Slow Broad Side Motion of a Flat Plate in a Viscous Liquid

By SUBHAS CHANDRA GUPTA, Agra, India[1])

1. Introduction

The problem of slow broad-side motion of a flat plate in viscous liquid has been solved in terms of a function φ which turns out to be the electrostatic potential of the plate kept at a constant potential in vacuum. The hydrodynamical problem thus reduces to a problem in electrostatics with known boundary conditions. The function φ satisfies LAPLACE's equation and the boundary conditions can be expressed in terms of φ. The problem corresponding to a disc of any arbitrary shape can be easily worked out by setting up an experiment based on this analogy.

The method has been applied to the case of a circular disc by introducing transforms and solving the problem as a boundary value problem in electrostatics. This problem has been also worked out by RAY[2]) by using an *ad-hoc* method, but the present method gives a systematic and logical approach to problems of this kind. It has a distinct advantage over RAY's method in which a correct form of the solution has to be guessed at the outset and later integral solutions have to be constructed which give desired discontinuity at the edges of the plate and this quite often is not easy.

2. The Equations of Motion, Their Solution and the Electrical Analogy

The hydrodynamical equations of slow viscous motion after neglecting the quadratic terms of inertia are:

$$\frac{\partial p'}{\partial x} = \mu \nabla^2 u , \tag{1}$$

$$\frac{\partial p'}{\partial y} = \mu \nabla^2 v , \tag{2}$$

$$\frac{\partial p'}{\partial z} = \mu \nabla^2 w , \tag{3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ,$$

[1]) Department of Mathematics, Agra College.

[2]) M. RAY, Phil. Mag. *21* (7), 553–558 (1936).

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 . \tag{4}$$

Let

$$u = z\,\frac{\partial \varphi}{\partial x}, \quad v = z\,\frac{\partial \varphi}{\partial y}, \quad w = z\,\frac{\partial \varphi}{\partial z} - \varphi . \tag{5}$$

Equation (4) now reduces to

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 . \tag{6}$$

From equations (1), (2) and (3) we find that

$$p' = p_0 + 2\,\mu\,\frac{\partial \varphi}{\partial z}, \tag{7}$$

p_0 being a constant. The plate coincides with the plane $z = 0$ and is bounded by the curve $z = 0$, $f(x, y) = 0$ and is moving in the direction of the z-axis with a small velocity V. The boundary conditions are

$$u = v = 0 , \quad w = V$$

on the plate, i.e., within

$$z = 0 , \quad f(x, y) = 0 ,$$

and

$$u = v = w = 0 ,$$

at infinity. Thus the conditions expressed in terms of φ are

$$\varphi = -V ,$$

on the plate, i.e., within

$$z = 0 , \quad f(x, y) = 0 ,$$

and φ must tend to zero at infinity in a manner such that

$$z\,\frac{\partial \varphi}{\partial x}, \quad z\,\frac{\partial \varphi}{\partial y}, \quad \text{and} \quad z\,\frac{\partial \varphi}{\partial z}$$

may also tend to zero. Also we see from (5) that on the plane $z = 0$, $u = v = 0$, i.e., there is no flow in the plane $z = 0$. The flow of the fluid is normal to the plane $z = 0$, and so the pressure throughout this plane must be constant. Therefore from (7)

$$\frac{\partial \varphi}{\partial z} = 0 , \quad \text{on} \quad z = 0 , \tag{8}$$

outside the plate. Thus the problem may be briefly restated as follows:

$$\nabla^2 \varphi = 0 , \tag{9}$$

$$\varphi = -V\,, \quad \text{within} \quad z = 0\,, \quad f(x, y) = 0\,, \tag{10}$$

$$\frac{\partial \varphi}{\partial z} = 0\,, \quad \text{outside} \quad z = 0\,, \quad f(x, y) = 0\,, \tag{11}$$

and

$$z\frac{\partial \varphi}{\partial x}\,, \quad z\frac{\partial \varphi}{\partial y}\,, \quad z\frac{\partial \varphi}{\partial z} \quad \text{and} \quad \varphi$$

must all tend to zero at infinity.

This problem is same as that of determining the electrostatic potential of a conducting electrified disc in the plane $z = 0$, bounded by the curve $f(x, y) = 0$, kept at a constant potential $-V$ in vacuum.

3. The Case of a Circular Disc

As an example of the above general treatment we take the case of a circular disc. Without loss of generality we take the radius of the disc to be unity and z-axis centrally perpendicular to it. Transforming the equation (9) in cylindrical coordinates (r, θ, z) we get

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \times \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0\,. \tag{12}$$

Here we have omitted the term $\partial^2\varphi/\partial\theta^2$ due to symmetry about the z-axis. The boundary conditions are

$$\left.\begin{aligned} \varphi &= -V \quad (0 \leqq r < 1)\,, \qquad (13)\\ \frac{\partial \varphi}{\partial z} &= 0 \quad (r > 1)\,. \qquad (14) \end{aligned}\right\} \text{ on } z = 0\,.$$

Introducing Hankel transform defined by the relation

$$\bar{\varphi}(p) = \int_0^\infty \varphi\, r\, J_0(p\, r)\, dr\,, \tag{15}$$

we find that

$$\int_0^\infty \nabla^2 \varphi\, r\, J_0(p\, r)\, dr = \left[\frac{d^2}{dz^2} - p^2\right] \bar{\varphi}(p)\,.$$

Equation (12) becomes

$$\frac{d^2\bar{\varphi}(p)}{dz^2} = p^2\, \bar{\varphi}(p)\,. \tag{16}$$

Since the boundary conditions (13) and (14) are of a mixed type, we invert $\bar{\varphi}$ for φ before satisfying them.

In view of the symmetry of the problem it will be sufficient to consider the region $z > 0$. Since the potential must vanish at infinity, the appropriate solution of (16) is

$$\bar{\varphi}(p) = A(p)\, e^{-p z} \quad (z > 0)\,, \tag{17}$$

$A(p)$ being an unknown function of p.

Inversion for φ gives

$$\varphi = \int_0^\infty A(p)\, e^{-p z}\, p\, J_0(r\, p)\, dp\,. \tag{18}$$

If we insert (18) in the boundary conditions (13) and (14) we get the following dual integral equations:

$$\int_0^\infty p\, A(p)\, J_0(r\, p)\, dp = -V \quad (0 \leqq r < 1)\,, \tag{19}$$

$$\int_0^\infty p^2\, A(p)\, J_0(r\, p)\, dp = 0 \quad (r > 1)\,. \tag{20}$$

TITCHMARSH[3]) and BUSBRIDGE[4]) have considered dual integral equations of this type and their solution is

$$A(p) = -\frac{2}{\pi} \times \frac{V}{p^2} \sin p\,. \tag{21}$$

Thus the required solution is

$$\varphi = -\frac{2\,V}{\pi} \int_0^\infty e^{-p z}\, J_0(r\, p)\, \frac{\sin p}{p}\, dp\,. \tag{22}$$

In this particular case Legendre transform can also be used employing oblate spheroidal coordinates. The result can be expressed in a neat form. Suppose (ξ, η) are oblate spheroidal coordinates related to the cylindrical coordinates (r, z) by

$$z = \xi\, \eta\,,$$

and

$$r = (1 - \xi^2)^{1/2}\, (1 + \eta^2)^{1/2}\,.$$

Transforming LAPLACE's equation into oblate spheroidal coordinates and employing Legendre transform of φ defined by

$$\bar{\varphi}(n) = \int_{-1}^{+1} \varphi\, P_n(\xi)\, d\xi\,,$$

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3) E. C. TITCHMARSH, *Theory of Fourier Integrals* (Oxford 1937), p. 335.

4) I. W. BUSBRIDGE, Proc. Lond. math. Soc. *44* (2), 115–129 (1938).

we find that[5])

$$\varphi = -\frac{2V}{\pi} \cot^{-1}\eta . \tag{23}$$

This solution can be shown to agree exactly with the more complicated form given by (22).

The resistance of the fluid to the motion of the disc calculated from either (22) or (23) is $16 V\mu$. If the radius of the disc be a the resistance can be shown to be $16 V a \mu$, which is identical with that obtained by RAY[2]).

I thank Dr. S. D. NIGAM for suggesting this problem to me.

Zusammenfassung

Das Problem der langsamen Querbewegung einer ebenen Platte in einer zähen Flüssigkeit ist zurückgeführt worden auf ein Problem der Elektrostatik. Die Geschwindigkeitskomponenten und der Druck können berechnet werden aus einer Funktion φ, die als die gleiche erscheint wie das elektrostatische Potential einer Platte unter konstantem Potential im Vakuum. Die Strömungsverhältnisse an einer Platte vom allgemeinen Umriss können mit Hilfe der Experimente auf Grund dieser Analogie bestimmt werden. Der Fall einer kreisrunden Platte wurde analytisch behandelt mit Hilfe der Abbildungen.

(Received: July 3, 1956.)

Charge and Discharge of a Non-Linear Condenser Through a Linear Non-Dissipative Inductance

By ASOKENDU MOZUMDER, Kharagpur, India[1])

Introduction

The capacity of a condenser with semi-conducting di-electric depends upon the voltage across it. This non-linearity arises from the processes of diffusion and recombination which occur within the di-electric, and is important at voltages below the breakdown value. At higher voltages, normal metallic conduction takes place and the non-linearity becomes inappreciable. Considering the building up of space charge at a blocking electrode by free charge carriers MACDONALD [1][2]) has deduced the formulae

$$C = C_0 \frac{\sinh \alpha V}{\alpha V} \text{ }^{3)} \tag{1a}$$

[5]) C. J. TRANTER, *Integral Transforms in Mathematical Physics* (Methuen, 1951), p. 100.

[1]) Indian Institute of Technology, Department of Physics and Meteorology.

[2]) Numbers in brackets refer to References, page 279.

[3]) Note that in [1] the differential capacity (dQ/dV) is deduced, whereas here the *corresponding* integral capacity (Q/V) is used since it is the latter capacity which appears in KIRCHHOFF's circuit equation (cf. [6]).