Duct Flow in Magnetohydrodynamics

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Introduction

This paper is an extension of the work of Hartmann [2][2]) and Shercliff [3, 4] on the steady flow of conducting fluids through ducts under transverse magnetic fields – the simplest class of magnetohydrodynamic problems. We are concerned here mainly with the boundary value problems associated with flow in ducts with conducting walls.

Equations and Boundary Conditions

The following set of vector equations appears to give an adequate description of the steady state interaction between electromagnetic and hydrodynamic forces:

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \boldsymbol{j} , \tag{1}$$

$$\operatorname{div} \boldsymbol{B} = 0 , \tag{2}$$

$$\operatorname{curl} \boldsymbol{E} = 0 , \tag{3}$$

$$\operatorname{div} \boldsymbol{E} = \varrho_e/\varepsilon_0 , \tag{4}$$

$$\boldsymbol{j} = \sigma (\boldsymbol{B} + \boldsymbol{V} \times \boldsymbol{B}) + \varrho_e \boldsymbol{V} , \tag{5}$$

$$\varrho \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\nabla p + \varrho \nu \nabla^2 \boldsymbol{V} + \boldsymbol{j} \times \boldsymbol{B} + \varrho_e E \tag{6}$$

$$\operatorname{div} \boldsymbol{V} = 0 . \tag{7}$$

It is assumed that the magnetic and dielectric properties of the medium are the same as in a vacuum; μ_0 and ε_0 are the magnetic permeability and dielectric constant in vacuum. In (5), Ohm's law, the electrical conductivity σ is assumed constant for a homogeneous medium. Equations (6) and (7) are the momentum and continuity equations which describe the steady motion of an incompressible fluid. If the system to be analyzed is composed partly of fluid and partly of solid or vacuum, the last two equations, (6) and (7), only have to be satisfied in the fluid, while the first five equations must be satisfied throughout all space. The description of the system is completed by specifying zero velocity at rigid boundaries and imposing continuity of tangential components of $\boldsymbol{E}$ and of

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[2]) Numbers in brackets refer to References, page 114.

normal and tangential components of $\boldsymbol{B}$ at interfaces, plus boundary conditions on $\boldsymbol{B}$ and $\boldsymbol{E}$ at infinity. It should be mentioned that continuity of tangential components of $\boldsymbol{B}$ implies no surface currents. This is the case, since in non-magnetic materials surface currents occur only in the presence of *unsteady* magnetic fields.

In equations (5) and (6) the free charge ϱ_e will be neglected. It will be retained in (4), which serves only to determine the free charge once $\boldsymbol{E}$ is known. This can be considered as the first step in an iterative process, the second step being to substitute the value of ϱ_e calculated in the first step into (5) and (6).

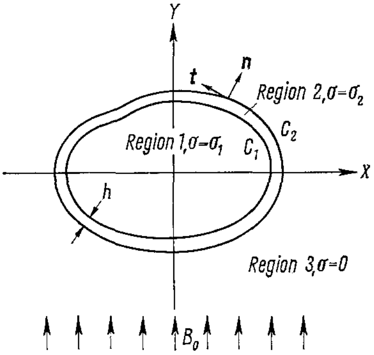

Figure 1
Cross-section of duct.

Figure 1 is a sketch of the system under study. An electrically conducting incompressible fluid (region 1) flows in the z-direction through a straight duct whose walls are of constant thickness (region 2). The electrical conductivity of the wall is σ_2. Outside of the duct (region 3) the conductivity is zero, and at infinity a uniform magnetic field B_0 acts in the y-direction.

If it is assumed that the velocity has only a z-component, that all physical quantities (except pressure) are independent of z, and that there is no net flow of current in the z-direction, then it can be shown that $B_x = 0$, $B_y = B_0$, and $j_z = E_z = 0$.

Using the results of the previous paragraph, the vector equations from (1) to (7) can be reduced to two second order linear partial differential equations, namely, the z-component of the momentum equation

$$\frac{\partial p}{\partial z} = \varrho\,\nu\,\nabla^2 V_z + \frac{B_0}{\mu_0}\,\frac{\partial B_z}{dy} \tag{8}$$

and the z-component of the curl of OHM's law,

$$\nabla^2 B_z + \sigma\,\mu_0\,B_0\,\frac{\partial V_z}{\partial y} = 0\,. \tag{9}$$

In the first of these $\partial p/\partial z$ is a constant, since from (6) ∇p is independent of z. In the fluid, region 1, both (8) and (9) are valid, while outside of the fluid, regions 2 and 3, only (9) is true with $V_z = 0$. In addition, since region 3 is non-conducting, the current must be zero there. This fact and (1) imply that B_z is constant. Therefore, since B_z goes to zero at infinity, B_z must be identically zero in region 3.

Across the boundaries of the several regions, B_z and the tangential component of electric field must be continuous. Using OHM's law, the condition on the electric field can be written

$$\left[\frac{j_t}{\sigma}\right] = 0$$

where j_t is the current in the direction of the tangent to the interface. (A square bracket around a quantity means the discontinuity in this quantity.) This is true since the velocity vanishes at the boundary, causing the $\boldsymbol{V} \times \boldsymbol{B}$ term in OHM's law to vanish. Observe now that the x- and y-components of (1) are

$$\mu_0\, j_x = \frac{\partial B_z}{\partial y}\,, \qquad \mu_0\, j_y = -\,\frac{\partial B_z}{\partial x}$$

so B_z can be considered as a 'stream function' for $\mu_0 \boldsymbol{j}$ – the current flows along the lines $B_z =$ constant. This shows that

$$\mu_0\, j_t = -\,\frac{\partial B_z}{\partial n}\,,$$

where $\boldsymbol{n}$ is the outward normal. Using this result, continuity of E_t becomes

$$\left[\sigma^{-1}\,\frac{\partial B_z}{\partial n}\right] = 0\,.$$

Before summing up the results of the last few paragraphs, let new dimensionless variables be introduced by

$$\left.\begin{aligned}
\xi &= \frac{x}{a}\,,\\
\eta &= \frac{y}{a}\,,\\
V &= -\,\frac{V_z}{(a^2/\nu\,\varrho)\,(\partial p/\partial z)}\,,\\
B &= -\,\frac{B_z}{(a^2/\nu\,\varrho)\,(\partial p/\partial z)\,\mu_0\,(\nu\,\varrho\,\sigma)^{1/2}}\,,\\
M &= B_0\,a\left(\frac{\sigma}{\nu\,\varrho}\right)^{1/2},
\end{aligned}\right\} \tag{10}$$

where a is a characteristic dimension of the duct and M is the Hartmann number. With this new notation and using subscripts 1 and 2 to denote the region, the problem is to solve

$$\nabla^2 V_1 + M \frac{\partial B_1}{\partial \eta} = -1\,, \tag{11}$$

$$\nabla^2 B_1 + M \frac{\partial V_1}{\partial \eta} = 0\,, \tag{12}$$

in region 1, and

$$\nabla^2 B_2 = 0 \tag{13}$$

in region 2. The boundary conditions are: $B_2 = 0$ on C_2, the boundary between 2 and 3; and $V_1 = 0$, $B_1 = B_2$, $\sigma_2\ \partial B_1/\partial n = \sigma_1\ \partial B_2/\partial n$ on C_1, the boundary between 1 and 2.

This problem is difficult since it involves two domains and two sets of boundary conditions. In special cases it can be simplified. If the duct wall is a perfect insulator ($\sigma_2 = 0$), then $B_2 = 0$ so that it is only necessary to solve (11) and (12) with $V_1 = 0$ and $B_1 = 0$ on C_1. If the duct wall is a perfect conductor ($\sigma_2 = \infty$) the boundary conditions become: $V_1 = 0$ and $\partial B_1/\partial n = 0$ on C_1. There is another limiting case discovered by SHERCLIFF [4], for which the problem reduces to solving (11) and (12) in region 1 with boundary conditions given on C_1. Suppose the thickness of the duct wall (h/a in the new notation) is much smaller than unity. To a good approximation the harmonic function B_2 is locally linear, that is, it varies linearly across the duct wall. This can be seen clearly by considering the membrane analogy for solutions of LAPLACE's equation. Then $\partial B_2/\partial n = -\,a\, B_1/h$ on C_1. The boundary conditions become $V_1 = 0$, $\partial B_1/\partial n + B_1/\varphi = 0$, where $\varphi = \sigma_2\, h/\sigma_1\, a$. It should be noted that the cases $\sigma_2 = 0$ and $\sigma_2 = \infty$ are included in the last boundary condition with $\varphi = 0$ and $\varphi = \infty$ respectively.

Parallel Sided Duct

There is one situation in which the approximate boundary condition becomes exact. This is the case of flow in a rectangular duct when the walls parallel to the applied magnetic field are at infinity. In this problem the harmonic function B_2 is independent of ξ, and hence must be a linear function of η.

With the dimensionless variables defined in the previous section, the problem is to solve

$$\frac{d^2 V}{d\eta^2} + M \frac{dB}{d\eta} = -\,1\,, \qquad \frac{d^2 B}{d\eta^2} + M \frac{dV}{d\eta} = 0\,,$$

with boundary conditions $V = 0$, $dB/d\eta \pm B/\varphi = 0$ when $\eta = \pm\, 1$ respectively. Here, half the duct height has been taken as the characteristic length, a. The solution of the above equations is easily found to be

$$V = \frac{1}{M} \frac{\varphi + 1}{M\,\varphi + \tanh M} \left(1 - \frac{\cosh M\,\eta}{\cosh M}\right), \tag{14}$$

$$B = -\frac{\eta}{M} + \frac{1}{M} \frac{\varphi + 1}{M\,\varphi + \tanh M} \frac{\sinh M\,\eta}{\cosh M}. \tag{15}$$

HARTMANN [2] gave this solution for the case $\varphi = 0$. For given φ, the velocity, which is parabolic for $M = 0$, becomes flatter in the center as M increases. When M is very large it tends to be uniform except in a boundary layer of thickness the order of $1/M$. Asymptotically,

$$V \sim \frac{1}{M} \frac{\varphi + 1}{M\,\varphi + 1}. \tag{16}$$

This shows that $V \sim 1/M$ for non-conducting walls while $V \sim 1/M^2$ for perfectly conducting walls: increasing the wall conductivity decreases the velocity.

Another quantity of interest is $M\,dB/d\eta$, the ratio of Lorentz force to the magnitude of the pressure gradient. This is also proportional to the current density. By an easy calculation,

$$M \frac{dB}{d\eta} = -1 + M \frac{\varphi + 1}{M\,\varphi + \tanh M} \frac{\cosh M\,\eta}{\cosh M}.$$

This is plotted in Figure 2 for $M = 3$, $\varphi = 1$, $h = 0{\cdot}5\,a$, to give a typical example. The value of this quantity in the wall,

$$\frac{a}{h}\,\varphi \left(\frac{M - \tanh M}{M\,\varphi + \tanh M}\right),$$

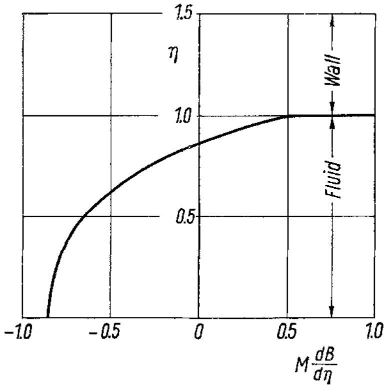

Figure 2

The current distribution across half of a parallel sided duct for $M = 3$, $\varphi = 1$, $h = 0{\cdot}5\,a$.

is also plotted. From this figure it is seen that near the center of the duct the current flows to the left. This gives a Lorentz force which tends to help the viscous forces balance the pressure gradient. Near the wall the current flows to the right giving a Lorentz force which opposes the viscous forces, so that the viscous forces have to be larger in order that the pressure gradient be balanced. For large M

$$M \frac{dB}{d\eta} \sim 1$$

except in a boundary layer of thickness the order of $1/M$, and in the wall. This shows that for large M, the Lorentz force completely balances the pressure gradient except in a thin boundary layer along the wall.

Stated differently, the current distribution tends to be uniform to the left and of a magnitude such as to make the Lorentz force balance the pressure gradient. Since the total current flow must be zero, part of the return current flows in the boundary layer and part in the wall. When the wall conductivity is larger a greater proportion of the return current flows in the wall, taking the path of least resistance. This shows that for fixed M (large), less current flows in the boundary layer when the wall conductivity is high, indicating that the Lorentz force opposing the viscous forces is less. Therefore the viscous forces must be smaller in order that the pressure gradient be balanced. Now if the viscous forces near the wall are smaller, the second derivatives of the velocity will be smaller; hence the velocity of the core will be smaller. This shows why the velocity becomes smaller when the wall conductivity increases.

Consider the dimensionless mass flow

$$Q = \int_{-1}^{1} V \, d\eta = \frac{\text{Mass flow per unit width}}{(a^3/\nu)\,(-\,\partial p/\partial z)}$$

$$= \frac{\varphi + 1}{M^2} \, \frac{M - \tanh M}{M\,\varphi + \tanh M} \, .$$

The reciprocal of Q is essentially the pressure gradient required to maintain a given mass flow. In Figure 3

$$\frac{Q(M=0)}{Q} = \frac{1}{3} \, \frac{M^2}{\varphi + 1} \, \frac{M\,\varphi + \tanh M}{M - \tanh M} \tag{17}$$

is plotted versus M for various values of φ. This can be interpreted as the ratio of the pressure gradient to the pressure gradient required to maintain a non-magnetic flow with the same mass flow. Notice that for a given mass flow a much larger pressure gradient is required to maintain flow through a perfectly conducting duct than through a non-conducting duct.

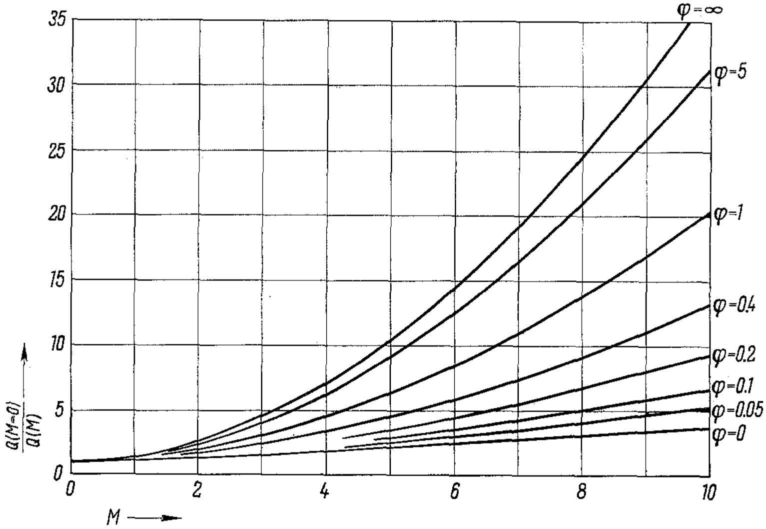

Figure 3
The pressure gradient versus M for various values of φ in a parallel sided duct.

Perfectly Conducting Rectangular Duct

In this section, the flow through a finite rectangular duct with perfectly conducting walls is considered. Regrettably, we have been unable to solve the problem for arbitrary wall conductivity.

The problem to be solved is

$$\nabla^2 V + M \frac{\partial B}{\partial \eta} = -1 , \qquad \nabla^2 B + M \frac{\partial V}{\partial \eta} = 0$$

in a rectangle of width $2\,l$ and height 2, with $V = 0$ on all the walls, $\partial B/\partial \xi = 0$ on the vertical walls, and $\partial B/\partial \eta = 0$ on the horizontal walls. The expansions

$$V = \sum_{j=0}^{\infty} v_j \cos\beta_j\, \eta , \qquad B = \sum_{j=0}^{\infty} b_j \sin\beta_j\, \eta , \qquad 1 = \sum_{j=1}^{\infty} a_j \cos\beta_j\, \eta ,$$

where $\beta_j = (j + 1/2)\, \pi$, satisfies the boundary conditions on the horizontal walls and allows the differential equations to be written as ordinary differential equations for v_j and b_j –

$$\frac{d^2 v_j}{d\xi^2} - \beta_j^2\, v_j + M\, \beta_j\, b_j = -\, a_j , \qquad \frac{d^2 b_j}{d\xi^2} - \beta_j^2\, b_j - M\, \beta_j\, v_j = 0 .$$

Solutions of these which satisfy the boundary conditions on the vertical walls are

$$v_j = \frac{a_j}{\beta_j^2 + M^2} \left(1 - \frac{r_{2j} \sinh r_{2j}\, l \cosh r_{1j} \sinh \xi + r_{1j} \sinh r_{1j}\, l \cosh r_{2j}\, \xi}{r_{2j} \cosh r_{1j}\, l \sinh r_{2j}\, l + r_{1j} \cosh r_{2j}\, l \sinh r_{1j}\, l}\right) \tag{18}$$

and

$$b_j = \frac{a_j}{\beta_j^2 + M^2} \left(\frac{M}{\beta_j} + i \, \frac{r_{2j} \sinh r_{2j}\, l \cosh r_{1j}\, \xi - r_{1j} \sinh r_{1j}\, l \cosh r_{2j}\, \xi}{r_{2j} \cosh r_{1j}\, l \sinh r_{2j}\, l + r_{1j} \cosh r_{2j}\, l \sinh r_{1j}\, l} \right) \tag{19}$$

where

$$r_{1j} = (\beta_j^2 + i\, M\, \beta_j)^{1/2}, \quad r_{2j} = (\beta_j^2 - i\, M\, \beta_j)^{1/2}$$

are complex. r_{1j} and r_{2j} may be separated into their real and imaginary parts, namely,

$$r_{1j} = \alpha_j + i\, \gamma_j$$

and

$$r_{2j} = \alpha_j - i\, \gamma_j\,,$$

where

$$\alpha_j = \left(\frac{\beta_j}{2}\right)^{1/2} [\beta_j + (\beta_j^2 + M^2)^{1/2}]^{1/2}, \quad \gamma_j = \left(\frac{\beta_j}{2}\right)^{1/2} [-\beta_j + (\beta_j + M^2)^{1/2}]^{1/2}.$$

After some algebra to express v_j and b_j in terms of the real quantities α_j and γ_j the final result becomes

$$V = \sum_{j=0}^{\infty} \frac{2\,(-1)^j}{\beta_j} \, \frac{\cos \beta_j\, \eta}{\beta_j^2 + M^2} \left\{ 1 - \frac{\alpha_j\, E_j(\xi) - \gamma_j\, F_j(\xi)}{\alpha_j \sinh 2\, \alpha_j\, l - \gamma_j \sin 2\, \gamma_j\, l} \right\}, \tag{20}$$

$$B = \sum_{j=0}^{\infty} \frac{2\,(-1)^j}{\beta_j} \, \frac{\sin \beta_j\, \eta}{\beta_j^2 + M^2} \left\{ \frac{M}{\beta_j} + \frac{\alpha_j\, F_j(\xi) + \gamma_j\, E_j(\xi)}{\alpha_j \sinh 2\, \alpha_j\, l - \gamma_j \sin 2\, \gamma_j\, l} \right\}, \tag{21}$$

where

$$E_j(\xi) = \frac{1}{4} \left[\cos\gamma_j\, (l - \xi) \sinh\alpha_j\, (l + \xi) + \cos\gamma_j\, (l + \xi) \sinh\alpha_j\, (l - \xi) \right],$$

$$F_j(\xi) = \frac{1}{4} \left[\sin\gamma_j\, (l - \xi) \cosh\alpha_j\, (l + \xi) + \sin\gamma_j\, (l + \xi) \cosh\alpha_j\, (l - \xi) \right].$$

The calculation of mass flow per unit pressure gradient is more easily accomplished by integrating the complex form of the velocity. The result is

$$Q = 8\, l \sum_{j=0}^{\infty} \frac{1}{\beta_j^2\, (\beta_j^2 + M^2)} \left\{ 1 - \frac{\beta_j}{l\, (\beta_j^2 + M^2)^{1/2}} \, \frac{\cosh 2\, \alpha_j\, l - \cos 2\, \alpha_j\, l}{\alpha_j \sinh 2\, \alpha_j\, l - \gamma_j \sin 2\, \gamma_j\, l} \right\}. \tag{22}$$

$Q(M = 0)/Q(M)$ is plotted versus M in Figure 4 for a square duct. The corresponding result for non-conducting walls, Shercliff [3], is included for comparison.

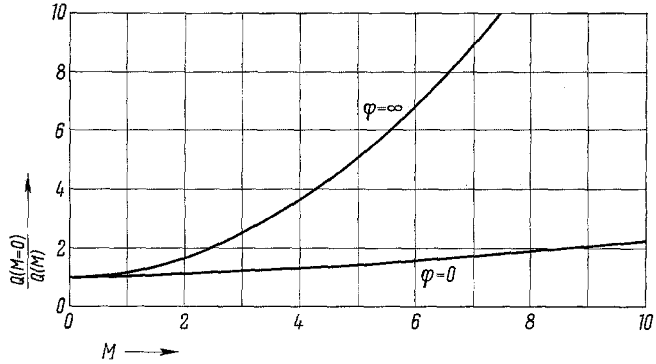

Figure 4
Pressure gradient versus M for $\varphi = 0$ and $\varphi = \infty$, square duct.

Flow in Arbitrary Symmetrical Duct for Large M

It has been seen for rectangular ducts that the velocity distribution for large M consists of a uniform core with a boundary layer near the walls in which the velocity changes rapidly. The purpose of the present section is to investigate how the cross-section form and wall conductivity affect the velocity distribution in the core. This question has been answered for non-conducting walls by Shercliff [3] and for circular ducts with walls of small conductivity by Shercliff [4].

Let the duct be as indicated in Figure 1, except symmetrical about the x-axis. Let the upper surface be described by $\eta = Y(\xi)$ the lower surface by $\eta = -Y(\xi)$. The problem is to solve

$$\nabla^2 V + M \frac{\partial B}{\partial \eta} = -1 , \tag{23}$$

$$\nabla^2 B + M \frac{\partial V}{\partial \eta} = 0 \tag{24}$$

with $V=0$ and $\partial B/\partial n + B/\varphi = 0$ at the wall. Let $Z_1 = M(V+B)$, $Z_2 = M(V-B)$. Adding and subtracting (23) and (24) gives

$$\nabla^2 Z_1 + M \frac{\partial Z_1}{\partial \eta} = - M , \tag{25}$$

$$\nabla^2 Z_2 - M \frac{\partial Z_2}{\partial \eta} = - M . \tag{26}$$

Singular perturbation theory, Levinson [1], says that at interior points and at points on the upper surface

$$Z_1 = Z_{1i} + O\left(\frac{1}{M^{1/2}}\right) ,$$

where Z_{1i} is the solution of

$$\frac{\partial Z_{1i}}{\partial \eta} = -1$$

which takes the given boundary values on the upper surface. There is a boundary layer near the lower surface and ends. Specifically, this is true when Z_1 is given on the boundary. In the present problem the value of Z_1 on the boundary is unknown but the above result should still be true, since if the exact solution of (23) and (24) were known the values which Z_1 takes on the boundary could be calculated and the problem formulated as a first boundary value problem. The analysis of Z_2 is similar, with Z_2 tending to the solution $\partial Z_{2i}/\partial\eta = 1$ which assumes the boundary values on the *lower* surface. This reasoning gives asymptotic solutions

$$Z_{1i} = Y(\xi) - \eta + Z_1(\xi,\ Y(\xi)) \,, \tag{27}$$

$$Z_{2i} = \eta + Y(\xi) + Z_2(\xi_1,\ -Y(\xi)) \,, \tag{28}$$

where $Z_1(\xi,\ Y(\xi))$ and $Z_2(\xi,\ -Y(\xi))$ are the values which the exact solutions of (25) and (26) assume on the boundaries. Since $Z_1 = M\ (V + B)$, $Z_2 = M\ (V - B)$ and V is zero on the boundaries, it must be the case that

$$Z_1(\xi,\ Y(\xi)) = M\ B(\xi,\ Y(\xi)) \,, \tag{29}$$

$$Z_2(\xi,\ -Y(\xi)) = -\ M\ B(\xi,\ -Y(\xi)) \,.$$

But by the symmetry of the boundaries, B is an odd function of η, therefore

$$Z_2(\xi,\ -Y(\xi)) = M\ B(\xi,\ Y(\xi)) \,. \tag{30}$$

Also, the velocity and induced field in the interior tend to

$$V_i = \frac{Z_{1i} + Z_{2i}}{2\ M} = B(\xi,\ Y(\xi)) + \frac{Y(\xi)}{M} \,, \tag{31}$$

$$B_i = \frac{Z_{1i} - Z_{2i}}{2\ M} = -\ \frac{\eta}{M} \,. \tag{32}$$

In order to find $B(\xi,\ Y(\xi))$ it is necessary to have a relation between quantities across the boundary layer. By integrating $\nabla^2 B + M\ \partial V/\partial\eta = 0$ over a small cylinder which extends through the boundary layer, and using Green's third identity, one finds

$$\frac{\partial B}{\partial n} - \frac{\partial B_i}{\partial n} + M\ (V - V_i)\cos(n,\ \eta) = 0 \,, \tag{33}$$

where n is the outward normal. This result is equivalent to imposing continuity of tangential components of the electric field across the boundary layer. Now using $V = 0$ and $\partial B/\partial n + B/\varphi = 0$ on the boundaries, it is found that

$$\frac{B}{\varphi} = -\frac{\partial B_i}{\partial n} - M_i \cos(n, \eta) \tag{34}$$

on the boundary. Observe that

$$\frac{\partial B_i}{\partial n} = \frac{\partial B_i}{\partial \xi} \cos(n, \xi) + \frac{\partial B_i}{\partial \eta} \cos(n, \eta) = -M^{-1} \cos(n, \eta) \tag{35}$$

whence

$$\frac{B}{\varphi} = [M^{-1} - (M\, B(\xi, Y(\xi)) + Y(\xi))] \cos(n, \eta) \tag{36}$$

on the boundary.

In particular this is true on the upper surface where $B = B(\xi, Y(\xi))$ and $\cos(n, \eta) = (1 + Y'^2)^{-1/2}$. Substituting these into (36) and solving for $B(\xi, Y(\xi))$ gives

$$B(\xi, Y(\xi)) = \frac{M^{-1} - Y}{M + \varphi^{-1} (1 + Y'^2)^{1/2}} . \tag{37}$$

With (37) and (31), V_i is solved –

$$V_i = \frac{1}{M} \left\{ \frac{Y + \varphi (1 + Y'^2)^{-1/2}}{1 + M \varphi (1 + Y'^2)^{-1/2}} \right\} . \tag{38}$$

This checks with (16) for the case $Y = 1$. Note, when the walls are non-conducting ($\varphi = 0$) that $V_i = Y(\xi)/M$ as shown by SHERCLIFF [3]. In this case the velocity distribution has the same shape as the cross-section of the duct. In fact this result is also true for non-symmetrical ducts with non-conducting walls, that is, if the upper surface is $\eta = Y_1(\xi)$ and the lower surface is $\eta = -Y_2(\xi)$ then $V_i = (Y_1(\xi) + Y_2(\xi))/2\,M$. On the other hand for perfectly conducting walls ($\varphi = \infty$), $V_i = 1/M^2$ which shows the velocity to be uniform in the core. Also, in this case, the dimensional velocity V_z is independent of viscosity – $V_z = -(\partial p/\partial z)/(\sigma B_0^2)$.

For a circular cross-section, $Y = (1 - \xi^2)^{1/2}$, a small calculation shows

$$V_i = \frac{1 + \varphi}{M} \frac{(1 - \xi^2)^{1/2}}{1 + M \varphi (1 - \xi^2)^{1/2}} .$$

This differs from SHERCLIFF [4] in the occurence of the factor $1 + \varphi$; SHERCLIFF restricted φ to be small. The volume flow through the circular duct is given by

$$\left.\begin{aligned}
Q &= \int_{-1}^{1} 2\,Y(\xi)\,V_i(\xi)\,d\xi \\
&= 4\int_{0}^{1} \frac{1+\varphi}{M}\,\frac{1-\xi^2}{1+M\,\varphi\,(1-\xi^2)^{1/2}}\,d\xi \\
&= 4\,\frac{1+\varphi}{M}\int_{0}^{\pi/2} \frac{\cos^3\theta}{1+M\,\varphi\cos\theta}\,d\theta \\
&= 4\,\frac{1+\varphi}{M}\left[\frac{\pi/4}{M\,\varphi} - \frac{1}{(M\,\varphi)^2} + \frac{\pi/2}{(M\,\varphi)^3} - \frac{1}{(M\,\varphi)^3}\,\frac{2}{1+M\,\varphi}\right. \\
&\qquad \left.\times \tan^{-1}\left(\frac{1-M\,\varphi}{1+M\,\varphi}\right)^{1/2} \Big/ \left(\frac{1-M\,\varphi}{1+M\,\varphi}\right)\right]^{1/2}.
\end{aligned}\right\} \tag{39}$$

Note that the expression for Q, (39), does *not* have a singularity at $M\,\varphi = 0$, the inverse powers are absorbed by the last term on the right. In fact when $M\,\varphi$ is small

$$Q = 4\,\frac{1+\varphi}{M}\left(\frac{2}{3} - \frac{3\,\pi}{16}\,M\,\varphi + \frac{8}{15}\,(M\,\varphi)^2 + \cdots\right).$$

In Figure 5 $(Q/\pi)\,[4\,M/(1+\varphi)]$ is plotted versus $M\,\varphi$. Q/π is the average (dimensionless) velocity. In Figure 6 V_i/V_{av} is plotted versus ξ with $M\,\varphi$ as parameter. The latter figure shows the effect of wall conductivity on the shape of the

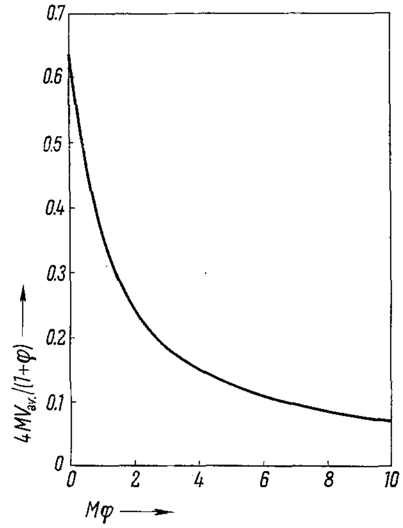

Figure 5
$4\,M\,V_{ar}/(1+\varphi)$ versus M for circular duct, large M.

velocity profile for the case of a circular duct. It should be noted in this case that the *shape* depends only on the product $M\varphi$.

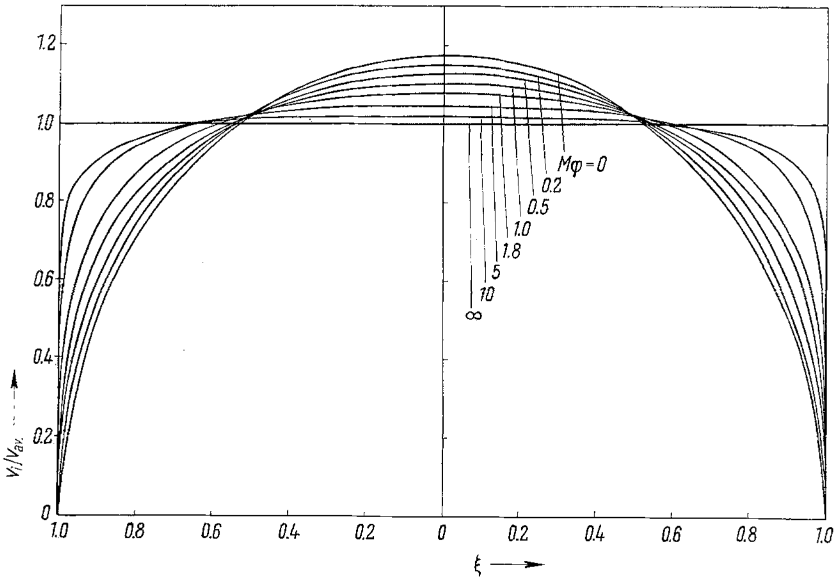

Figure 6
Velocity distribution in a circular duct with $M\varphi$ as parameter.

The Effect of the Free Charge on the Electric Field

In the introduction we stated that the free charge would be neglected in OHM's law and in the momentum equation, but would be retained in (4). We have seen that B_z and V_z can be determined without using (4), therefore $\boldsymbol{E}$ can be determined from OHM's law leaving (4) for the determination of ϱ_e. This is done simply by taking the divergence of (5), with the result

$$\varrho_e = -\varepsilon_0 \operatorname{div}(\boldsymbol{V} \times \boldsymbol{B}) . \tag{40}$$

We shall now check to see that ϱ_e was negligible in (5) and (6) for the class of problems which we have considered. We note that in this case (40) becomes

$$\varrho_e = \varepsilon_0 B_0 \frac{\partial V_z}{\partial x} . \tag{41}$$

For large M it is apparent that ϱ_e will be largest in the boundary layer. To be more specific it will be largest near that part of the boundary which is parallel to the applied magnetic field. SHERCLIFF [3] has shown that the boundary layer has thickness of the order of $a/\sqrt{M}$ on walls parallel to the applied magnetic field. This allows us to estimate $\partial V_z/\partial x$ by $V_{z\,av} \sqrt{M}/a$ at this point ($V_{z\,av}$ is the average or core velocity) which makes

$$\varrho_e = O\left(\varepsilon_0\, B_0\, V_{z\,av}\, \frac{\sqrt{M}}{a}\right). \tag{42}$$

We note that in equations (5) and (6) the terms with ϱ_e will be small compared to $\sigma\, V \times B$ and $j \times B$ respectively provided the inequality $\varrho_e/\sigma\, B_0 \ll 1$ is satisfied. With (42) this inequality can be put in the form

$$\frac{V_{z\,av}}{c} \ll \frac{\lambda\, R_c}{\sqrt{M}}, \tag{43}$$

where c is the speed of light, $\lambda = \mu_0\, \sigma\, \nu$ and $R_c = a\, c/\nu$ is a 'Reynolds' number based on the speed of light. As an example consider the case of mercury where $\lambda \simeq 10^{-7}$, $M = 100$ (the experimental upper limit) and $R_c = O(10^{15})$ (with $a = 1$ cm). In this case (43) becomes $V_{z\,av}/c \ll 10^7$ which is obviously always satisfied.

It should not be concluded from the previous paragraph that the effect of ϱ_e is completely negligible, for it has a dominant effect on the electric field. The free charge in the boundary causes the electric field to make a large increase on passing through this layer. This is best illustrated by a concrete example. Consider a long rectangular duct with non-conducting walls. Except in the boundary layer near the ends the solution is essentially that given by equations (14) and (15), with $\varphi = 0$. Using this solution we can calculate $\boldsymbol{E}$ from equation (5), that is

$$E_x = \frac{1}{\sigma\, \mu_0}\, \frac{\partial B_z}{\partial y} + V_z\, B_0, \tag{44}$$

$$E_y = -\frac{1}{\sigma\, \mu_0}\, \frac{\partial B_z}{\partial x}.$$

From (14) and (15) we get

$$E_x = \frac{a}{\sqrt{\nu\, \varrho\, \sigma}}\, \frac{\partial p}{\partial z} \left\{\frac{1}{M} - \frac{1}{\tanh M}\right\}, \tag{45}$$

$$E_y = 0.$$

It is seen that E_x is constant. One might suppose that E_x is constant all the way to the end of the duct, taking a jump through the surface of the wall. This is *not* the case, since for nonconducting walls B_z is zero at the wall, therefore $\partial B_z/\partial y$ is zero on the vertical wall, and equation (44) then shows that $E_x = 0$ on the vertical wall. We conclude that E_x varies from zero at the wall to the value given by (45) as we go through a boundary layer of thickness $a/\sqrt{M}$. This is due to the free charge in this boundary layer. (There is some question as to whether the free charge causes the variation in electric field or whether the variation in electric field causes the free charge.)

Conclusion

In the flow of conducting fluids through ducts, in general, the domain of the equations describing the flow is not the same as the domain of the fluid. It was found that the problem reduced to two sets of equations, one set, (11) and (12), in the fluid, and one set, (13), in the wall, with boundary conditions specified on the outer boundary of the wall and across the inner boundary of the wall. The known special cases (SHERCLIFF [3, 4]) where this problem reduces to an ordinary boundary value problem with boundary conditions given at the boundary of the fluid, follow as limiting cases of the above formulations. These are $\varphi \to 0$, $\varphi \to \infty$ and $h \to 0$.

In the bulk of the paper the flow through rectangular ducts was considered. The essential conclusion is that increasing the wall conductivity tends to decrease the average velocity if the pressure gradient is the same in the two cases. Stated differently, when the wall conductivity is increased, the pressure gradient must be increased in order to maintain the same mass flow.

The flow through arbitrarily shaped symmetrical ducts was considered for large M and arbitrary wall conductivity. It was found that when $\varphi \to \infty$ the velocity tends to be uniform except in a thin boundary layer along the wall.

In the final section the effect of the neglected free charge was considered. We found that free charge tends to accumulate near walls which are parallel to the applied field, but not enough to effect the velocity distribution.

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Zusammenfassung

In dieser Arbeit wird die Strömung einer elektrisch-leitenden Flüssigkeit durch ein gerades Rohr mit einem gleichförmigen, querlaufenden magnetischen Feld betrachtet. Das Problem wird unter Berücksichtigung des elektromagnetischen Feldes sowohl innerhalb als ausserhalb der Flüssigkeit formuliert. Besondere Aufmerksamkeit wird auf die Ableitung der Randbedingungen gerichtet. Es wird klargemacht, dass, wenn die Wände des Rohres endliche Leitfähigkeit haben, das Problem, abgesehen von Einzelfällen, kein gewöhnliches Randwertproblem darstellt. Eine Lösung wird für die Strömung durch ein rechteckiges Rohr mit unendlich fernen, dem äusseren Feld parallelen Wänden bei beliebiger Wandleitfähigkeit gefunden, ebenso für die Strömung durch ein endliches rechteckiges Rohr mit idealleitenden Wänden. Es gelingt, die asymptotische Geschwindigkeitsverteilung bei grosser Hartmannscher Zahl für ein dünnwandiges symmetrisches Rohr bei beliebiger Wandleitfähigkeit anzugeben.

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