Minimum Weight Design of Circular Plates Under Arbitrary Loading[1])

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To Professor F. K. G. ODQVIST *on the occasion of his 60th anniversary*

1. Introduction

The direct design procedures developed by DRUCKER and SHIELD [1, 2][4]) have been used recently [3] to obtain the minimum weight design of circular plates under rotationally symmetric pressure distributions. The sandwich plate and the solid or homogeneous plate were considered and designs for both simply supported and built-in edge conditions were derived. As an extension of this work, the minimum weight design is obtained in the following for a circular sandwich plate loaded by an *arbitrary* distribution of pressure. The deformation modes used to obtain the designs are the same as those used in [3] for rotationally symmetric pressure distributions.

2. Definition of the Problem

The ideal sandwich plate is composed of a core of constant thickness H between two identical face sheets of variable thickness h, where $h \ll H$. The core carries shear stress only while the face sheets carry direct stresses and so provide the bending moment across a section. The material of the face sheets is assumed to be elastic-perfectly plastic and to obey TRESCA's yield condition, with yield stress σ_0 in tension or compression. It is convenient to represent the principal bending moments M_1, M_2 at a generic point of the plate by a point with rectangular coordinates (M_1, M_2) (see Figure 1). With TRESCA's yield criterion, the stress point must then be within or on the hexagon in the figure [4], the maximum bending moment M_0 being given by

$$M_0 = \sigma_0 H h . \tag{1}$$

For a point on the hexagon, plastic bending of the plate can occur. The principal curvature rates $\varkappa_1$, $\varkappa_2$ associated with a mode of plastic deformation can be represented in Figure 1 by a vector with components proportional to $\varkappa_1$, $\varkappa_2$ attached to the stress point in question. The flow rule requires the curvature rate vector to be normal to the hexagon for points on a side, and at the corners of the hexagon the vector must lie in the fan bounded by normals to adjacent sides. The rate of dissipation of energy D_A per unit area of the middle surface due to plastic

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4) Numbers in brackets refer to References, page 425.

action is given by

$$D_A = M_1 \varkappa_1 + M_2 \varkappa_2 \tag{2}$$

and is uniquely determined by the curvature rates $\varkappa_1$, $\varkappa_2$.

It is required to determine the thickness h of the face sheets so that the plate can just carry a given distribution of pressure and so that minimum weight of material is involved. The material is assumed to be homogeneous and minimum weight coincides with minimum volume. The minimum weight design is achieved

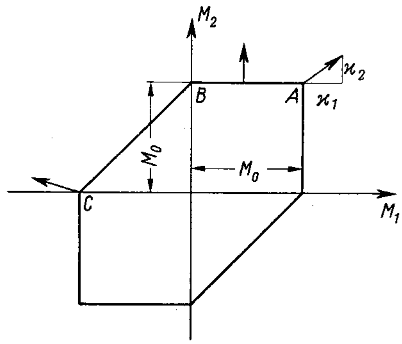

Figure 1
Yield condition.

by a plate designed to collapse in a mode for which the condition

$$\frac{D_A}{h} = \text{const} \tag{3}$$

is satisfied over the plate [1, 2], if body forces are neglected. Condition (3) is a condition on the rate of deflection w of the middle surface of the plate and the thickness h does not enter into condition (3) because of the linear dependence of D_A on h. To this extent, the condition is independent of the pressure distribution over the plate but the form of the condition depends upon the position of the stress point on the hexagon, which in turn is influenced by the loading.

For a circular plate with conditions of simple support or built-in support at the edge of the plate, the deflection mode w satisfying (3) is rotationally symmetric. With polar coordinates (r, θ), the edge of the plate being $r = R$, the curvature rate $\varkappa_{r\theta}$ is zero and isotropy then requires the bending moment $M_{r\theta}$ to be zero. For equilibrium, the bending moments M_r, M_θ satisfy the equation

$$\frac{\partial^2}{\partial r^2}(r\, M_r) - \frac{\partial M_\theta}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 M_\theta}{\partial \theta^2} = -r\, p\,, \tag{4}$$

where $p = p(r, \theta)$ is the pressure over the plate. For definiteness, the plate is taken to be horizontal. Positive values of M_r, M_θ stress the lower surface of the plate in tension and the pressure is applied to the upper surface. The shear forces Q_r, Q_θ are given by

$$Q_r = \frac{\partial M_r}{\partial r} + \frac{1}{r}(M_r - M_\theta)\,, \quad Q_\theta = \frac{1}{r} \cdot \frac{\partial M_\theta}{\partial \theta}\,. \tag{5}$$

3. Simply Supported Plate

When the edge $r = R$ of the plate is simply supported, the minimum weight design is obtained by assuming that the whole of the plate is stressed to state A of Figure 1. The deflection rate w (measured in the downwards direction), associated with point A and satisfying (3), and the condition $w = 0$ at $r = R$, is given by [3]

$$w = \alpha \,(R^2 - r^2)\,, \tag{6}$$

where α is a positive constant. The bending moments M_r, M_θ are both equal to the maximum moment M_0 and substitution in the equation of equilibrium gives

$$\frac{\partial^2 M_0}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial M_0}{dr} + \frac{1}{r^2} \cdot \frac{\partial^2 M_0}{\partial \theta^2} = -p\,. \tag{7}$$

This equation and the condition $M_0 = 0$ at $r = R$ are sufficient to determine M_0 over the plate.

For a concentrated load P at the point (ϱ, φ), the solution of (7) satisfying $M_0 = 0$ at $r = R$ is

$$\sigma_0\, H\, h = M_0 = \frac{P}{2\,\pi} \ln\left[\frac{r_2}{r_1} \cdot \frac{\varrho}{R}\right], \tag{8}$$

where r_1, r_2 are the distances of the point (r, θ) from the point (ϱ, φ) and its inverse with respect to the circle $r = R$, that is the point $(R^2/\varrho, \varphi)$, respectively:

$$r_1^2 = \varrho^2 + r^2 - 2\,\varrho\, r \cos(\theta - \varphi)\,, \quad r_2^2 = \frac{R^4}{\varrho^2} + r^2 - 2\,\frac{R^2}{\varrho}\, r \cos(\theta - \varphi)\,. \tag{9}$$

The solution for a distributed load $p(r, \theta)$ can be obtained from the fundamental integral (8) by integration:

$$\sigma_0\, H\, h = \frac{1}{4\,\pi} \int_0^{2\pi} \int_0^R p(\varrho, \varphi) \ln\left\{\frac{R^2 + r^2\, \varrho^2/R^2 - 2\,\varrho\, r \cos(\theta - \varphi)}{\varrho^2 + r^2 - 2\,\varrho\, r \cos(\theta - \varphi)}\right\} \varrho\, d\varrho\, d\varphi\,. \tag{10}$$

4. Built-in Plate

For the plate with a built-in edge, stress point C of Figure 1 applies near the edge of the plate and stress point A applies in the center of the plate. The velocity field

$$w = \alpha \left(\frac{3}{2}\, a^2 - r^2\right) \quad (0 \leqq r \leqq a)\,, \quad w = 2\,\alpha\,(R - r)^2 \quad (a \leqq r \leqq R)\,, \tag{11}$$

where $a = 2/3\, R$, satisfies the condition (3) over the plate, together with the conditions $w = \partial w/\partial r = 0$ at the clamped edge $r = R$ and w and $\partial w/\partial r$ continuous at $r = a$ [3]. The deformation mode (11) is associated with stress point A for $0 \leqq r \leqq a$ and stress point C for $a \leqq r \leqq R$. The moments M_r, M_θ are therefore given by

$$M_r = M_\theta = M_0 \quad (0 \leqq r < a)\,, \quad M_r = -M_0\,, \quad M_\theta = 0 \quad (a < r \leqq R)\,. \tag{12}$$

For equilibrium, it follows from equations (4) and (12) that M_0 must satisfy equation (7) for $0 \leqq r < a$, and for $a < r \leqq R$

$$\frac{\partial^2}{\partial r^2}\,(r\, M_0) = r\, p\,. \tag{13}$$

At the junction $r = a$, M_r is continuous so that $M_0 = 0$ at $r = a$. In addition, the shear force Q_r must be continuous at $r = a$ (in the absence of concentrated load) and this requires $|\partial M_0/\partial r|$ to be continuous. Equation (7) and the condition $M_0 = 0$ at $r = a$ are sufficient to determine M_0 for $0 \leqq r < a$. Equation (13) and the condition $M_0 = 0$ at $r = a$ together with the known derivative $(\partial M_0/\partial r)_{r=a}$ are then sufficient to determine M_0 for $a < r < R$.

We first consider a concentrated load P at a point (ϱ, φ) within the circle $r = a$, so that $\varrho < a$. As in the previous section, the solution of equation (7) with $p = 0$ and having the appropriate singularity at (ϱ, φ) and $M_0 = 0$ at $r = a$ is

$$\sigma_0 H h = M_0 = \frac{P}{2\pi} \ln\left[\frac{r_2}{r_1} \cdot \frac{\varrho}{a}\right] \quad (r \leqq a), \tag{14}$$

where now

$$r_1^2 = \varrho^2 + r^2 - 2 \varrho r \cos(\theta - \varphi), \quad r_2^2 = \frac{a^4}{\varrho^2} + r^2 - 2 \frac{a^2}{\varrho} r \cos(\theta - \varphi). \tag{15}$$

For $r \geqq a$, M_0 satisfies equation (13) with $p = 0$ and as $M_0 = 0$ at $r = a$, we obtain

$$M_0 = f(\theta)\left(1 - \frac{a}{r}\right) \quad (a \leqq r \leqq R).$$

The positive function $f(\theta)$ entering into this equation is determined by the condition that the magnitude of the derivative $\partial M_0/\partial r$ is continuous at $r = a$. This requires

$$\sigma_0 H h = M_0 = \frac{P}{2\pi} \cdot \frac{a^2 - \varrho^2}{a^2 + \varrho^2 - 2 \varrho a \cos(\theta - \varphi)} \left(1 - \frac{a}{r}\right) \quad (a \leqq r \leqq R). \tag{16}$$

The design for a load $p(r, \theta)$ distributed inside the circle $r = a$ and zero load over the annulus $a \leqq r \leqq R$ can be obtained from the design (14), (16) by integration:

$$\sigma_0 H h = \frac{1}{4\pi} \int_0^{2\pi} \int_0^a p(\varrho, \varphi) \ln\left\{\frac{a^2 + r^2 \varrho^2/a^2 - 2 \varrho r \cos(\theta - \varphi)}{\varrho^2 + r^2 - 2 \varrho r \cos(\theta - \varphi)}\right\} \varrho \, d\varrho \, d\varphi \tag{17}$$

for $r \leqq a$, and for $a \leqq r \leqq R$

$$\sigma_0 h H = \frac{1}{2\pi} \left(1 - \frac{a}{r}\right) \int_0^{2\pi} \int_0^a p(\varrho, \varphi) \frac{a^2 - \varrho^2}{a^2 + \varrho^2 - 2 \varrho a \cos(\theta - \varphi)} \varrho \, d\varrho \, d\varphi. \tag{18}$$

When there is no load inside the circle $r = a$, M_0 is harmonic for $r \leqq a$ and since $M_0 = 0$ at $r = a$, it follows that M_0 is zero identically inside $r = a$. We assume that the plate is loaded in the annular region by a line load $\tilde{\omega}$ per unit length along the arc element $\varphi \leqq \theta \leqq \varphi + \Delta\theta$ of the circle $r = \varrho$, where $\varrho > a$ (Figure 2). The total load on the plate is then $P = \tilde{\omega} \varrho \Delta\theta$. Because $p = 0$ in equation (13) except for $r = \varrho$, $\varphi \leqq \theta \leqq \varphi + \Delta\theta$, and also $M_0 = \partial M_0/\partial r = 0$ at $r = a$, it follows from equation (13) that M_0 is zero except in the truncated sector $\varrho \leqq r \leqq R$, $\varphi \leqq \theta \leqq \varphi + \Delta\theta$, shown shaded in Figure 2. Thus the load is carried by a 'cantilever' from the point on the support circle $r = R$ nearest to the region of application of the load. In order to determine M_0 in the shaded region, we note that M_0 is zero at $r = \varrho$ and the shear force Q_r, and therefore $\partial M_0/\partial r$, has a discontinuity of amount $\tilde{\omega}$ across the loaded element of arc. The result is

$$\sigma_0 H h = M_0 = \varrho \tilde{\omega} \left(1 - \frac{\varrho}{r}\right) \tag{19}$$

in the region $\varrho \leqq r \leqq R$, $\varphi \leqq \theta \leqq \varphi + \Delta\theta$.

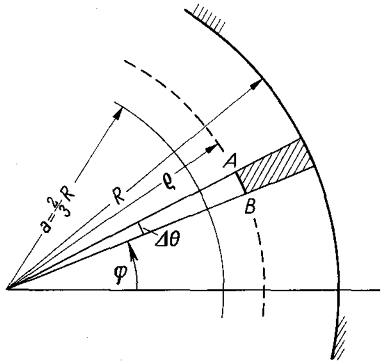

Figure 2
Line load on arc element AB.

Integration of the design (19) leads to the design for a distributed load $p(r, \theta)$ which is zero inside the circle $r = a$:

$$\sigma_0 H h = \int_a^r p(\varrho, \theta) \varrho \left(1 - \frac{\varrho}{r}\right) d\varrho \tag{20}$$

for $a \leqq r \leqq R$ and the thickness h is zero for $r \leqq a$. The upper limit of integration in equation (20) is r because the pressure $p(\varrho, \theta)$ at the point (ϱ, θ) adds material only to points (r, θ) for which $\varrho \leqq r$.

When the pressure distribution $p(r, \theta)$ is non-zero in both $r \leqq a$ and $a \leqq r \leqq R$, the thickness distributions given by equations (17), (18) and equation (20) are added.

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Zusammenfassung

Das Dimensionierungsverfahren von DRUCKER und SHIELD [1, 2] wurde kürzlich auf die Dimensionierung rotationssymmetrisch belasteter Kreisplatten für Mindestgewicht angewandt [3], wobei sowohl gelenkig gestützte als auch eingespannte Sandwichplatten betrachtet wurden. In der vorliegenden Arbeit werden diese Untersuchungen auf kreisförmige Sandwichplatten unter nicht rotationssymmetrischer Belastung ausgedehnt. Es zeigt sich, dass auch für solche Lasten der Dimensionierung für Mindestgewicht diejenigen Verformungszustände zugrunde gelegt werden können, welche schon zur Lösung der entsprechenden Aufgabe bei rotationssymmetrischer Belastung konstruiert wurden. Für Belastung einer gelenkig gestützten oder eingespannten Kreisplatte durch eine Einzellast mit beliebigem Angriffspunkt werden explizite Dimensionierungsformeln gewonnen, für beliebig verteilte Belastung werden Integraldarstellungen der optimalen Dimensionierung gegeben.

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A Note on Addition Theorems for Mathieu Functions

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It is the purpose of this note to point out the existence of an addition theorem for Mathieu functions besides the one given by MEIXNER and SCHÄFKE [1–3][2]). The latter states – in the notation of [1] – the following. We consider two elliptical cylinder coordinate systems A and B:

$$A: \; x_1 \pm i\, x_2 = c \cosh (z \pm i\, t);$$

$$B: \; x_1' \pm i\, x_2' = c_0 \cosh (z_0 \pm i\, t_0),$$

connected by

$$c \cosh (z \pm i\, t) = l\, e^{\pm i u} + c_0\, e^{\pm i v} \cosh (z_0 \pm i\, t_0)\,.$$

(We consider only real, positive values of c, c_0, z, z_0, t and t_0; see figure.) Let z_1 and z_2 be the elliptical coordinates – in the A-system – of the focal points of the B-system and z_{max} the largest one of z_1, z_2, then

$$\left. \begin{aligned} M_p^{(j)}(z_0;\, h_0)\, me_p(t_0;\, h_0^2) &= \sum_{s=\div\infty}^{\infty} A_s\, M_{p+s}^{(j)}(z;\, h)\, me_{p+s}(t;\, h^2) \\ &\qquad (j = 1,\, 2,\, 3 \text{ or } 4;\; p = \text{integer} \gtreqless 0) \end{aligned} \right\} \quad (1)$$

will be valid in the region

$$z > z_{max} \quad (0 \leqq t \leqq 2\,\pi)\,.$$

In (1) $M_p^{(j)}(z;\, h)\, me_p(t;\, h^2)$ and $M_p^{(j)}(z_0;\, h_0)\, me_p(t_0;\, h_0^2)$ are solutions of the wave-

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[2]) Numbers in brackets refer to References, page 428.