks, becomes

$$k\,\nabla^2 T + \nabla k \cdot \nabla T = 0\,,$$

with ∇ and ∇^2 denoting a gradient and a Laplace operator, respectively, and the dot denoting the scalar product.

In the polarly symmetrical case we get in polar coordinates simply

$$\frac{d}{dr}\left(k\,r^2\,\frac{dT}{dr}\right) = 0\,, \tag{27}$$

or after integration

$$\int k\,dT = -\frac{K_1}{r} + K_2 \tag{28}$$

with K_1 and K_2 as constants of integration.

For most practical situations the nonuniformity of k can be expressed without any substantial error by a linear function

$$k = k_0\,(1 - \beta\,T)\,, \tag{29}$$

where k_0 is the known conductivity at the ambient temperature and β is a coefficient, in general, positive for homogeneous metals.

To be specific, let us assume the isothermal boundary conditions at the surfaces of the container discussed,

$$r = a\,, \quad T = T_a\,, \quad r = b\,, \quad T = T_b\,. \tag{30}$$

Then integration of (28) and solution of the quadratic equation in T yields, by virtue of (30), the following distribution of temperature:

$$T = \frac{1}{\beta} - \sqrt{\frac{1}{b-a}\left[\left(\frac{1}{\beta} - T_b\right)^2 b\left(1 - \frac{a}{r}\right) - \left(\frac{1}{\beta} - T_a\right)^2 a\left(1 - \frac{b}{r}\right)\right]}\,. \tag{31}$$

Following the procedure of [3] it is easy to show that using Taylor's formula for the right-hand member of equation (31) and passing to the limit, with $\beta \to 0$, yields the well-known temperature distribution for uniform conductivity:

$$T = \frac{1}{b-a}\left[T_b\, b\left(1 - \frac{a}{r}\right) - T_a\, a\left(1 - \frac{b}{r}\right)\right]. \tag{32}$$

It will be shown later that for moderate temperature gradients, such as assumed in the following numerical example involving prescribed steady temperature values on the surfaces of the container, no significant difference exists between the temperature fields for temperature dependent and independent conductivities[5]). In view of this, the simple temperature distribution (32) instead of the complex one (31) can sometimes be adopted.

Numerical Example

Let us consider a hollow sphere under steady-state radial temperature gradient and no surface tractions. Using our previous notation we assume that $a = 50$ [cm], $b = 100$ [cm], $T_a = 500$ [°C], $T_b = 400$ [°C]. For a steel container we take

$$\left.\begin{aligned} \alpha &= (1200 + T\,°\mathrm{C})\,10^{-8} \quad [°\mathrm{C}^{-1}]\,,\\ E &= [1758{\cdot}2 - 3{\cdot}573\,(T\,°\mathrm{C} - 400)] \quad [\mathrm{kg\,cm^{-2}}]\,. \end{aligned}\right\} \tag{33}$$

The last expression has been computed for the interval $750 \leqq T\,°\mathrm{F} \leqq 1000$, which includes the interval $400 \leqq T\,°\mathrm{C} \leqq 500$, from the experimental data given in [5] for the S.A.E. 1095 steel.

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[5]) It was shown, however, in [3] for a circular tube, that this is not always the case if instead of the prescribed boundary temperatures a heat exchange between a surface of the body and the surrounding medium is assumed.

Assume in (29) that $k_0 = 0{\cdot}11$ [cal cm^{-1} s^{-1} °C^{-1}] and $\beta = 0{\cdot}0007$ [°C^{-1}]. Then equations (32) and (31) yield the following temperature fields:

$$\left.\begin{aligned} T_1 &= 300 + \frac{10\,000}{r}, \\ T_2 &= 1428{\cdot}57 - \sqrt{2{,}115{,}912\left(1 - \frac{50}{r}\right) - 862{,}242\left(1 - \frac{100}{r}\right)}, \end{aligned}\right\} \quad (34)$$

which for specific values of $r = 60$ and 80 become

r	T_1	T_2	Difference per cent
60	466·67	465·51	1·16
80	425·00	424·07	0·93

In view of an insignificant difference between the respective values of temperature just obtained we shall use the simpler equation (32). By substituting its right-hand member in the second relation (33) we get finally YOUNG's modulus in terms of the position vector,

$$E = \left(2{\cdot}116 - \frac{35{\cdot}73}{r}\right) 10^6 .$$

Consequently, we compute Φ from (4) and then the stresses from (10) and (11). They are shown in Figures 1 and 2 being denoted by (temp. dep.). Along with

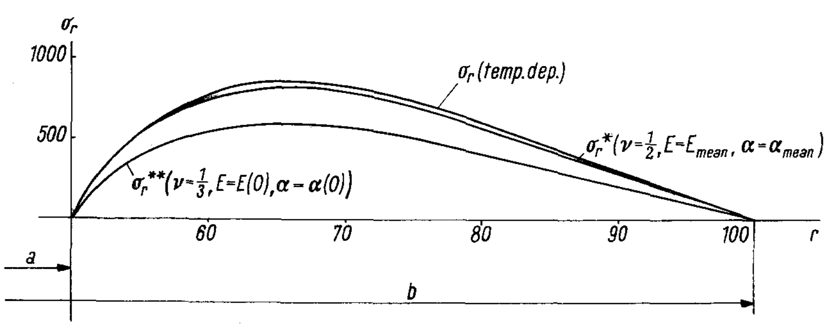

Figure 1
The radial stress.

the curves related to the latter stresses two other pairs of curves are plotted in the figures marked by one or two asterisks. Those marked with one asterisk correspond to the well-known formulae for a temperature insensitive material, given, for instance, in [7]. There have been computed for POISSON's ratio equal to 0·5 and for the values of α and E from (33) corresponding to the arithmetic mean of the boundary temperatures, that is to 450°C. It may be seen that, at least in the particular case considered, the difference between the corresponding stress fields is meaningless as far as the accuracy usually required is concerned.

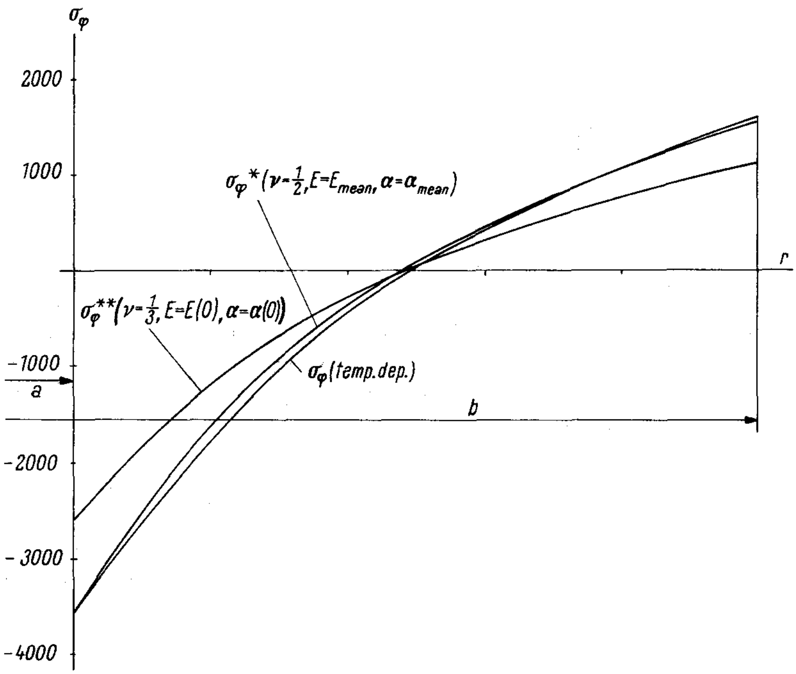

Figure 2
The hoop stress.

Hence, in this instance, we could calculate with orthodox equations for temperature independent material using mean values of α and E. For the sake of comparison, in Figures 1 and 2 the stress field for temperature independent material has also been shown assuming $\nu = 1/3$ and the values of E and α corresponding to the ambient conditions. Clearly, in this case the actual influence of the temperature on the intensity of the stress field would be considerably underestimated. Thus, for instance, the maximum compressive hoop stress on the inner surface of the sphere would be almost 27% less than that computed from the more accurate equation.

It should be noted that in the foregoing numerical example the temperature field is of such an intensity that it induces stress locally exceeding the proportional limit of the material. Consequently, the numerical values of stress in these regions may serve only as an illustration of the influence of the temperature sensitive properties of the material on the magnitude of thermal stress.

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Zusammenfassung

Es werden Wärmespannungen infolge polarsymmetrischer stationärer Temperaturfelder in einer isotropen Kugel mit temperaturabhängigen Stoffwerten untersucht. Das Material wird als inkompressibel vorausgesetzt, was einem oberen Grenzwert der viskoelastischen Spannungen in einem Maxwellschen Körper entspricht. Diese Annahme erlaubt, die Lösung in einer geschlossenen Form und für jede Temperaturverteilung und Temperaturabhängigkeit der Materialkonstanten und des Wärmeausdehnungskoeffizienten aufzustellen. Es werden allgemeine Formeln für die Spannungs- und Verschiebungsfelder in einer vollen Kugel und in einer kugelförmigen Schale gegeben und die Einwirkung der variablen Wärmeleitzahl auf das Temperaturfeld untersucht. Ein numerisches Beispiel für lineare Temperaturabhängigkeit des Elastizitätsmoduls und des Wärmeausdehnungskoeffizienten wird berechnet.

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